A new elementary algorithm for proving q-hypergeometric identities

Bao-Yin Zhang*

*Center for Combinatorics, The Key Laboratory of Pure Mathematics and Combinatorics of Ministry of
Education, Nankai University, Tianjin 300071, People's Republic of China*

Abstract

We give a fast elementary algorithm to get a small number n_1 for an admissible q-proper-hypergeometric identity

$$\sum_k F(n, k) = G(n), \qquad n \geq n_0$$

such that we can prove the identity by checking its correctness for n ($n_0 \leq n \leq n_1$). For example, we get $n_1 = 191$ for the q-Vandermonde-Chu identity, $n_1 = 70$ for a finite version of Jacobi's triple product identity and $n_1 = 209$ for an identity due to L.J. Rogers.
© 2003 Published by Elsevier Science Ltd.

1. Introduction and definitions

The idea that one can prove a hypergeometric identity by checking a finite number of special cases was presented by Zeilberger in 1982.

In 1996, Yen (1996) first gave an estimate of such a number for q-hypergeometric identities, as a polynomial of degree 24 in the parameters of $F(n, k)$. Although she gave a specific a priori formula, her estimate is very large and is not a practical-sized computation (Petkovšek et al., 1996, p. 70).

Here we give a fast elementary algorithm to get a small number n_1 for an admissible q-proper-hypergeometric identity

$$\sum_k F(n, k) = G(n), \qquad n \geq n_0 \tag{1.1}$$

where $G(n)$ is a q-hypergeometric term, such that we can prove the identity by checking its correctness for n ($n_0 \le n \le n_1$). This makes the idea of proving the q-hypergeometric identities by simply checking them for finite initial values much closer to practice from theory. However, in general proving the q-hypergeometric identities by checking them for n_1 initial values is still slower than proving them by q-Zeilberger algorithm.

This idea is implemented by proving that both sides of the identity satisfy the same recurrence, and by giving both an estimate of the order of the recurrence and an estimate of the integer m_1 such that the leading coefficient of the recurrence does not vanish, when $n \ge m_1$.

Let K be a computable field of characteristic zero, q is transcendental over K, $L = K(q^{1/2})$, and we are considering polynomials in $K[q^n, q^{1/2}]$, $K[q^n, q^k, q^{1/2}]$, $L[q^n]$ or $L[q^n, q^k]$, and rational functions in $L(q^n)$ or $L(q^n, q^k)$.

In this paper, we prove that the left-hand side $l(n)$ and the right-hand side $r(n)$ of the identity (1.1) satisfy the same recurrence of the form

$$l(n) = \frac{c_1(q^n)l(n-1) + c_2(q^n)l(n-2) + \cdots + c_J(q^n)l(n-J)}{c_0(q^n)},$$

$$r(n) = \frac{c_1(q^n)r(n-1) + c_2(q^n)r(n-2) + \cdots + c_J(q^n)r(n-J)}{c_0(q^n)},$$

where $c_i(q^n) \in L[q^n]$. At the same time, we give both an estimate of J which is the order of the recurrence, and an estimate of m_1 such that $c_0(q^n) \ne 0$ for all $n \ge m_1$. Then it is clear that we can prove the identity by checking its correctness for $n \in \{n_0, \dots, n_1\}$, where $n_1 \ge \max\{J, m_1\}$.

An upper bound of the order J was given by Wilf and Zeilberger (1992), and it is very small.

In the next section we generalize Sister Celine's technique and obtain an upper bound of the degree in $q^{1/2}$ in the coefficients of the recurrence for the left-hand side of the identity. Next, with the elimination method and with a similar observation for the polynomial in $K[q^n, q^{1/2}]$ by Yen (1996, Proposition 3.1), we get the same recurrence for the two sides of the identity, and m_1 such that the leading coefficient of the recurrence is not equal to zero for all $n > m_1$. The algorithm and a fast implementation of it are shown in Section 3. We present three examples in the final section.

In the following, we introduce some definitions and notations for q-series.

