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A blossoming algorithm for tree volumes of composite digraphs

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Abstract

A weighted composite digraph is obtained from some weighted digraph by replacing each vertex with a weighted digraph. In this paper, we give a beautiful combinatorial proof of the formula for forest volumes of composite digraphs obtained by Kelmans et al. [DIMACS Technical Report 2000-03, 2000]. Moreover, a generalization of this formula is present.

Keywords: Composite digraph; Tree volume; Oriented tree volume; Bijection

1. Introduction

The composite graph $G(H_1, \ldots, H_n)$ is obtained from the graph G on $[n]$ by replacing each vertex i by a graph H_i. It is a generalization of multipartite graph, whose complexity was studied by Knuth [8] and Kelmans [6]. The number of spanning trees of $G(H_1, \ldots, H_n)$ was obtained by Pak and Postnikov [9]. These papers generalized the encoding of Prüfer [10]. Recently, Kelmans et al. [7] investigated the tree and forest volumes of weighted digraphs with algebraic methods, and deduced a nice formula for forest volumes of composite weighted digraphs.

To give a combinatorial interpretation of Kelmans–Pak–Postnikov's formula, we notice that the composite digraph can always be obtained from a series of digraph-substitutions. This fact discloses the essence of a composite digraph. Based on it, we construct a bijection between two sets related to oriented trees. The bijection uses the ideas of Joyal [5] and Goulden and Jackson [3], who independently constructed an elegant encoding for bi-rooted trees. The Joyal encoding was also used by Stanley [12], Eğecioğlu and Remmel [1,2], and

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Guo [4] in their approaches to bipartite or multipartite trees. Although a bijective proof of Kelmans–Pak–Postnikov's formula may be given by using the Prüfer code, the proof applying the Joyal code is much more simple and straightforward.

We introduce the concept of the *oriented tree volume* of a weighted digraph, and obtain a formula for oriented tree volumes of weighted composite digraphs, which is equivalent to Kelmans–Pak–Postnikov's formula. Moreover, a generalized form of this formula is present.

2. Notation and terminology

A *directed graph* or simply a *digraph* G is a graph G with each edge uv endowed with a direction from u to v or v to u. The edge set and vertex set of G are denoted by $V(G)$ and $E(G)$, respectively. The edge (or called arc) with direction from u to v is denoted by (u,v), and we call u the *initial vertex* and v the *final vertex*. The *outdegree* of a vertex v, denoted $\deg_G^+(v)$, is the number of edges of G with initial vertex v. Similarly, the *indegree* of v, denoted $\deg_G^-(v)$, is the number of edges with final vertex v. An *oriented tree* (or *in-tree*) with root v is a digraph T with v as one of its vertices, such that there is a unique directed path from any vertex u to v. The set of spanning oriented trees of G is denoted by $Sp(G)$.

A *weighted digraph* G is a digraph G, such that each edge e of G is associated with an indeterminate t_e (or an element of a commutative ring).

For each vertex v of a weighted digraph G, we associate an indeterminate x_v. Let T be a spanning oriented tree of G. Define the *weight* $\omega(T)$ and *oriented weight* $\overline{\omega}(T)$ of T by

$$\omega(T) := \prod_{v\in V(T)} x_v^{\deg_T(v)-1} \cdot \prod_{e\in E(T)} t_e, \tag{2.1}$$

$$\overline{\omega}(T) := \prod_{v\in V(T)} x_v^{\deg_T^-(v)} \cdot \prod_{e\in E(T)} t_e, \tag{2.2}$$

where t_e is the weight of the edge e. Then define the *tree volume* $f_G(x,t)$ and *oriented tree volume* $\bar{f}_G(x,t)$ of G to be the polynomials in the variables $(x_v)_{v\in V(G)}$ and $(t_e)_{e\in E(G)}$ by

$$f_G(x,t) := \sum_{T\in Sp(G)} \omega(T), \tag{2.3}$$

$$\bar{f}_G(x,t) := \sum_{T\in Sp(G)} \overline{\omega}(T). \tag{2.4}$$

Let $G=(V,E)$ be a weighted digraph, $\mathcal{G}=\{G_v\colon\ v\in V\}$ and $\mathcal{H}=\{H_v\colon\ v\in V\}$ be two families of disjoint weighted digraphs such that H_v is a subgraph of G_v for every $v\in V$. We construct a new digraph $\Gamma=G(\mathcal{G},\mathcal{H})$ as follows:

(i) $V(\Gamma) = \bigcup_{v \in V} V(G_v)$;
(ii) $E(\Gamma) = \bigcup_{v \in V} E(G_v) \cup \{(x, y)\colon\ x \in V(H_u),\ y \in V(H_v),\ \text{and } (u, v) \in E(G)\}$;
(iii) for $x \in V(H_u)$, $y \in V(H_v)$, and $(u, v) \in E(G)$, the edge (x, y) is endowed with the weight $t_{(x,y)} = t_{(u,v)}$.

We call Γ the *composition of G through $(\mathcal{G}, \mathcal{H})$*. Denote by $G(\mathcal{H}) = G(\mathcal{H}, \mathcal{H})$.

