Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Discrete Mathematics 271 (2003) 303–310
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

On a tree graph defined by a set of cycles

Xueliang Li[a], Víctor Neumann-Lara[b], Eduardo Rivera-Campo[c,1]

[a] *Center for Combinatorics, Nankai University, People's Republic of China*
[b] *Instituto de Matemáticas, Universidad Nacional Autónoma de México, Mexico*
[c] *Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa, Mexico*

Received 30 January 2002; received in revised form 21 January 2003; accepted 7 February 2003

Abstract

For a set C of cycles of a connected graph G we define $T(G,C)$ as the graph with one vertex for each spanning tree of G, in which two trees R and S are adjacent if $R \cup S$ contains exactly one cycle and this cycle lies in C. We give necessary conditions and sufficient conditions for $T(G,C)$ to be connected.

Keywords: Tree graph; Cycle space; Δ^*-dense

1. Introduction

The *tree graph* of a connected graph G is the graph $T(G)$ whose vertices are the spanning trees of G, in which two trees R and S are adjacent if S can be obtained from R by deleting an edge r of R and adding another edge s of S. In [2] Cummins proved that $T(G)$ is Hamiltonian and therefore connected, and in [5] Holzmann and Harary proved the analogue result for matroids.

Later, two variations of the tree graph were studied: The *adjacency tree graph* $T_a(G)$ and the *leaf-exchange tree graph* $T_l(G)$ which are spanning subgraphs of $T(G)$. Let R and S be adjacent trees in $T(G)$ and r and s be edges of G such that $S=(R-r)+s$. In $T_a(G)$, R and S are adjacent only if r and s are adjacent edges of G, whereas in $T_l(G)$, R and S are adjacent only if r and s are leaf edges of R and S, respectively. Zhang and Chen [6] proved that if G is a connected graph, not necessarily simple but with no loops, then $T_a(G)$ is ρ-connected, where ρ is the dimension of the cycle space of

[1] Part of the research was done while this author was on sabbatical leave at the Universitat Politècnica de Catalunya with grants by MECD (Spain) and CONACYT (Mexico).

E-mail address: erc@xanum.uam.mx (E. Rivera-Campo).

0012-365X/03/$ - see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/S0012-365X(03)00132-8

G, and Heinrich and Liu [4] proved that $T_a(G)$ is 2ρ-connected for any simple graph G. In [3], Harary et al. proved that $T_l(G)$ is connected for every 2-connected graph G, and in [1] Broersma and Li characterized those graphs for which $T_l(G)$ is connected.

For any set of cycles C of a connected graph G we define $T(G,C)$ as the spanning subgraph of $T(G)$, in which two trees R and S are adjacent if they are adjacent in $T(G)$ and the unique cycle σ contained in $R\cup S$ lies in C. In this article, we present necessary conditions and sufficient conditions for $T(G,C)$ to be connected.

Throughout this article we shall denote with the same characters graphs and their sets of edges. The symmetric difference $\sigma\Delta\tau$ of two cycles σ and τ of a graph G is the subgraph of G induced by the edge set $(\sigma\cup\tau)\backslash(\sigma\cap\tau)$.

2. Necessary conditions

For any graph G we denote by $\Gamma(G)$ the cycle space of G, and for any set C of cycles of G we denote by $\operatorname{Span} C$ the subspace of $\Gamma(G)$ spanned by C. A cycle σ is *cyclically spanned* by C if there are cycles $\tau_1,\tau_2,\dots,\tau_m\in C$ such that $\sigma=\tau_1\Delta\tau_2\Delta\cdots\Delta\tau_m$ and for $j=1,2,\dots,m$, $\tau_1\Delta\tau_2\Delta\cdots\Delta\tau_j$ is a cycle of G.

Lemma 2.1. *Let C be a set of cycles of a connected graph G and let $Q_0,Q_1,\dots,Q_n$ be a path in $T(G,C)$ with length $n\geqslant 1$. For $i=1,2,\dots,n$, denote by τ_i the unique cycle contained in $Q_{i-1}\cup Q_i$. If σ is a cycle of G such that $\sigma\subset Q_0+e$ for some $e\in Q_n$, then σ is cyclically spanned by $\{\tau_1,\tau_2,\dots,\tau_n\}$.*

Proof. If $n=1$, then $e\in Q_1$ and $\sigma\subset Q_0\cup Q_1$ which implies $\sigma=\tau_1$. We proceed by induction assuming $n>1$ and that the result holds for any path in $T(G,C)$ with length less than n.