Definition 1.1. For any $a \in L$ and any integer n, let the nth q-factorial of a be given by

$$(a; q)_n := \begin{cases} (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{for } n > 0; \\ 1 & \text{for } n = 0; \\ \dfrac{1}{(aq^n; q)_{-n}} & \text{for } n < 0. \end{cases}$$

Definition 1.2. A term $F(n, k)$ in the discrete variables n and k is q-hypergeometric if $F(n+1, k)/F(n, k)$ and $F(n, k+1)/F(n, k)$ are both rational functions belonging to $L(q^n, q^k)$.

Definition 1.3. A term $F(n, k)$ is q-proper-hypergeometric if

$$F(n, k) = P(q^n, q^k) \frac{\prod_{s=1}^{p} (c_s q^{\beta_s}; q)_{a_s n + b_s k}}{\prod_{r=1}^{h} (w_r q^{\delta_r}; q)_{u_r n + v_r k}} \xi^k q^{an^2 + bnk + ck^2 + dk + en}, \tag{1.2}$$

where $P(q^n, q^k) \in K[q^n, q^k, q^{1/2}]$, p and h are positive integers, a_s, b_s, u_r, v_r, β_s, δ_r are integers, a, b, c, d, and e are integers or half integers, and $c_s, w_r, \xi \in K$.

The definitions of $F(n, k)$ of the form (1.2) being well-defined at a point (n, k), and of $F(n, k)$ satisfying a k-free recurrence for some (n_0, k_0) are the same as the definitions in Yen (1996).

Owing to the definition of admissible q-hypergeometric terms $F(n, k)$ in Wilf and Zeilberger (1992), we can obtain a non-trivial recurrence for $f(n) := \sum_k F(n, k)$ from a non-trivial k-free recurrence for $F(n, k)$. We proceed to give this definition.

For a fixed integer n, let $B(n) = [a(n), b(n)]$ denote a maximal interval of integer values of k for which $F(n, k)$ is well-defined and non-zero. Just outside the interval $B(n)$ we suppose that there are intervals $\alpha(n) \leq k < a(n)$ and $b(n) < k \leq \beta(n)$ in which F is well-defined and is equal to 0. We call the interval $B(n)$ the natural support of F.

Definition 1.4. An admissible q-hypergeometric term $F(n, k)$ is one in which for all sufficiently large n there is a natural support $B(n)$ such that $B(n)$ is compact and

$$B(n) \subseteq B(n + 1) \subseteq B(n + 2) \subseteq \cdots \qquad (n > n_0)$$

and such that the intervals of zero values which surround $B(n)$ satisfy

$$\beta(n - j) \geq b(n) + I \qquad \text{and} \qquad \alpha(n - j) \leq a(n) - I$$

for $0 \leq j \leq J$ and $n > n_0$, where I and J are the orders of a k-free recurrence that F satisfies.

2. The coefficients of the recurrence

Wilf and Zeilberger (1992) prove the existence of a non-trivial k-free recurrence for q-proper-hypergeometric terms and give an upper bound of the order of the recurrence with Sister Celine's technique. In the following we generalize Sister Celine's technique and give an upper bound of the degree in $q^{1/2}$ in the coefficients of the k-free recurrence.

We rewrite the k-free recurrence for $F(n, k)$

$$\sum_{i=0}^{I} \sum_{j=0}^{J} \alpha(i, j, n) \frac{F(n - j, k - i)}{F(n, k)} = 0$$

as

$$\sum_{i=0}^{I} \sum_{j=0}^{J} \alpha(i, j, n) \frac{D_{ij}(n, k)}{D(n, k)} = 0 \tag{2.1}$$

where $\alpha(i, j, n) \in K[q^n, q^{1/2}]$, $D(n, k) \in L(q^n, q^k)$ is the common denominator, $D_{ij}(n, k) \in K[q^n, q^k, q^{1/2}]$, such that $D_{ij}(n, k)/D(n, k) = F(n - j, k - i)/F(n, k)$.

