q-Identities from Lagrange and Newton Interpolation

Amy M. Fu
Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P.R. China
Email: fmu@eyou.com

Alain Lascoux
Nankai University, Tianjin 300071, P.R. China
Email: Alain.Lascoux@univ-mlv.fr
CNRS, IGM Université de Marne-la-Vallée
77454 Marne-la-Vallée Cedex, France

ABSTRACT: Combining Newton and Lagrange interpolation, we give q-identities which generalize results of Van Hamme, Uchimura, Dilcher and Prodinger.

1. introduction

Van Hamme [7] gave the following identity involving Gauss polynomials, see also Andrews [1],

$$\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1-q^i} = \sum_{i=1}^{n} \frac{q^i}{1-q^i}, \tag{1.1}$$

where $\begin{bmatrix} n \\ i \end{bmatrix}$ is the Gauss polynomials defined by $\begin{bmatrix} n \\ i \end{bmatrix} = (q;q)_n \left((q;q)_i (q;q)_{n-i}\right)^{-1}$ with $(z;q)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$.

Uchimura [6] generalize (1.1) as following:

$$\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1-q^{i+m}} = \sum_{i=1}^{n} \frac{q^i}{1-q^i} \Big/ \begin{bmatrix} i+m \\ i \end{bmatrix}, \ m \geq 0 \tag{1.2}$$

and by Dilcher [2]:

$$\sum_{1\le i\le n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^{i-1} \frac{q^{\binom{i}{2}+mi}}{(1-q^i)^m} = \sum_{1\le i_1\le i_2\le\cdots\le i_m\le n} \frac{q^{i_1}}{1-q^{i_1}} \cdots \frac{q^{i_m}}{1-q^{i_m}}. \tag{1.3}$$

Prodinger [5] mentioned the following identity as a q-analogue of Kirchenhofer's [3] formula,

$$\sum_{i=0, i\ne M} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1-q^{i-M}} = (-1)^M q^{\binom{M+1}{2}} \begin{bmatrix} n \\ M \end{bmatrix} \sum_{i=0, i\ne M} \frac{q^{i-M}}{1-q^{i-M}} \tag{1.4}$$

and explained how to obtain all these formulas by using Cauchy residues.

We shall show that in fact, all the above formulas are a direct consequence of Newton and Lagrange interpolation.

Given two finite sets of variables $\mathbb{A}$ and $\mathbb{B}$, we denote by $R(\mathbb{A}, \mathbb{B})$ the product $\prod_{a\in\mathbb{A}, b\in\mathbb{B}}(a-b)$, and by $\mathbb{A}\setminus\mathbb{B}$ the set difference of $\mathbb{A}$ and $\mathbb{B}$.

Let $\mathbb{A} = \{x_1, x_2, \cdots\}$, $\mathbb{A}_n = \{x_1, x_2, \cdots, x_n\}$, for any $n \ge 0$. Lagrange wrote the following summation:

$$f(x) = \sum_{i=1}^{n} f(x_i) \frac{R(x, \mathbb{A}\setminus x_i)}{R(x_i, \mathbb{A}\setminus x_i)}, \ \mathrm{mod} R(x, \mathbb{A}_n). \tag{1.5}$$

On the other hand, Newton's development is:

$$f(x) = f(x_1) + f\partial_1 R(x, \mathbb{A}_1) + f\partial_1\partial_2 R(x, \mathbb{A}_2) + \cdots, \tag{1.6}$$

where ∂_i, $i \ge 1$, operating on its left, is defined by

$$f(x_1, x_2, \cdots, x_i, x_{i+1}, \cdots)\partial_i = \frac{f(\cdots, x_i, x_{i+1}, \cdots) - f(\cdots, x_{i+1}, x_i, \cdots)}{x_i - x_{i+1}}.$$

Taking $x_{n+1} = x$, Newton's summation terminates:

$$f(x) = f(x_1) + f\partial_1 R(x, \mathbb{A}_1) + f\partial_1\partial_2 R(x, \mathbb{A}_2) + \cdots + f\partial_1\cdots\partial_n R(x, \mathbb{A}_n).$$

Newton's and Lagrange's expressions imply the same remainder $f\partial_1\cdots\partial_n R(x, \mathbb{A}_n)$, and the polynomial $g_n(x) = f(x) - f\partial_1\cdots\partial_n R(x, \mathbb{A}_n)$ is the only polynomial of degree $\le n-1$ such that $f(x_i) = g(x_i)$, $1 \le i \le n$.

Taking now $f(x) = (y-x)^{-1}$, since

$$\frac{1}{y-x_1}\partial_1 \cdots \partial_{n-1} = \frac{1}{(y-x_1)\cdots(y-x_n)},$$

one gets, by comparing Newton's and Lagrange's expressions, the identity:

$$\sum_{i=0}^{n-1} \frac{R(x,\mathbb{A}_i)}{R(y,\mathbb{A}_{i+1})} = \sum_{i=1}^{n} \frac{f(x_i)R(x,\mathbb{A}\setminus x_i)}{R(x_i,\mathbb{A}\setminus x_i)} = \frac{1}{y-x} - \frac{R(x,\mathbb{A}_n)}{R(y,\mathbb{A}_n)(y-x)}. \quad (1.7)$$

Letting $x=1$, we derive:

$$\sum_{i=0}^{n-1} \frac{R(1,\mathbb{A}_i)}{R(y,\mathbb{A}_{i+1})} = \sum_{i=1}^{n} \frac{f(x_i)R(1,\mathbb{A}\setminus x_i)}{R(x_i,\mathbb{A}\setminus x_i)}. \quad (1.8)$$