If $G_u = G_1$, $H_u = H_1$, and $G_v = H_v = \{v\}$ for $v \neq u$, then $G(\mathcal{G}, \mathcal{H})$ is denoted by $G[G_1, H_1, u]$, and we call $G[G_1, H_1, u]$ a *substitution of (G_1, H_1) into G in place of u*. When $V(H_1) = V(G_1)$, $G[G_1, H_1, u]$ is denoted by $G[G_1, u]$.

Let G be a weighted digraph, we construct a weighted digraph G^* as follows:

(i) $V(G^*) = V(G) \cup *$, where $* \notin V(G)$;
(ii) $E(G^*) = E(G) \cup \{(v, *)\colon\ v \in V(G)\}$;
(iii) each edge $(v, *)$ is weighted by $t_{(v,*)} = 1$.

We call G^* the *cone* of G. For convenience, we also write $G^\star$ for the cone of G with the new vertex $\star$.

Now define the *forest volume* of $G(\mathcal{H})$ to be the tree volume of the digraph $G(\mathcal{H})^\star$.

Theorem 2.1 (Kelmans et al. [7, Theorem 11.2]). *We have*

$$f_{G(\mathcal{H})^\star}(x, t) = f_{G^\star}(x, t)|_{x_v = h_v(x)} \prod_{v \in V} f_{H_v^*}(x, t)|_{x_* = x_\star + g_v(x,t)}, \tag{2.5}$$

where $h_v(x) = \sum_{u \in V(H_v)} x_u$ *and* $g_v(x, t) = \sum_{(v,a) \in E(G)} h_a(x) t_{(v,a)}$.

We will prove the above theorem in the last section.

3. The complete bipartite digraph

Let $R = \{1, 2, \ldots, r\}$, $S = \{r+1, r+2, \ldots, r+s$, and let $K_{r,s}^{p,q}$ be the complete bipartite digraph with vertex set partitioned into $R \cup S$, and for any $i \in R$ and $j \in S$, the arcs (i, j) and (j, i) are weighted by $t_{(i,j)} = p_i$ and $t_{(j,i)} = q_i$, respectively.

Lemma 3.1. *We have*

$$\begin{aligned} \bar{f}_{K_{r,s}^{p,q}}(x, t) = {} & p_1 \cdots p_r (q_1 x_1 + \cdots + q_r x_r)^{s-1} (x_{r+1} + \cdots + x_{r+s})^{r-1} \\ & \cdot (x_1 q_1/p_1 + \cdots + x_r q_r/p_r + x_{r+1} + \cdots + x_{r+s}). \end{aligned} \tag{3.1}$$

We will give a bijective proof of Lemma 3.1. The bijection established here is a little different from Eğecioğlu and Remmel [1] and the second solution of Stanley [12, pp. 125–126].

The following definitions and lemma play an important role.

Definition 3.2. An element a_i is called a *left-to-right maximum* of a permutation $a_1 a_2 \cdots a_n$, if $a_i > a_j$ for every $j < i$.

Let $\mathfrak{S}_n$ denote the set of all permutations on $[n] := \{1, 2, \ldots, n\}$. It is well known that a permutation can be written as a product of its distinct cycles.

Definition 3.3. A *standard representation* of a permutation is a product of its distinct cycles satisfying that

(a) each cycle is written with its largest element first, and
(b) the cycles are written in increasing order of their largest element.

Definition 3.4. For a permutation π, we define $\phi(\pi)$ to be the permutation obtained from π by writing it in standard form and erasing the parentheses.

Lemma 3.5 (cf. Stanley [11, Proposition 1.3.1]). *The map $\phi : \mathfrak{S}_n \mapsto \mathfrak{S}_n$ defined above is a bijection. If $\pi \in \mathfrak{S}_n$ has k cycles, then $\phi(\pi)$ has k left-to-right maxima.*

Example 3.6. For $\pi = 24718635$ with standard form $\pi = (412)(6)(73)(85)$, we have $\phi(\pi) = 41267385$ with left-to-right maxima $4, 6, 7, 8$.

Note that ϕ^{-1} (or ϕ) is well-defined for any permutation $\pi = a_1 a_2 \cdots a_k$ over $[n]$ for $k \leqslant n$. For instance, if $\pi = 517249$, then $\phi^{-1}(\pi) = (51)(724)(9)$.

Proof of Lemma 3.1. Recall that $R = \{1, 2, \ldots, r\}$ and $S = \{1', 2', \ldots, s'\} = \{r+1, r+2, \ldots, r+s\}$. We linearly order R and S by $1 < 2 < \cdots < r$ and $1' < 2' < \cdots < s'$, respectively, and we would not compare the elements between R and S.

Suppose $T \in Sp(K_{r,s}^{p;q})$ with root v. There is a unique directed path $\pi = a_1 b_1 a_2 b_2 \cdots a_m b_m$ from v_0 to v, where $v_0 = 1'$ if $v \in R$, and $v_0 = 1$ if $v \in S$. By Lemma 3.5, we obtain $\phi^{-1}(\pi') = C_1' \cdots C_\ell'$ from $\pi' = b_1 b_2 \cdots b_{m-1}$, where ϕ is defined in Definition 3.4 and $C_1', \ldots, C_\ell'$ are cycles. Then define C_k to be the cycle obtained from C_k' by replacing each b_i with $b_i a_{i+1}$. Specifically, the cycle $C_k' = (b_i b_{i+1} \cdots b_j)$ corresponds to $C_k = (b_i a_{i+1} b_{i+1} a_{i+2} \cdots b_j a_{j+1})$.