Let

$$D_{q^k} := \max\{\deg_{q^k} D_{ij}(n, k), \ i = 0, \ldots, I, \ j = 0, \ldots, J\}$$
$$D_{q^{1/2}} := \max\{\deg_{q^{1/2}} D_{ij}(n, k), \ i = 0, \ldots, I, j = 0, \ldots, J\}$$
$$D_{q^n} := \max\{\deg_{q^n} D_{ij}(n, k), \ i = 0, \ldots, I, \ j = 0, \ldots, J\}.$$

Theorem 2.1. *Let $F(n, k)$ be a q-proper-hypergeometric term, then there exist positive integers I, J, M, T, and $\beta(i, j, m, t) \in K$ which are not all zero for $i = 0, \ldots, I$; $j = 0, \ldots, J, m = 0, \ldots, M, t = 0, \ldots, T$, such that the recurrence*

$$\sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{m=0}^{M} \sum_{t=0}^{T} \beta(i, j, m, t) q^{t/2} q^{mn} F(n - j, k - i) = 0 \tag{2.2}$$

holds at every point (n, k) at which $F(n, k) \neq 0$ and all of the values of F that occur in (2.2) are well-defined. Furthermore, when $(I + 1)(J + 1) > 2(D_{q^k} + 1) + 1$, T is at most $D_{q^{1/2}}$ and M is at most $2(D_{q^k} + 1)D_{q^n}$.

Proof. Since $F(n, k) \neq 0$, we divide both sides of (2.2) by it and get

$$\sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{m=0}^{M} \sum_{t=0}^{T} \beta(i, j, m, t) q^{t/2} q^{mn} \frac{F(n - j, k - i)}{F(n, k)} = 0.$$

We rewrite it in the form (2.1)

$$\sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{m=0}^{M} \sum_{t=0}^{T} \beta(i, j, m, t) q^{t/2} q^{mn} \frac{D_{ij}(n, k)}{D(n, k)} = 0.$$

We generalize Sister Celine's technique to zero the coefficients of all the powers of $q^{t/2} q^{mn} q^{lk}$ that appear in the numerator of the left-hand side of the formula above. This gives at most $(D_{q^k} + 1)(D_{q^n} + M + 1)(D_{q^{1/2}} + T + 1)$ linear equations, and the number of the variables $\beta(i, j, m, t)$ is $(I + 1)(J + 1)(M + 1)(T + 1)$. From the knowledge of linear algebra, we know that a non-trivial solution exists if

$$(I + 1)(J + 1)(M + 1)(T + 1) > (D_{q^k} + 1)(D_{q^n} + M + 1)(D_{q^{1/2}} + T + 1). \tag{2.3}$$

We claim that (2.3) holds when $(I + 1)(J + 1) > 2(D_{q^k} + 1) + 1$, $T = D_{q^{1/2}}$ and $M = 2(D_{q^k} + 1)D_{q^n}$.

If $M = 2(D_{q^k} + 1)D_{q^n}$, then

$$(2(D_{q^k} + 1) + 1)(M + 1) > 2(D_{q^k} + 1)(D_{q^n} + M + 1),$$

and if $T = D_{q^{1/2}}$, then

$$2(T + 1) > (D_{q^{1/2}} + T + 1),$$

so we get

$$(2(D_{q^k} + 1) + 1)(M + 1)(T + 1) > (D_{q^k} + 1)(D_{q^n} + M + 1)(D_{q^{1/2}} + T + 1).$$

And from $(I + 1)(J + 1) > 2(D_{q^k} + 1) + 1$, it is clear that (2.3) holds, which completes the proof of the theorem. $\square$

Note that the condition $(I + 1)(J + 1) > 2(D_{q^k} + 1) + 1$ is always reached if I, J are large enough.