Case 1: $e\in Q_t$ for some $t<n$.

Since $Q_0,Q_1,\dots,Q_t$ is a path in $T(G,C)$ with length $t<n$, $\sigma\subset Q_0+e$ and $e\in Q_t$, then by induction σ is cyclically spanned by $\{\tau_1,\tau_2,\dots,\tau_t\}$.

Case 2: $\sigma\subset Q_t+e$ for some $t\geqslant 1$.

Since $Q_t,Q_{t+1},\dots,Q_n$ is a path in $T(G,C)$ with length $n-t<n$, $\sigma\subset Q_t+e$ and $e\in Q_n$, then by induction σ is cyclically spanned by $\{\tau_{t+1},\tau_{t+2},\dots,\tau_n\}$.

Case 3: $\sigma\not\subseteq Q_t+e$ for $t=1,2,\dots,n$ and $e\notin Q_t$ for $t=0,1,\dots,n-1$.

Let a and b be the edges of G such that Q_1 is obtained from Q_0 by deleting a and adding b. Clearly $a,b\in\tau_1$, and since σ is contained in Q_0+e but not in $Q_1+e=((Q_0-a)+b)+e$, then a also lies in σ.

Since $\sigma\subset Q_0+e$ and $\tau_1\subset Q_0\cup Q_1$, then $\sigma\cup\tau_1\subset(Q_0\cup Q_1)+e=(Q_1+a)+e$. Since $a\in\sigma\cap\tau_1$, then $\sigma\Delta\tau_1\subset Q_1+e$.

In this case $Q_1,Q_2,\dots,Q_n$ is a path in $T(G,C)$ with length $n-1$, $\sigma\Delta\tau_1\subset Q_1+e$ and $e\in Q_n$. By induction $\sigma\Delta\tau_1$ is cyclically spanned by $\{\tau_2,\tau_3,\dots,\tau_n\}$. Since $\sigma=(\sigma\Delta\tau_1)\Delta\tau_1$, then σ is cyclically spanned by $\{\tau_1,\tau_2,\dots,\tau_n\}$. $\square$

Theorem 2.2. *Let C be a set of cycles of a connected graph G. If $T(G,C)$ is connected, then C spans the cycle space of G.*

Proof. Let σ be any cycle of G and e be an edge of σ. Let R and S be spanning trees of G such that $\sigma \subset R + e$ and $e \in S$. Since $T(G, C)$ is connected, there is a path $R = Q_0, Q_1, \ldots, Q_n = S$ connecting R and S in $T(G, C)$. By Lemma 2.1, $\sigma \in \mathrm{Span}\{\tau_1, \tau_2, \ldots, \tau_n\}$, where for $i = 1, 2, \ldots, n$, $\tau_i \in C$ is the unique cycle of G contained in $Q_{i-1} \cup Q_i$. □

Consider the complete graph G with vertex set $\{u_1, u_2, u_3, u_4\}$. Let C consist of the cycles $u_1u_2u_3u_4$, $u_1u_2u_4$ and $u_1u_4u_3$. The set C is a basis of the cycle space of G, nevertheless the path $u_1u_3u_2u_4$ is an isolated vertex in $T(G, C)$. This shows that the condition on Theorem 2.2 is not a sufficient condition for $T(G, C)$ to be connected.

3. Sufficient conditions

A *unicycle* of a connected graph G is a connected spanning subgraph of G that contains exactly one cycle.

Let C be a set of cycles of a connected graph G. A cycle σ of G satisfies property Δ^* (with respect to C) if for any unicycle U of G containing σ there are two cycles $\tau, \upsilon \in C$ contained in $U + e$ for some edge e of G such that $\sigma = \tau \Delta \upsilon$.