Let D_π be the disjoint union of the directed cycles $C_1, \ldots, C_\ell$. When we remove all the edges of the path π, we obtain a disjoint union of oriented trees. Merge these oriented trees and D_π by identifying vertices with the same label. Then we obtain a weighted digraph $\theta_v(T)$ by endowing each arc e with initial (respectively, final) vertex $i \in R$ with weight $t_e = p_i$ (respectively, $t_e = q_i$). It is clear that $\theta_v : T \mapsto \theta_v(T)$ is a bijection.

If $v \in R$, then we define the *weight* of $\theta_v(T)$ by

$$\varpi\big(\theta_v(T)\big) = q_v x_v \prod_{e \in E(\theta_v(T))} t_e \cdot \prod_{i \in R \cup S} x_i^{\deg^-(i)},$$

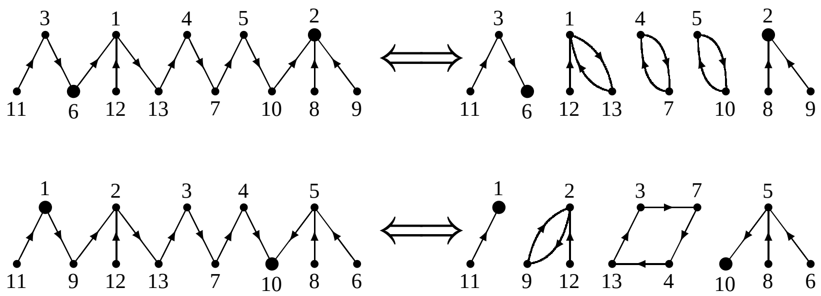


Fig. 1. The two cases for Lemma 3.1.

while if $v \in S$, then we define the *weight* of $\theta_v(T)$ by

$$\overline{\omega}\big(\theta_v(T)\big) = p_1 x_v \prod_{e \in E(\theta_v(T))} t_e \cdot \prod_{i \in R \cup S} x_i^{\deg^-(i)}.$$

It is easy to see $\overline{\omega}(T) = \overline{\omega}(\theta_v(T))$, and $\overline{\omega}(\theta_v(T))$ is a term in the expansion of $p_1 \cdots p_r (q_1 x_1 + \cdots + q_r x_r)^{s-1} (x_{r+1} + \cdots + x_{r+s})^{r-1} x_v q_v / p_v$ if $v \in R$, or a term in $p_1 \cdots p_r (q_1 x_1 + \cdots + q_r x_r)^{s-1} (x_{r+1} + \cdots + x_{r+s})^{r-1} x_v$ if $v \in S$.

The lemma follows by summing over all v. □

Examples for $r = 5, s = 8$ are shown in Fig. 1.

4. The bijections for substituted digraphs

The substituted digraph plays an important part in composite digraphs. We modify the bijection in the argument of Lemma 3.1 to count oriented tree volumes of weighted substituted digraphs. The key idea is to visualize each small tree as a vertex, and a spanning oriented tree of a weighted substituted digraph is then something like a bipartite oriented tree.

Let H be a subgraph of G. Define $Sp(G, H)$ to be the set of spanning oriented trees of G with root in H, and denote by

$$\bar{f}_{G,H}(x,t) := \sum_{T \in Sp(G,H)} \overline{\omega}(T).$$

Lemma 4.1. *Let G, G_1 be disjoint weighted digraphs, $H_1 \subseteq V(G_1)$, and $u \in V(G)$. We have*

$$\bar{f}_{G[G_1,H_1,u],\,G[H_1,u]}(x,t) = \bar{f}_G(x,t)|_{x_u = h_1(x)} \cdot f_{G_1 \cup H_1^*}(x,t)|_{x_* = g_u(x,t)}, \tag{4.1}$$

where $h_1(x) = \sum_{v \in H_1} x_v$ and $g_u(x,t) = \sum_{(u,v) \in E(G)} x_v t_{(u,v)}$.

Proof. Let $G_2 = G \setminus \{u\}$ and H_2 be the set of vertices v such that $(u,v) \in E(G)$, and let Ω be the set of four-tuples $(T_1, T_2, w^{(1)}, w^{(2)})$ such that

(i) $T_1 \in Sp(G_1 \cup H_1^*)$ and $T_2 \in Sp(G)$;
(ii) $w^{(1)}$ and $w^{(2)}$ are words on H_1 and H_2, respectively;
(iii) $|w^{(1)}| = \deg^-_{T_2}(u)$ and $|w^{(2)}| = \deg_{T_1}(*) - 1$.

We want to construct a bijection between $Sp(G[G_1, H_1, u], G[H_1, u])$, and Ω.

First linearly order H_1 and $V(G_2)$, respectively.

Suppose T is a spanning oriented tree of $G[G_1, H_1, u]$ with root r in $G[H_1, u]$. Deleting the edges of T between H_1 and G_2, we obtain weighted oriented forests F_1 and F_2, which are contained in G_1 and G_2, respectively. The oriented tree in F_1 or F_2 with root v is denoted by R_v, and the root sets of F_1 and F_2 are denoted by M_1 and M_2, respectively.