Using the same method as in the proof of Wilf and Zeilberger (1992, Theorems 3.2B and 3.2C) and from (2.2), we can obtain a non-trivial recurrence for the left-hand side $f(n) := \sum_k F(n, k)$ of the identity (1.1)

$$a_0(q^n)f(n) + a_1(q^n)f(n - 1) + \cdots + a_J(q^n)f(n - J) = 0, \tag{2.4}$$

where the coefficients $a_i(q^n) \in K[q^n, q^{1/2}]$. At the same time, from Theorem 2.1, the degree in $q^{1/2}$ in $a_i(q^n)$ is at most $D_{q^{1/2}}$.

For the right side $G(n)$ of (1.1), it is easy to see that it satisfies a recurrence of order 1.

In the following we will use the elimination method to prove a theorem which not only gets the same recurrence for the two sides of the identity but also gives the number m_1.

First we proceed to introduce the concept of the linear recurrence operator in Petkovšek and Zeilberger (1996). We define

$$N : Ng(n) = g(n - 1).$$

Let

$$A := \sum_{j=0}^{J} a_j(q^n)N^j,$$

so (2.4) can be rewritten as $Af(n) = 0$.

Because $G(n)$ is a q-hypergeometric term, we have

$$\frac{G(n)}{G(n - 1)} = \frac{r(q^n)}{s(q^n)},$$

where $r(q^n), s(q^n) \in K[q^n, q^{1/2}]$. Let

$$B := s(q^n) - r(q^n)N,$$

so $BG(n) = 0$.

The following proposition varying slightly from Yen (1996, Proposition 3.1) gives a condition for the non-vanishing of the polynomial $P(q^n)$ in terms of the degree in $q^{1/2}$ in $P(q^n)$.

Proposition 2.2. *Let $P(q^n) \in K[q^n, q^{1/2}]$ be a non-zero polynomial, m be the maximal degree in $q^{1/2}$ in $P(q^n)$, then $P(q^n)$ is not equal to zero for all $n > \lfloor m/2 \rfloor$.*

The proof of the above proposition is analogous to the proof of Yen (1996, Proposition 3.1), we omit it here.

Theorem 2.3. *Let A be a linear recurrence operator of order J ($J \geq 1$) which has coefficients in $K[q^n, q^{1/2}]$ and B be a linear recurrence operator of order 1 which also has coefficients in $K[q^n, q^{1/2}]$, the maximal degree in $q^{1/2}$ in the coefficients of A is $D^A_{q^{1/2}}$, the maximal degrees in $q^{1/2}$ and q^n in the coefficients of B are $D^B_{q^{1/2}}$ and $D^B_{q^n}$ respectively, then there exist linear recurrence operators C of order $J+1$, P and Q which have coefficients in $L[q^n]$, such that $C = PA = QB$, and the coefficient of N^0 in C is not equal to zero for all*

$$n > \max \left\{ \binom{J}{2} D^B_{q^n} + \lfloor (J D^B_{q^{1/2}} + D^A_{q^{1/2}})/2 \rfloor + 1, J \right\}. \tag{2.5}$$

Proof. Let

$$A = \sum_{j=0}^{J} a_j^0(q^n) N^j, \qquad B = b_0(q^n) + b_1(q^n) N,$$

we eliminate N^J with A and B, and have

$$A^1 := q^{(J-1)D^B_{q^n}} b_1(q^{n-J+1}) A - q^{(J-1)D^B_{q^n}} a_J^0(q^n) N^{J-1} B$$

$$= \sum_{j=0}^{J-1} q^{(J-1)D^B_{q^n}} b_1(q^{n-J+1}) a_j^0(q^n) N^j - q^{(J-1)D^B_{q^n}} a_J^0(q^n) b_0(q^{n-J+1}) N^{J-1}.$$

It is clear that the order of A^1 is at most $J - 1$, the coefficients of A^1 belong to $K[q^n, q^{1/2}]$, and the maximal degree in $q^{1/2}$ in the coefficients of A^1 is at most $2(J - 1)D^B_{q^n} + D^B_{q^{1/2}} + D^A_{q^{1/2}}$.