Lemma 3.1. *Let C be a set of cycles of a connected graph G and σ be a cycle of G satisfying property Δ^*. The graph $T(G, C)$ is connected if and only if $T(G, C \cup \{\sigma\})$ is connected.*

Proof. Clearly $T(G, C \cup \{\sigma\})$ is connected whenever $T(G, C)$ is connected.

Let R and S be spanning trees of G, adjacent in $T(G, C \cup \{\sigma\})$ and $\omega \in C \cup \{\sigma\}$ be the unique cycle contained in $R \cup S$. If $\omega \in C$, then R and S are also adjacent in $T(G, C)$.

Suppose now $\omega = \sigma$. Since σ satisfies property Δ^* and $R \cup S$ is a unicycle of G that contains σ, there are two cycles $\tau, \upsilon \in C$ contained in $(R \cup S) + e$ for some edge e of G such that $\sigma = \tau \Delta \upsilon$.

Let a and b be the edges of σ such that $S = (R - a) + b$. Four cases are considered.

Case 1: $a \in \tau \backslash \upsilon$ and $b \in \upsilon \backslash \tau$.

Let Q be the spanning tree of G obtained from R by deleting the edge a and adding the edge e. Since $S = (Q - e) + b$, both pairs of trees R and Q and Q and S are adjacent in $T(G)$.

Since $\tau \subset (R \cup S) + e = (R + b) + e$ and $b \notin \tau$, then $\tau \subset R + e = R \cup Q$; therefore R and Q are adjacent in $T(G, \{\tau\})$.

Since $\upsilon \subset (R \cup S) + e = (S + a) + e$ and $a \notin \upsilon$, then $\upsilon \subset S + e = S \cup Q$; therefore S and Q are adjacent in $T(G, \{\upsilon\})$.

In this case R and S are connected in $T(G, \{\tau, \upsilon\}) \subset T(G, C)$ by a path of length two.

Case 2: $b \in \tau \backslash \upsilon$ and $a \in \upsilon \backslash \tau$.

Interchange τ and υ in Case 1.

Case 3: $a, b \in \tau \backslash \upsilon$.

Let c be an edge in $\upsilon\backslash\tau$ and let $Q_1=(R-c)+e$ and $Q_2=(S-c)+e$. Since $Q_2=(Q_1-a)+b$, all three pairs R and Q_1, Q_1 and Q_2 and Q_2 and S are adjacent in $T(G)$.

Since $\upsilon\subset(R\cup S)+e=(R+b)+e$ and $b\notin\upsilon$, then $\upsilon\subset R+e=R\cup Q_1$; therefore R and Q_1 are adjacent in $T(G,\{\upsilon\})$.

Since $\tau\subset(R\cup S)+e=(Q_1\cup Q_2)+c$ and $c\notin\tau$, then $\tau\subset Q_1\cup Q_2$; therefore Q_1 and Q_2 are adjacent in $T(G,\{\tau\})$.

Since $\upsilon\subset(R\cup S)+e=(S+a)+e$ and $a\notin\upsilon$, then $\upsilon\subset S+e=Q_2\cup S$; therefore Q_2 and S are adjacent in $T(G,\{\upsilon\})$.

In this case, R and S are connected in $T(G,\{\tau,\upsilon\})\subset T(G,C)$ by a path of length three.

Case 4: $a,b\in\upsilon\backslash\tau$.

Interchange τ and υ in Case 3. □

Let G be a connected graph. For any set C of cycles of G, we define the closure $cl_G(C)$ of C in G as the set of cycles obtained from C by recursively adding new cycles of G that satisfy property Δ^* until no such cycle remains.

Theorem 3.2. *For any connected graph G and any set C of cycles of G, the closure of C in G is well defined.*

Proof. Suppose the result is false and let C' and C'' be two different sets of cycles of G obtained from C by recursively adding new cycles of G that satisfy property Δ^* until no such cycle remains. Let $\sigma_1,\sigma_2,\ldots,\sigma_n$ and $\tau_1,\tau_2,\ldots,\tau_m$ denote the sequences of cycles added to C while obtaining C' and C'', respectively.