If $r \in M_1$, then we define v_0 to be $\min M_2$, while if $r \in M_2$, then we define v_0 to be $\min M_1$. Assume that r_1 is the first vertex on the path from v_0 to r in T such that $r_1 \in V(R_r)$.

We may obtain an oriented tree T_1 from F_1 by adding the vertex $*$, and edges $(v,*)$ $(v \in M_1)$, each weighted by 1. To obtain the oriented tree T_2 from F_2, we add the vertex u. If $r \in M_1$, then we add edges (v,u) $(v \in M_2)$ with weights in G, while if $r \in M_2$, then we add edges (v,u) $(v \in M_2 \setminus \{r\})$ and (u, r_1) with weights in G. It remains to find out the words $w^{(1)}$ and $w^{(2)}$.

Identifying each R_v with v in T, we obtain a weighted oriented tree T' rooted at r. Assume that the directed path from v_0 to r in T' is $\pi = p_1 p_2 \cdots p_{2m}$, and $p_{i_1} < p_{i_2} < \cdots < p_{i_\ell}$ are all the left-to-right maxima of $p_2 p_4 \cdots p_{2m-2}$. Let $a_k = p_{i_k - 1}$ $(1 \leqslant k \leqslant \ell)$, and $a_{\ell+1} = p_{2m-1}$.

For any vertex $v \neq v_0, r$ of T', let $D(v)$ denote the vertex z such that

$$\begin{cases} (a_k, z) \in E(T), & \text{if } v = a_{k+1} \ (1 \leqslant k \leqslant \ell), \\ (v, z) \in E(T), & \text{otherwise.} \end{cases}$$

Write $M_1 \setminus \{v_0, r\} = \{u_1, u_2, \ldots, u_q\}$ and $M_2 \setminus \{v_0, r\} = \{v_1, v_2, \ldots, v_s\}$ in increasing order. Put

$$w^{(1)} = \begin{cases} D(v_1)D(v_2)\cdots D(v_s)r_1, & \text{if } r \in M_1, \\ D(v_1)D(v_2)\cdots D(v_s), & \text{if } r \in M_2, \end{cases}$$
$$w^{(2)} = D(u_1)D(u_2)\cdots D(u_q).$$

It is clear that $|w^{(1)}| = \deg^-_{T_2}(u)$ and $|w^{(2)}| = \deg_{T_1}(*) - 1$.

The procedure from Ω to $Sp(G[G_1, H_1, u], G[H_1, u])$ is as follows:

Given $\widetilde{T} = (T_1, T_2, w^{(1)}, w^{(2)}) \in \Omega$. Deleting the vertices $*$ and u of T_1 and T_2, we get oriented forests F_1 and F_2, respectively. The oriented tree in F_1 or F_2 with root v is denoted by R_v, and the root sets of F_1 and F_2 are denoted by M_1 and M_2, respectively. Suppose the root of T_2 is r'.

If $r' = u$, then we define v_0 to be $\min M_2$, r_1 the entry in the last position of $w^{(1)}$, and r the root of the oriented tree in F that contains r_1. Otherwise, $(u, r_1) \in E(T_2)$ for some vertex r_1, we denote $r = r'$, and define v_0 to be $\min M_1$.

Write $M_1 \setminus \{v_0, r\} = \{u_1, u_2, \ldots, u_q\}$ and $M_2 \setminus \{v_0, r\} = \{v_1, v_2, \ldots, v_s\}$ in increasing order. For the words $w^{(1)}$ and $w^{(2)}$, if we identify each letter (vertex) with the root of the oriented tree in F_1 or F_2 that contains it, then we obtain words $\overline{w}^{(1)}$ and $\overline{w}^{(2)}$ on M_1 and M_2, respectively. By the definition of Ω, we have $|\overline{w}^{(1)}| = s + 1$ or s, according to $r' = u$ or not, and $|\overline{w}^{(2)}| = q$. Regard the following function

$$D_0 = \begin{pmatrix} v_1, \ldots, v_s, u_1, \ldots, u_q \\ \overline{w}_1^{(1)}, \ldots, \overline{w}_s^{(1)}, \overline{w}_1^{(2)}, \ldots, \overline{w}_q^{(2)} \end{pmatrix}$$

as a digraph on $M_1 \cup M_2$, such that $\deg^+(v_0) = \deg^+(r) = 0$. We may recover the oriented tree $T' = \theta_r^{-1}(D_0)$ with root r by Lemma 3.1.

Let

$$D = \begin{pmatrix} v_1, \ldots, v_s, u_1, \ldots, u_q \\ w_1^{(1)}, \ldots, w_s^{(1)}, w_1^{(2)}, \ldots, w_q^{(2)} \end{pmatrix}.$$

Assume that the path from v_0 to r in T' is $\pi = p_1 p_2 \cdots p_{2m}$, and $p_{i_1} < p_{i_2} < \cdots < p_{i_\ell}$ are all the left-to-right maxima of $p_2 p_4 \cdots p_{2m-2}$. Let $a_k = p_{i_k - 1}$ $(1 \leqslant k \leqslant \ell)$, and $a_{\ell+1} = p_{2m-1}$. We now connect all the oriented trees in F_1 or F_2 by drawing the following edges:

- $(a_k, D(a_{k+1}))$ $(1 \leqslant k \leqslant \ell)$, and (p_{2m-1}, r_1);
- $(v, D(v))$, for $v \in V(T') \setminus \{a_1, a_2, \ldots, a_{\ell+1}, r\}$.