Let $A^1 = \sum_{j=0}^{J-1} a_j^1(q^n) N^j$, we use A^1 and B to eliminate N^{J-1}, then

$$A^2 := q^{(J-2)D^B_{q^n}} b_1(q^{n-J+2}) A^1 - q^{(J-2)D^B_{q^n}} a_{J-1}^1(q^n) N^{J-2} B$$

$$= q^{(J-1)D^B_{q^n}} q^{(J-2)D^B_{q^n}} b_1(q^{n-J+1}) b_1(q^{n-J+2}) A$$

$$\quad - (q^{(J-1)D^B_{q^n}} q^{(J-2)D^B_{q^n}} b_1(q^{n-J+2}) a_J^0(q^n) N^{J-1}$$

$$\quad + q^{(J-2)D^B_{q^n}} a_{J-1}^1(q^n) N^{J-2}) B.$$

We also have that the order of A^2 is at most $J - 2$, the coefficients of A^2 belong to $K[q^n, q^{1/2}]$, the maximal degree in $q^{1/2}$ in the coefficients of A^2 is at most $2(J-1)D^B_{q^n} + 2(J-2)D^B_{q^n} + 2D^B_{q^{1/2}} + D^A_{q^{1/2}}$.

So we can do it until the order of A^J is 0, and have

$$A^J := b_1(q^n) A^{J-1} - a_1^{J-1}(q^n) B = \left(\prod_{j=0}^{J-1} q^{j D^B_{q^n}} b_1(q^{n-j}) \right) A$$

$$\quad - \left(\sum_{j=0}^{J-1} \left(\prod_{i=0}^{j} q^{i D^B_{q^n}} \right) \left(\prod_{h=0}^{j-1} b_1(q^{n-h}) \right) a_{j+1}^{J-j-1}(q^n) N^j \right) B, \tag{2.6}$$

where $a_{j+1}^{J-j-1}(q^n)$ is the coefficient of A^{J-j-1} for $j = 0, \ldots, J - 1$.

Similarly, A^J belongs to $K[q^n, q^{1/2}]$, the maximal degree in $q^{1/2}$ in A^J is at most $2\binom{J}{2}D_{q^n}^B + JD_{q^{1/2}}^B + D_{q^{1/2}}^A$.

For A^J and B, we have

$$A' := a_0^J(q^{n-1})B - b_1(q^n)NA^J = a_0^J(q^{n-1})b_0(q^n), \tag{2.7}$$

so we get

$$a_0^J(q^n)A' - a_0^J(q^{n-1})b_0(q^n)A^J = 0. \tag{2.8}$$

Replacing A' and A^J in (2.8) with (2.6) and (2.7), we get

$$
(a_0^J(q^n)b_1(q^n)N + a_0^J(q^{n-1})b_0(q^n))\left(\prod_{j=0}^{J-1}q^{jD_{q^n}^B}b_1(q^{n-j})\right)A
$$
$$
= \left((a_0^J(q^n)b_1(q^n)N + a_0^J(q^{n-1})b_0(q^n))\sum_{j=0}^{J-1}\left(\prod_{i=0}^{j}q^{iD_{q^n}^B}\right)\right.
$$
$$
\left. \times \left(\prod_{h=0}^{j-1}b_1(q^{n-h})\right)a_{j+1}^{J-j-1}(q^n)N^j + a_0^J(q^n)a_0^J(q^{n-1})\right)B.
$$

Let

$$
C := (a_0^J(q^n)b_1(q^n)N + a_0^J(q^{n-1})b_0(q^n))\left(\prod_{j=0}^{J-1}q^{jD_{q^n}^B}b_1(q^{n-j})\right)A,
$$
$$
P := (a_0^J(q^n)b_1(q^n)N + a_0^J(q^{n-1})b_0(q^n))\left(\prod_{j=0}^{J-1}q^{jD_{q^n}^B}b_1(q^{n-j})\right),
$$
$$
Q := (a_0^J(q^n)b_1(q^n)N + a_0^J(q^{n-1})b_0(q^n))
$$
$$
\times \sum_{j=0}^{J-1}\left(\prod_{i=0}^{j}q^{iD_{q^n}^B}\right)\left(\prod_{h=0}^{j-1}b_1(q^{n-h})\right)a_{j+1}^{J-j-1}(q^n)N^j + a_0^J(q^n)a_0^J(q^{n-1}),
$$

then

$$C = PA = QB.$$

It is easy to see that the order of C is $J + 1$, and the coefficients of C, P and Q belong to $L[q^n]$.