Without loss of generality we assume $C'\subsetneq C''$ and let σ_k be the first cycle in the sequence $\sigma_1,\sigma_2,\ldots,\sigma_n$ which is not in C''. Let $D=C\cup\{\sigma_1,\sigma_2,\ldots,\sigma_{k-1}\}$; since σ_k satisfies property Δ^* with respect to D and $D\subset C''$, then σ_k satisfies property Δ^* with respect to C'' which is not possible since C'' is Δ^*-closed and $\sigma_k\notin C''$. □

Theorem 3.3. *Let C be a set of cycles of a connected graph G. The graph $T(G,C)$ is connected if and only if $T(G,cl_G(C))$ is connected.*

Proof. Let $\sigma_1,\sigma_2,\ldots,\sigma_n$ be the sequence of cycles added to C while obtaining $cl_G(C)$. Set $C_0=C$ and for $i=0,1,\ldots,n-1$, let $C_{i+1}=C_i\cup\{\sigma_{i+1}\}$. Clearly $C_n=cl_G(C)$ and by Lemma 3.1, $T(G,C_i)$ is connected if and only if $T(G,C_{i+1})$ is connected. □

A set of cycles C of a connected graph G is *Δ^*-dense in G* if $cl_G(C)$ is the set of all cycles of G.

Corollary 3.4. *If C is a Δ^*-dense set of cycles in a connected graph G, then $T(G,C)$ is connected.*

Proof. If C is Δ^*-dense, then $T(G,cl_G(C))=T(G)$ which is always connected. By Theorem 3.3, $T(G,C)$ is connected. □

We know no example of a connected graph G and a set of cycles C of G such that $T(G,C)$ is connected but C is not Δ^*-dense in G.

4. Δ^*-dense sets of cycles

In this section, we present two examples of sets of cycles which are Δ^*-dense.

Theorem 4.1. *For any* 2*-connected plane graph* G, *the set* C *of internal faces of* G *is* Δ^**-dense in* G.

Proof. Let σ be a cycle of G and k be the number of edges of G contained in the interior of σ. If $k=0$, then $\sigma \in C \subset cl_G(C)$. We proceed by induction assuming $k \geqslant 1$ and that if α is a cycle of G whose interior contains fewer than k edges of G, then $\alpha \in cl_G(C)$.

Let U be a unicycle of G containing σ. For each vertex w of G let U_w be the minimal path contained in U that connects w to σ and denote by $w(\sigma)$ the unique vertex of U_w that lies in σ.

Since $k \geqslant 1$ and G is 2-connected, there is an edge $e=uv$ of G, contained in the interior of σ, such that $u(\sigma) \neq v(\sigma)$. Let L and R the two paths contained in σ, joining $u(\sigma)$ and $v(\sigma)$ and let $\tau=(U_u \cup U_v \cup L)+e$ and $\upsilon=(U_u \cup U_v \cup R)+e$. Since $u(\sigma) \neq v(\sigma)$, then U_u and U_v are disjoint paths and therefore τ and υ are cycles of G contained in $U+e$. Moreover, since G is a plane graph and e is contained in the interior of σ, all edges of $U_u \cup U_v$ are also contained in the interior of σ; this implies that the interiors of τ and υ are contained in the interior of σ and therefore contain fewer than k edges of G. By induction τ and υ must be in $cl_G(C)$.

Since $\sigma = L \cup R = ((U_u \cup U_v \cup L)+e)\Delta((U_u \cup U_v \cup R)+e) = \tau \Delta \upsilon$, then σ satisfies property Δ^* with respect to $cl_G(C)$ and therefore $\sigma \in cl_G(C)$. □

Corollary 4.2. *If* C *is the set of internal faces of a* 2*-connected plane graph* G, *then* $T(G,C)$ *is connected.*

Theorem 4.3. *Let* $e=uv$ *be an edge of a* 2*-connected graph* G. *If* C_e *is the set of cycles of* G *that contain the edge* e, *then* C_e *is* Δ^**-dense in* G.

Proof. For every path L in G we denote by $l(L)$ the length of L. Assume the result is false and for each cycle α of G, not in $cl_G(C_e)$, let L_α and R_α be disjoint paths of G connecting u and v to α, respectively, and such that $l(L_\alpha) < l(L)$, or $l(L_\alpha) = l(L)$ and $l(R_\alpha) \leqslant l(R)$ for any pair L and R of disjoint paths of G that connect u and v to α, respectively.