Finally, endow the above edges with weights in $G[G_1, H_1, u]$. Thus we obtain a spanning oriented tree T of $G[G_1, H_1, u]$ with root r (in $G[H_1, u]$).

It is not difficult to see that the above two procedures are inverse to each other, therefore we obtain a bijection between $Sp(G[G_1, H_1, u], G[H_1, u])$ and Ω. We now define the *weight* of $\widetilde{T} = (T_1, T_2, w^{(1)}, w^{(2)}) \in \Omega$ by

$$\omega(\widetilde{T}) = \omega(T_1)\overline{\omega}(T_2) x_*^{1-\deg_{T_1}(*)} x_u^{-\deg_{T_2}^-(u)} \prod_{k \in w^{(1)}} x_k \cdot \prod_{v \in w^{(2)}} x_v t_{(u,v)}.$$

It is straightforward to see that the above bijection $T \mapsto \widetilde{T}$ is weight-preserving, that is, $\overline{\omega}(T) = \omega(\widetilde{T})$. Clearly, $\overline{\omega}(T)$ is a term of $f_{G[G_1, H_1, u], G[H_1, u]}(x)$, while $\omega(\widetilde{T})$ is a term in the expansion of $\bar{f}_G(x, t)|_{x_u = \sum_{v \in H_1} x_v} \cdot f_{G_1 \cup H_1^*}(x, t)|_{x_* = \sum_{v \in H_2} x_v t_{(u,v)}}$. This completes the proof of (4.1). $\square$

Examples for Lemma 4.1 are given in Figs. 2–5, where $H_1 = \{1', 2', \ldots, 14'\} = \{12, 13, \ldots, 25\}$ and $N = \{1, 2, \ldots, 11\}$ is the set of vertices adjacent to u in G. We leave out labels of those vertices in $V(G_1) \setminus H_1$ or $V(G_2) \setminus N$. But the root is labeled by r if necessary.

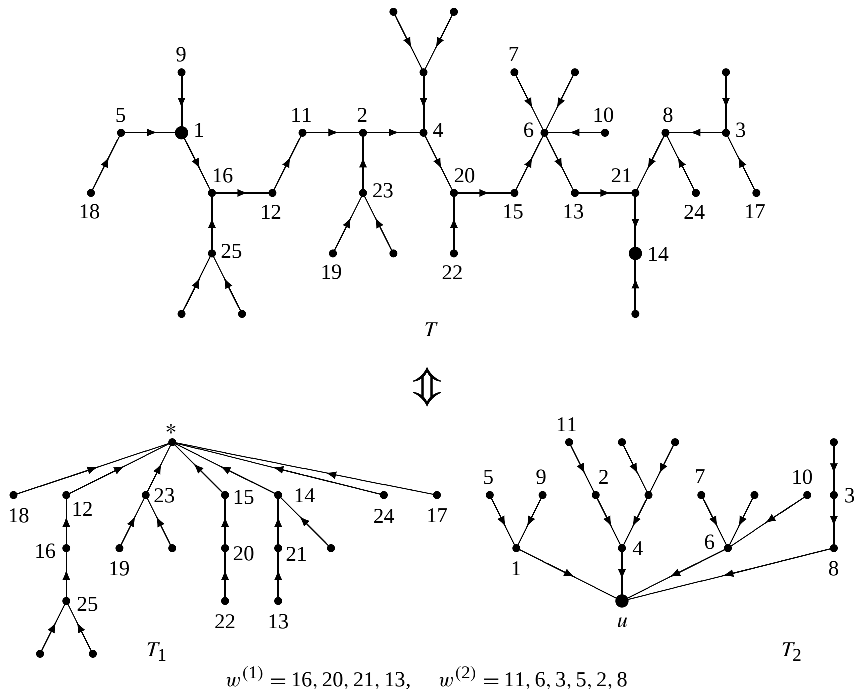


Fig. 2. Example for Lemma 4.1 in the case $r \in H_1$.