The coefficient of N^0 in C is

$$
a_0^J(q^{n-1})b_0(q^n)a_0^0(q^n)\prod_{j=0}^{J-1}q^{jD_{q^n}^B}b_1(q^{n-j}).
$$

From Proposition 2.2, $a_0^J(q^n)$ is not equal to zero for all

$$n > \binom{J}{2} D_{q^n}^B + \lfloor (J D_{q^{1/2}}^B + D_{q^{1/2}}^A)/2 \rfloor,$$

so it is easy to see that $a_0^J(q^{n-1})$ is not equal to zero for all

$$n > \binom{J}{2} D_{q^n}^B + \lfloor (J D_{q^{1/2}}^B + D_{q^{1/2}}^A)/2 \rfloor + 1.$$

Similarly, we have

$$
\begin{aligned}
b_0(q^n) &\neq 0, &&\text{for all } n > \lfloor D_{q^{1/2}}^B/2 \rfloor, \\
a_0^0(q^n) &\neq 0, &&\text{for all } n > \lfloor D_{q^{1/2}}^A/2 \rfloor, \\
b_1(q^{n-j}) &\neq 0, &&\text{for all } n > \lfloor D_{q^{1/2}}^B/2 \rfloor + j.
\end{aligned}
$$

So all the factors of the coefficient of N^0 in C are not equal to zero for all

$$n > \max \left\{ \binom{J}{2} D_{q^n}^B + \lfloor (J D_{q^{1/2}}^B + D_{q^{1/2}}^A)/2 \rfloor + 1, J \right\},$$

which completes the proof of the theorem. $\quad\square$

3. The algorithm

In this section we introduce the elementary algorithm in detail and give a fast implementation.

1. Fix trial values of I and J, say $I = J = 1$.
2. Simplify $\sum_{i=0}^I \sum_{j=0}^J \frac{F(n-j,k-i)}{F(n,k)}$ to the form $\sum_{i=0}^I \sum_{j=0}^J \frac{D_{ij}(n,k)}{D(n,k)}$ and compute D_{q^k}.
3. If $(I+1)(J+1) \leq 2(\mathrm{D}_{q^k}+1)+1$, increase I by 1 or increase J by 1, go to step 2; otherwise, compute $\mathrm{D}_{q^{1/2}}$ and J.
4. Simplify $G(n)/G(n-1)$, and compute the maximal degrees in q^n and $q^{1/2}$ in the numerator and the denominator, denoted by $D_{q^n}^B$ and $D_{q^{1/2}}^B$ respectively.
5. Let $D_{q^{1/2}}^A := \mathrm{D}_{q^{1/2}}$, from (2.5) of Theorem 2.3, output n_1.

Comparing this algorithm with Sister Celine's algorithm (see Petkovšek et al., 1996, p. 59), we avoid the time consuming aspects of getting the linear equations and solving the equations in Sister Celine's algorithm; in our algorithm, the time consuming computing is simplifying $\sum_{ij} \frac{F(n-j,k-i)}{F(n,k)}$ to the form $\sum_{ij} \frac{D_{ij}(n,k)}{D(n,k)}$ for obtaining D_{q^k} and $\mathrm{D}_{q^{1/2}}$. In the following, we show a fast method to get D_{q^k} and $\mathrm{D}_{q^{1/2}}$.