Choose $\sigma \in \Gamma(G) \backslash cl_G(C_e)$ such that $l(L_\sigma) < l(L_\alpha)$, or $l(L_\sigma) = l(L_\alpha)$ and $l(R_\sigma) \leqslant l(R_\alpha)$ for any cycle $\alpha \in \Gamma(G) \backslash cl_G(C_e)$. Let $u = u_0, u_1, \ldots, u_n$ be the path L_σ and $v = v_0, v_1, \ldots, v_m$ be the path R_σ.

Let U be a unicycle of G containing σ. As in Theorem 4.1, for each vertex w of G let U_w be the minimal path contained in U that connects w to σ and denote by $w(\sigma)$ the unique vertex of U_w that lies in σ.

Case 1: $u(\sigma) \neq v(\sigma)$.

Denote by A and B the two paths contained in σ joining $u(\sigma)$ and $v(\sigma)$ and let $\tau = (U_u \cup U_v \cup A) + e$ and $\upsilon = (U_u \cup U_v \cup B) + e$. Since $u(\sigma) \neq v(\sigma)$, then U_u and U_v are disjoint paths and therefore τ and υ are cycles of G contained in $U + e$. Since the edge e belongs to both cycles τ and υ, then $\tau, \upsilon \in C_e$ and since $\tau \Delta \upsilon = ((U_u \cup U_v \cup A) + e)\Delta((U_u \cup U_v \cup B) + e) = A \cup B = \sigma$, then σ satisfies property Δ^* with respect to C which is a contradiction.

Case 2: $u(\sigma) = v(\sigma)$.

Since L_σ and R_σ are disjoint paths, either $u_n \neq u(\sigma)$ or $v_m \neq v(\sigma)$.

Subcase 2.1: $u_n \neq u(\sigma)$.

Since $u_n(\sigma) = u_n \neq u(\sigma) = u_0(\sigma)$, there is an edge $f = u_i u_{i+1}$ in L_σ such that $u_i(\sigma) = u(\sigma)$ and $u_{i+1}(\sigma) \neq u(\sigma)$. In this case let $\tau = (U_{u_i} \cup U_{u_{i+1}} \cup Q) + f$ and $\upsilon = (U_{u_i} \cup U_{u_{i+1}} \cup R) + f$, where Q and R are the two paths contained in σ, joining $u_i(\sigma)$ and $u_{i+1}(\sigma)$. Since U_{u_i} and $U_{u_{i+1}}$ are disjoint paths, τ and υ are cycles of G.

Since $u = u_0, u_1, \ldots, u_i$ is a path in G, with length $i < n = l(L_\sigma)$, joining u to both cycles τ and υ, then $l(L_\tau) < l(L_\sigma)$ and $l(L_\upsilon) < l(L_\sigma)$. By the choice of σ, both cycles τ and υ are in $cl_G(C_e)$. Since τ and υ are contained in $U + f$ and $\sigma = \tau \Delta \upsilon$, then σ satisfies property Δ^* with respect to $cl_G(C_e)$ which is a contradiction.

Subcase 2.2: $u_n = u(\sigma)$ and $v_m \neq v(\sigma)$.

Since $v_m(\sigma) = v_m \neq v(\sigma) = v_0(\sigma)$, there is an edge $g = v_i v_{i+1}$ in R_σ such that $v_i(\sigma) = v(\sigma)$ and $v_{i+1}(\sigma) \neq v(\sigma)$. In this case, let $\tau = (U_{v_i} \cup U_{v_{i+1}} \cup Q) + g$ and $\upsilon = (U_{v_i} \cup U_{v_{i+1}} \cup R) + g$, where Q and R are the two paths contained in σ, joining $v_i(\sigma)$ and $v_{i+1}(\sigma)$. Since U_{v_i} and $U_{v_{i+1}}$ are disjoint paths, τ and υ are cycles of G.