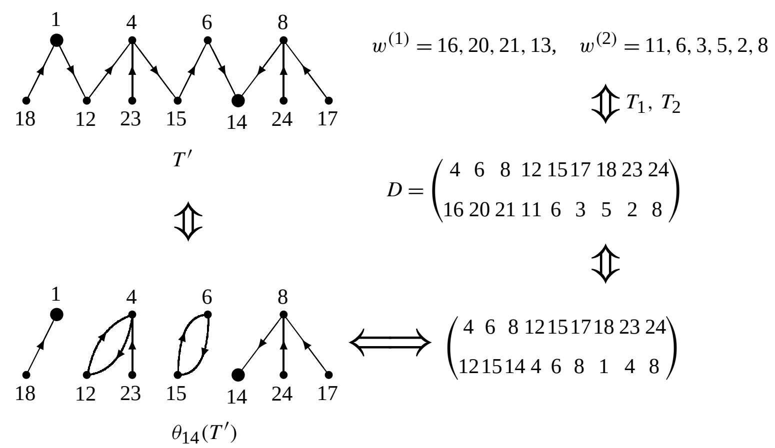
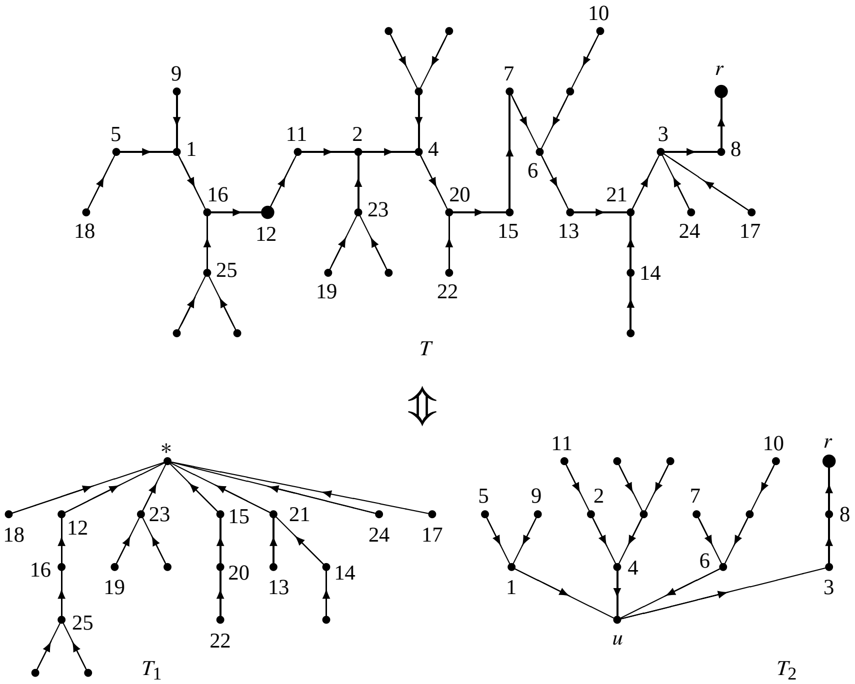


Fig. 3. Explanation of $w^{(1)}$ and $w^{(2)}$ in Fig. 2.



$w^{(1)} = 16, 20, 13, \quad w^{(2)} = 11, 3, 5, 7, 2, 3$

Fig. 4. Example for Lemma 4.1 in the case $r \in V(G_2)$.

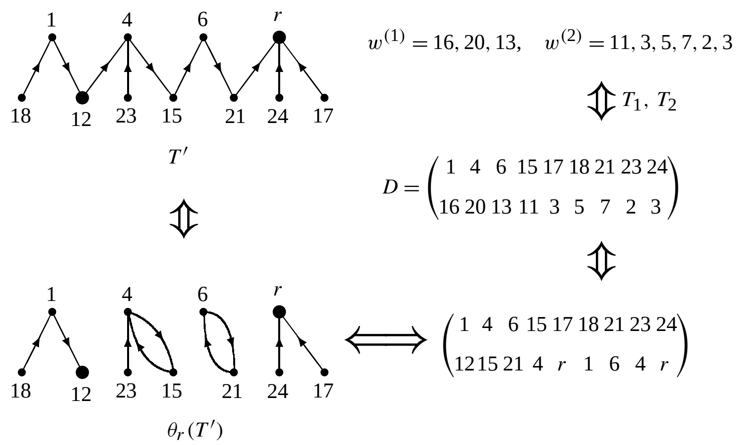


Fig. 5. Explanation of $w^{(1)}$ and $w^{(2)}$ in Fig. 4.

Note that, in the proof of Lemma 4.1, the bijection $T \leftrightarrow (T_1, T_2, w^{(1)}, w^{(2)})$ transforms the root of T into the root of T_2 when it belongs to $V(G_2)$, and vice versa. Hence we actually prove the following assertion:

Lemma 4.2. *Let G, G_1 be disjoint weighted digraphs, G_0 a subgraph of G, $u \in V(G)$, and $H_1 \subseteq V(G_1)$. If $u \in V(G_0)$, then we have*

$$\bar{f}_{G[G_1,H_1,u],\,G_0[H_1,u]}(x,t) = \bar{f}_{G,\,G_0}(x,t)|_{x_u=h_1(x)} \cdot f_{G_1\cup H_1^*}(x,t)|_{x_*=g_u(x,t)}, \tag{4.2}$$

while if $u \notin V(G_0)$, we have

$$\bar{f}_{G[G_1,H_1,u],G_0}(x,t) = \bar{f}_{G,\,G_0}(x,t)|_{x_u=h_1(x)} \cdot f_{G_1\cup H_1^*}(x,t)|_{x_*=g_u(x,t)}, \tag{4.3}$$

where $h_1(x) = \sum_{v\in H_1} x_v$, and $g_u(x,t) = \sum_{(u,v)\in E(G)} x_v t_{(u,v)}$.

5. The blossoming theorem

In the previous section, we obtain a formula for counting the oriented tree volume of a substituted digraph. By using Lemmas 4.1, 4.2, and a series of substitutions of digraphs and variables, we can deduce the main theorem of this paper.