Let $x^+ := \max\{x, 0\}$, $\phi(i,j) = -2aj - ib$ and $\psi(i,j) = aj^2 + bij + ci^2 - id - je$. Then

$$
\begin{aligned}
&\frac{F(n-j, k-i)}{F(n,k)} \\
&= \frac{P(q^{n-j}, q^{k-i})}{P(q^n, q^k)} \frac{\prod_r (w_r q^{\delta_r + u_r n + v_r k - u_r j - v_r i}; q)_{u_r j + v_r i}}{\prod_s (c_s q^{\beta_s + a_s n + b_s k - a_s j - b_s i}; q)_{a_s j + b_s i}} \xi^{-i} \frac{q^{\phi(i,j)n + \psi(i,j)}}{q^{k(bj + 2ci)}}.
\end{aligned}
$$

By a similar case analysis (see Yen, 1996, p. 8), we have

$$\frac{F(n-j, k-i)}{F(n, k)} = \frac{D_{ij}^*(n, k)}{D^*(n, k)},$$

where

$$
\begin{aligned}
D_{ij}^*(n, k) = {} & P(q^{n-j}, q^{k-i})\xi^{-i}q^{\phi(i,j)n+\psi(i,j)}q^{k(b^+J+2c^+I-bj-2ci)} \\
& \times \prod_r (w_r q^{\delta_r+u_r n+v_r k-u_r j-v_r i}; q)_{(u_r j+v_r i)^+} \\
& \times (w_r q^{\delta_r+u_r n+v_r k+(-u_r j-v_r i)^+}; q)_{(-u_r)^+J+(-v_r)^+I-(-u_r j-v_r i)^+} \\
& \times \prod_s (c_s q^{\beta_s+a_s n+b_s k}; q)_{(-a_s j-b_s i)^+} \\
& \times (c_s q^{\beta_s+a_s n+b_s k-(a_s)^+J-(b_s)^+I}; q)_{(a_s)^+J+(b_s)^+I-(a_s j+b_s i)^+}, \tag{3.1}
\end{aligned}
$$

and

$$
\begin{aligned}
D^*(n, k) = {} & P(q^n, q^k)q^{k(b^+J+2c^+I)} \prod_r (w_r q^{\delta_r+u_r n+v_r k}; q)_{(-u_r)^+J+(-v_r)^+I} \\
& \times \prod_s (c_s q^{\beta_s+a_s n+b_s k-(a_s)^+J-(b_s)^+I}; q)_{(a_s)^+J+(b_s)^+I}.
\end{aligned}
$$

We cannot guarantee $D_{ij}^*(n, k) \in K[q^n, q^k, q^{1/2}]$, but we have $D_{ij}^*(n, k)$ is the Laurent polynomial in q^n, q^k and $q^{1/2}$. Hence

$$
\begin{aligned}
D_{ij}(n, k) &= D_{ij}^*(n, k)(q^n)^{-d_{q^n}}(q^k)^{-d_{q^k}}(q^{1/2})^{-d_{q^{1/2}}} \\
D(n, k) &= D^*(n, k)(q^n)^{-d_{q^n}}(q^k)^{-d_{q^k}}(q^{1/2})^{-d_{q^{1/2}}}
\end{aligned}
$$

where d_{q^n}, d_{q^k} and $d_{q^{1/2}}$ are the lowest degrees (including negative exponents) of q^n, q^k and $q^{1/2}$ in $\sum_{ij} D_{ij}^*(n, k)$ respectively.

Thus

$$\mathrm{D}_{q^k} = \theta_{q^k} - d_{q^k}, \qquad \mathrm{D}_{q^{1/2}} = \theta_{q^{1/2}} - d_{q^{1/2}},$$

where θ_{q^k} and $\theta_{q^{1/2}}$ are the maximal degrees of q^k and $q^{1/2}$ in $\sum_{ij} D_{ij}^*(n, k)$ respectively.

From formula (3.1), we can get θ_{q^k}, $\theta_{q^{1/2}}$, d_{q^k} and $d_{q^{1/2}}$ quite quickly. Consequently, we can obtain D_{q^k} and $\mathrm{D}_{q^{1/2}}$ quickly.

We have implemented this algorithm in Maple, in the next section we give some examples.