Since $u_n = u(\sigma) = v(\sigma) = v_i(\sigma)$, then u_n lies in $U_{v_i} \subset \tau \cap \upsilon$. This implies that $u = u_0, u_1, \ldots, u_n$ is a path of length $n = l(L_\sigma)$ that joins u to τ and to υ. Therefore, $l(L_\tau) \leqslant l(L_\sigma)$ and $l(L_\upsilon) \leqslant l(L_\sigma)$. Since $v = v_0, v_1, \ldots, v_i$ is a path in G, with length $i < m$, joining v to both cycles τ and υ, then $l(R_\tau) \leqslant i < m = l(R_\sigma)$ and $l(R_\upsilon) \leqslant i < m = l(R_\sigma)$. By the choice of σ, both cycles τ and υ must be in $cl_G(C_e)$. Since τ and υ are contained in $U + g$ and $\sigma = \tau \Delta \upsilon$, then σ satisfies property Δ^* with respect to $cl_G(C_e)$ which, again, is a contradiction. □

Corollary 4.4. *Let e be an edge of a 2-connected graph G. If C_e is the set of cycles of G that contain the edge e, then $T(G, C)$ is connected.*

5. The basis graph of a binary matroid

A binary matroid is a matroid M such that for any two circuits τ and υ, the symmetric difference $\tau \Delta \upsilon$ contains a circuit. A matroid is loopless if it has no circuit consisting of a single element.

The *basis graph* of a binary matroid M is the graph $B(M)$ whose vertices are the basis of M, in which two basis R and S are adjacent if S can be obtained from R by deleting an element r of R and adding an another element s of S.

For any set C of circuits of a binary matroid M, we define a graph $B(M,C)$ in which two basis R and S are adjacent if they are adjacent in $B(M)$ and the unique circuit of M contained in $R\cup S$ lies in C.

A *unicircuit* of a loopless binary matroid M is a set obtained from a basis of M by adding a new element. Let C be a set of circuits of a loopless binary matroid M. A circuit σ of M satisfies property Δ^* (with respect to C) if for any unicircuit U of M containing σ, there are two circuits $\tau,\upsilon\in C$ contained in $U+e$ for some element e of M such that $\sigma=\tau\Delta\upsilon$.

As for a set of cycles in a graph, we can define the closure $cl_M(C)$ of a set of circuits C in a loopless binary matroid M as the set of circuits of M obtained from C by adding new circuits of M that satisfy property Δ^* until no such circuit remains. A set of circuits C is *Δ^*-dense* in M if $cl_M(C)$ contains every circuit of M.

The following results can be proved in an analogous way as the corresponding results for graphs.

Theorem 5.1. *Let C be a set of circuits of a loopless binary matroid M. If $B(M,C)$ is connected, then C spans the circuit space of M.*

Theorem 5.2. *If C is a Δ^*-dense set of circuits in a loopless binary matroid M, then $B(M,C)$ is connected.*

Let F be graph and X be a set of vertices of F; we denote by $F[X]$ the subgraph of F induced by X. For any disjoint sets X and Y of vertices of F let $[X,Y]$ denote the set of edges of F joining a vertex in X with a vertex in Y. A *bond* of a connected graph F is a set $[X,\bar{X}]$ such that both graphs $F[X]$ and $F[\bar{X}]$ are connected, where $\bar{X}=V(F)\backslash X$.

Another example of a Δ^*-dense set of circuits worth to mention is the following: Let G be a 2-connected graph and $M(G)$ be the cographic matroid of G, where the circuits are the bonds of G and the basis are the complements of the spanning trees of G.

Let $v_1,v_2,\dots,v_n$ denote the vertices of G and for $i=1,2,\dots,n$ let τ_i be the set of edges incident with v_i. Since G is 2-connected, τ_i is a bond of G for $i=1,2,\dots,n$; let $C=\{\tau_1,\tau_2,\dots,\tau_n\}$. The graph $B(M(G),C)$ is isomorphic to the leaf exchange graph $T_l(G)$ and therefore it is connected. We claim that C is Δ^*-dense in $M(G)$. Moreover, the set $C_n=\{\tau_1,\tau_2,\dots,\tau_{n-1}\}$ is Δ^*-dense in $M(G)$.