Theorem 5.1 (The blossoming theorem). *Let $G = (V,E)$ be a weighted digraph, and let $\mathcal{G} = \{G_v\colon v \in V\}$, $\mathcal{H} = \{H_v\colon v \in V\}$ be two families of disjoint weighted digraphs such that H_v is a subgraph of G_v for every $v \in V$. We have*

$$\bar{f}_{G(\mathcal{G},\mathcal{H}),\,G(\mathcal{H})}(x,t) = \bar{f}_G(h,t)|_{h_v=h_v(x)} \times \prod_{v\in V} f_{G_v\cup H_v^*}(x,t)|_{x_*=g_v(x,t)}, \tag{5.1}$$

where $h_v(x) = \sum_{u\in V(H_v)} x_u$ and $g_v(x,t) = \sum_{(v,a)\in E(G)} h_a(x) t_{(v,a)}$. In particular,

$$\bar{f}_{G(\mathcal{H})}(x,t) = \bar{f}_G(h,t)|_{h_v=h_v(x)} \times \prod_{v\in V} f_{H_v^*}(x,t)|_{x_*=g_v(x,t)}. \tag{5.2}$$

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$. We introduce the following notation:

$$\begin{aligned}
\mathcal{G}_k &:= \{G_{v_i}\colon 1 \leqslant i \leqslant k\} \cup \bigl\{(\{v_j\}, \emptyset)\colon k+1 \leqslant j \leqslant n\bigr\},\\
\mathcal{H}_k &:= \{H_{v_i}\colon 1 \leqslant i \leqslant k\} \cup \bigl\{(\{v_j\}, \emptyset)\colon k+1 \leqslant j \leqslant n\bigr\},\\
h_v^{(k)}(x) &:= \begin{cases} h_v(x) = \displaystyle\sum_{u\in V(H_v)} x_u, & \text{if } v \in \{v_1,\ldots,v_k\},\\ x_v, & \text{otherwise,}\end{cases}\\
g_v^{(k)}(x,t) &:= \sum_{(v,a)\in E(G)} h_a^{(k)}(x) t_{(v,a)}, \quad \forall v \in V(G).
\end{aligned}$$

By the definition of composite digraphs, it is easy to see

$$G(\mathcal{G}_k,\mathcal{H}_k)=G(\mathcal{G}_{k-1},\mathcal{H}_{k-1})[G_{v_k},H_{v_k},v_k],\qquad G(\mathcal{H}_k)=G(\mathcal{H}_{k-1})[H_{v_k},v_k].$$

As we call (G_{v_k},H_{v_k}) a flower, the process from $G=G(\mathcal{G}_0,\mathcal{H}_0)$ to $G(\mathcal{G},\mathcal{H})=G(\mathcal{G}_n,\mathcal{H}_n)$ is something like blossoming, and any linear ordering of $V(G)$ leads to the same result $G=G(\mathcal{G},\mathcal{H})$. Hence, it is proper to call the following proof the *blossoming algorithm*.

By Lemma 4.2, for $k\leqslant m$, we have

$$\bar{f}_{G(\mathcal{G}_k,\mathcal{H}_k),G(\mathcal{H}_k)}=\bar{f}_{G(\mathcal{G}_{k-1},\mathcal{H}_{k-1}),G(\mathcal{H}_{k-1})}(x,t)|_{x_{v_k}=h_{v_k}(x)}\times f_{G_{v_k}\cup H^*_{v_k}}(x,t)|_{x_*=g^{(k)}_{v_k}(x,t)}.$$

It follows that

$$\begin{aligned}
&\bar{f}_{G(\mathcal{G}_k,\mathcal{H}_k),G(\mathcal{H}_k)}(x,t)|_{x_{v_i}=h_{v_i}(x),\,k+1\leqslant i\leqslant n}\\
&\quad=\bar{f}_{G(\mathcal{G}_{k-1},\mathcal{H}_{k-1}),G(\mathcal{H}_{k-1})}(x,t)|_{x_{v_i}=h_{v_i}(x),\,k\leqslant i\leqslant n}\times f_{G_{v_k}\cup H^*_{v_k}}(x,t)|_{x_*=g_{v_k}(x,t)}.
\end{aligned}$$

By iteration of the above equation, we have

$$\begin{aligned}
&\bar{f}_{G(\mathcal{G},\mathcal{H}),\,G(\mathcal{H})}(x,t)\\
&\quad=\bar{f}_{G(\mathcal{G}_n,\mathcal{H}_n),G(\mathcal{H}_n)}(x,t)\\
&\quad=\bar{f}_{G(\mathcal{G}_{n-1},\mathcal{H}_{n-1}),G(\mathcal{H}_{n-1})}(x,t)|_{x_{vn}=h_{vn}(x)}\times f_{G_{vn}\cup H^*_{vn}}(x,t)|_{x_*=g_{vn}(x,t)}\\
&\quad=\bar{f}_{G(\mathcal{G}_{n-2},\mathcal{H}_{n-2}),G(\mathcal{H}_{n-2})}(x,t)|_{x_{v_j}=h_{v_j}(x),n-1\leqslant j\leqslant n}(x,t)\\
&\qquad\times f_{G_{v_{n-1}}\cup H^*_{v_{n-1}}}(x,t)|_{x_*=g_{v_{n-1}}(x,t)}\times f_{G_{vn}\cup H^*_{vn}}(x,t)|_{x_*=g_{vn}(x,t)}\\
&\quad=\cdots\\
&\quad=\bar{f}_{G(\mathcal{G}_0,\mathcal{H}_0),G(\mathcal{H}_0)}(x,t)|_{x_{v_i}=h_{v_i}(x),1\leqslant i\leqslant n}\times\prod_{1\leqslant j\leqslant n}f_{G_{v_j}\cup H^*_{v_j}}(x,t)|_{x_*=g_{v_j}(x,t)}\\
&\quad=\bar{f}_G(h,t)|_{h_u=h_u(x)}\times\prod_{v\in V}f_{G_v\cup H^*_v}(x,t)|_{x_*=g_v(x,t)}.
\end{aligned}$$