4. Examples

As our first example, we compute n_1 for the q-Vandermonde-Chu identity

$$\sum_k q^{k^2} \binom{n}{k}_q^2 = \binom{2n}{n}_q.$$

The identity can be expressed in the form

$$\sum_k q^{k^2} \frac{(q;q)_n^2}{(q;q)_k^2 (q;q)_{n-k}^2} = \frac{(q;q)_{2n}}{(q;q)_n^2}.$$

Hence $a = b = d = e = 0, c = 1, P(q^n, q^k) = 1, p = 2, a_s = 1, b_s = 0$ and $\beta_s = 1$ for $s \in \{1, 2\}, h = 4, u_1 = u_2 = 1, v_1 = v_2 = -1, u_3 = u_4 = 0, v_3 = v_4 = 1$, and $w_r = c_r = 1$ and $\delta_r = 1$ for $r \in \{1, 2, 3, 4\}$. We input these parameters to the Maple program, obtain $I = 6$, $J = 7$ and get $n_1 = 191$.

Next, we compute n_1 for a finite version of Jacobi's triple product identity.

It is well known (see e.g. Andrews, 1976) that Jacobi's triple product identity

$$\sum_{k=-\infty}^{\infty} q^{\binom{k}{2}} x^k = \prod_{j=1}^{\infty} (1 - q^j)(1 + x^{-1}q^j)(1 + xq^{j-1})$$

can be deduced, for instance, as a limiting case $n \to \infty$ of the following finite variant of the q-binomial formula

$$\sum_k \binom{2n}{n+k}_q q^{\binom{k}{2}} x^k = (-x^{-1}q; q)_n (-x; q)_n.$$

We express the identity in the form

$$\sum_k \frac{(q;q)_{2n}}{(q;q)_{n+k}(q;q)_{n-k}} q^{k^2/2 - k/2} x^k = (-x^{-1}q; q)_n (-x; q)_n.$$

Hence $a = 0, b = 0, c = 1/2, d = -1/2, e = 0, P(q^n, q^k) = 1, p = 1, a_1 = 2, b_1 = 0, c_1 = 1, \beta_1 = 1$, and $h = 2, u_1 = 1, v_1 = 1, u_2 = 1, v_2 = -1, w_1 = w_2 = 1, \delta_1 = \delta_2 = 1$. We input these parameters to the Maple program, obtain $I = 5$, $J = 4$ and get $n_1 = 70$.

Finally, we compute n_1 for an identity due to L.J. Rogers

$$\sum_k \frac{(-1)^k (q;q)_n q^{k(3k-1)/2}}{(q;q)_{n+k}(q;q)_{n-k}} = 1.$$

This identity is a finite version of Euler's pentagonal number theorem.

Hence $a = 0, b = 0, c = 3/2, d = -1/2, e = 0, P(q^n, q^k) = 1, p = 1, a_1 = 1, b_1 = 0, c_1 = 1, \beta_1 = 1$, and $h = 2, u_1 = 1, v_1 = 1, u_2 = 1, v_2 = -1, w_1 = w_2 = 1, \delta_1 = \delta_2 = 1$. We input these parameters to the Maple program, obtain $I = 8$, $J = 9$ and get $n_1 = 209$.

Acknowledgements

References

Andrews, G.E., 1976. In: Rota, G.-C. (Ed.), The Theory of Partitions, Encyclopedia of Mathematics and its Applications, vol. 2, Addison-Wesley, Reading, MA (Reissued: Cambridge University Press, London and New York, 1985).

Petkovšek, M., Wilf, H.S., Zeilberger, D., 1996. "$A = B$", Wellesley, MA.

Wilf, H.S., Zeilberger, D., 1992. An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. Invent. Math. 108, 575–633.

Yen, L., 1996. A two-line algorithm for proving q-hypergeometric identities. J. Math. Anal. Appl. 213, 1–14.

Zeilberger, D., 1982. Sister Celine's technique and its generalizations. J. Math. Anal. Appl. 85, 114–145.