Proof of claim. Let $\sigma=[X,\bar{X}]$ be any bond of G and assume without loss of generality that $v_n\in\bar{X}$. If $|X|=1$, then $\sigma\in C_n\subset cl_{M(G)}(C_n)$. We proceed by induction assuming $|X|>1$ and that if $\alpha=[Y,\bar{Y}]$ is any bond of G with $v_n\in\bar{Y}$ and $|Y|<|X|$, then $\alpha\in cl_{M(G)}(C_n)$.

Let U be a unicircuit of $M(G)$ containing σ and let B be a basis of $M(G)$ such that $U=B\cup\{x\}$ for some edge x of G. Then $U=\bar{T}+x$ for some edge x of T, where T is the spanning tree of G such that $B=\bar{T}$. Since $\sigma\subset\bar{T}+x$, then x is the only edge of T contained in σ and therefore $T[X]$ and $T[\bar{X}]$ are spanning trees of $G[X]$ and $G[\bar{X}]$, respectively.

For each edge $c \in \sigma$ let u_c denote the end of c in X. Since G is 2-connected and $|X| > 1$, there are two edges $a, b \in \sigma$ such that $u_a \neq u_b$. Let P be the unique path contained in $T[X]$ that joins u_a and u_b and let e be any edge of P.

Let X_a and X_b denote the sets of vertices in X which are connected in $T[X] - e$ to u_a and to u_b, respectively and let $\tau = [X_a, \bar{X} \cup X_b]$ and $\upsilon = [X_b, \overline{X} \cup X_a]$. Since $\bar{X} \cup X_b = \overline{X_a}$, $\bar{X} \cup X_a = \overline{X_b}$ and $T[X_a]$, $T[X_b]$, $(T[\bar{X}] \cup T[X_a]) + a$ and $(T[\bar{X}] \cup T[X_b]) + b$ are spanning trees of $G[X_a]$, $G[X_b]$, $G[\bar{X} \cup X_a]$ and $G[\bar{X} \cup X_b]$, respectively, then τ and υ are bonds of G.

Since $\tau = [X_a, \bar{X} \cup X_b] = [X_a, \bar{X}] \cup [X_a, X_b]$, $[X_a, \bar{X}] \subset \sigma \subset \bar{T} + x$ and $[X_a, X_b] \subset \overline{T[X]} + e \subset \bar{T} + e$, then $\tau \subset (\bar{T} + x) + e = U + e$. Analogously $\upsilon \subset U + e$. By induction $\tau, \upsilon \in cl_{M(G)}(C_n)$, since $|X_a| < |X|$ and $|X_b| < |X|$. Notice that

$$\begin{aligned}
\tau \Delta \upsilon &= [X_a, \bar{X} \cup X_b] \Delta [X_b, \bar{X} \cup X_a] \\
&= ([X_a, \bar{X}] \cup [X_a, X_b]) \Delta ([X_b, \bar{X}] \cup [X_b, X_a]) \\
&= [X_a, \bar{X}] \cup [X_b, \bar{X}] \\
&= [X_a \cup X_b, \bar{X}] \\
&= [X, \bar{X}] \\
&= \sigma,
\end{aligned}$$

hence σ satisfies property Δ^* with respect to $cl_{M(G)}(C_n)$ and therefore $\sigma \in cl_{M(G)}(C_n)$. □

References

[1] H.J. Broersma, X. Li, The connectivity of the leaf-exchange spanning tree graph of a graph, Ars Combin. 43 (1996) 225–231.

[2] R.L. Cummins, Hamilton circuits in tree graphs, IEEE Trans. Circuit Theory CT-13 (1966) 82–90.

[3] F. Harary, R.J. Mokken, M.J. Plantholt, Interpolation theorem for diameters of spanning trees, IEEE Trans. Circuits Systems 30 (7) (1983) 429–432.

[4] K. Heinrich, G. Liu, A lower bound on the number of spanning trees with k endvertices, J. Graph Theory 12 (1) (1988) 95–100.

[5] C.A. Holzmann, F. Harary, On the tree graph of a matroid, SIAM J. Appl. Math. 22 (1972) 187–193.

[6] F.J. Zhang, Z. Chen, Connectivity of (adjacency) tree graphs, J. Xinjiang Univ. Natur. Sci. 3 (4) (1986) 1–5.