This completes the proof of (5.1). □

An illustration of the blossoming process is shown in Fig. 6, where the directed connection of H_i and H_j (or j) is meant that every vertex of H_i is connected to every vertex of H_j (or the vertex j) with the same direction and weight as the edge ij.

Proof of Theorem 2.1. The proof follows from (5.2) by replacing G with $G^\star$ and $\mathcal{H}$ with $\mathcal{H}\cup\{\star\}$, and the following obvious relations:

$$\begin{aligned}
\bar{f}_{G(\mathcal{H})^\star}(x,t)&=f_{G(\mathcal{H})^\star}(x,t)\cdot x_\star,\\
\bar{f}_{G^\star}(x,t)|_{x_v=h_v(x)}&=f_{G^\star}(x,t)|_{x_v=h_v(x)}\cdot x_\star. \qquad\square
\end{aligned}$$

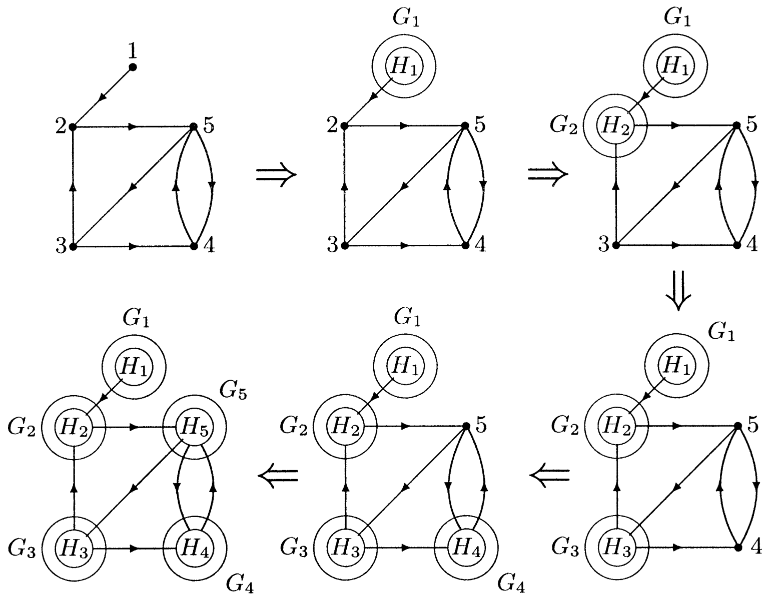


Fig. 6. The blossoming procedure.

It is less obvious to see that (2.5) also leads to (5.2) by comparing the terms independent of $x_\star$ on both sides of (2.5). Thus they are in fact equivalent to each other.

For a (weighted) graph G, G^* is understood to be an undirected graph, and the tree volume $f_G(x,t)$ is defined as (2.3), where $Sp(G)$ denotes the set of spanning trees of G. Eq. (5.2) has many conclusions, we would mention the following:

Corollary 5.2. *Let $G=(V,E)$ be a weighted graph, and let $\mathcal{H}=\{H_v\colon v\in V\}$ be a family of disjoint weighted graphs. Then*

$$f_{G(\mathcal{H})}(x,t)=f_G(h,t)|_{h_v=h_v(x)}\times\prod_{v\in V}f_{H_v^*}(x,t)|_{x_*=g_v(x,t)},$$

where $h_v(x)=\sum_{u\in V(H_v)}x_u$ and $g_v(x,t)=\sum_{(v,a)\in E(G)}h_a(x)t_{(v,a)}$.

Corollary 5.3 (Kelmans [6, Theorem 11]). *Let H_v be the empty graph on v_n vertices. Then*

$$f_{G(\mathcal{H})}(x)=f_G(h)|_{h_u=h_u(x)}\prod_{v\in V(G)}\big(g_v(x)\big)^{n_v-1},$$

where $f_G(x)=f_G(x,t)|_{t_e=1}$.

Corollary 5.4 (Pak and Postnikov [9]). *The number of spanning trees of $G(H_1, \dots, H_n)$ is equal to*

$$\left(\sum_{T \in Sp(G)} \prod_{v \in V} |H_v|^{\deg_T(v)-1}\right)\left(\prod_{v \in V} \sum_{i=1}^{|H_v|} f_i(H_v) g(v)^{i-1}\right),$$

where $f_i(H_v)$ is the number of forests in H_v with i roots and $g(v) = \sum_{(v,a)\in E} |H_a|$.

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