Expanding $(y-x)^{-1} = \sum_i y^{-i-1}x^i$ and taking the coefficient of y^{-m-1}, we have:

$$x_1^m \partial_1 \cdots \partial_{n-1} = \sum_{i=1}^{n} \frac{x_i^m}{\prod_{j\neq i}(x_i - x_j)} = h_{m-n+1}(x_1, x_2, \cdots, x_n). \quad (1.9)$$

Recall that complete functions [4] h_k are defined by

$$h_k(x_1, x_2, \cdots, x_n) = \sum_{1\le i_1\le i_2\le\cdots\le i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k}.$$

2. General identities

The identities that we present just correspond to taking $\mathbb{A} = \{\frac{a-bq}{c-zq}, \frac{a-bq^2}{c-zq^2}, \cdots\}$ in Lagrange or Newton interpolation. In that case, the products $R(x_i, \mathbb{A}\setminus x_i)$ are immediate to write.

Proposition 2.1 *Let $m,n\in\mathbb{N}$, $\tau = m-n+1$, a,b,z,q be variables , and $\mathbb{A} = \{\frac{a-bq}{c-zq}, \frac{a-bq^2}{c-zq^2}, \cdots, \frac{a-bq^n}{c-zq^n}\}$. We have:*

$$\sum_{1\le i_1\le i_2\le\cdots\le i_\tau\le n} \frac{a-bq^{i_1}}{c-zq^{i_1}}\frac{a-bq^{i_2}}{c-zq^{i_2}}\cdots\frac{a-bq^{i_\tau}}{c-zq^{i_\tau}} =$$

$$\frac{c^n(zq/c;q)_n}{(q;q)_n(az-bc)^{n-1}} \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1}q^{\binom{i+1}{2}-ni}(1-q^i)(a-bq^i)^m}{(c-zq^i)^{\tau+1}}. \quad (2.10)$$

In particular, for $m = n$:

$$\sum_{i=1}^{n} \frac{a - bq^i}{c - zq^i} = \frac{c^n (zq/c; q)_n}{(q; q)_n (az - bc)^{n-1}} \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2} - ni} (1 - q^i)(a - bq^i)^n}{(c - zq^i)^2} \tag{2.11}$$

Proof. For our choice of $\mathbb{A}$, then $R(x_i, \mathbb{A} \setminus x_i) = \prod_{j \neq i} \frac{(az-bc)(q^i-q^j)}{(c-zq^i)(c-zq^j)}$, and

$$\begin{aligned} &\sum_{i=1}^{n} \frac{((a - bq^i)/(c - zq^i))^m}{\prod_{j \neq i}(\frac{a-bq^i}{c-zq^i} - \frac{a-bq^j}{c-zq^j})} \\ &= \frac{c^n (zq/c; q)_n}{(q; q)_n (az - bc)^{n-1}} \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2} - ni} (1 - q^i)(a - bq^i)^m}{(c - zq^i)^{m-n+2}}. \end{aligned}$$

∎

Consider the case $a = 0$, $b = -1$, $c = 1$ and $z = 1$, i.e. $\mathbb{A} = \{\frac{q}{1-q}, \frac{q^2}{1-q^2}, \cdots \frac{q^n}{1-q^n}\}$. Eq (2.11) implies Van Hamme identity :

$$\begin{aligned} \sum_{i=1}^{n} \frac{q^i}{1 - q^i} &= \frac{(q; q)_n}{(q; q)_n} \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2} - ni} (1 - q^i) q^{ni}}{(1 - q^i)^2} \\ &= \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1 - q^i}. \end{aligned}$$

From (2.10), we have,

$$\sum_{1 \le i_1 \le i_2 \le \cdots \le i_{m-n+1} \le n} \frac{q^{i_1}}{1 - q^{i_1}} \frac{q^{i_2}}{1 - q^{i_2}} \cdots \frac{q^{i_{m-n+1}}}{1 - q^{i_{m-n+1}}} = \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i+1} q^{\binom{i+1}{2} - ni} q^{mi}}{(1 - q^i)^{m-n+1}},$$

replacing $m - n + 1$ by m, we derive (1.3).

Take now $a = 1$, $b = 0$, $c = 0$, $z = -1$, then $\mathbb{A} = \{q^{-1}, q^{-2}, \cdots, q^{-n}\}$, and Eq.(1.8) implies:

$$R.H.S = \sum_{i=0}^{n-1} \frac{R(1, \mathbb{A}_i)}{R(y, \mathbb{A}_{i+1})} = \sum_{i=0}^{n-1} \frac{\prod_{j=1}^{i} (1 - q^{-j})}{\prod_{j=1}^{i+1} (y - q^{-j})} = -\sum_{i=0}^{n-1} \frac{q^{i+1} (q; q)_i}{(yq; q)_{i+1}},$$

$$L.H.S = \sum_{i=1}^{n} \frac{f(x_i)R(1, \mathbb{A} \setminus x_i)}{R(x_i, \mathbb{A} \setminus x_i)}$$

$$= \sum_{i=1}^{n} \frac{(y - q^{-i})^{-1} \prod_{j \neq i}(1 - q^{-j})}{\prod_{j \neq i}(q^{-i} - q^{-j})} = \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^i q^{\binom{i+1}{2}}}{1 - yq^i}.$$

Clearly:

$$\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+1}{2}}}{1 - yq^i} = \sum_{i=0}^{n-1} \frac{q^{i+1}(q;q)_i}{(yq;q)_{i+1}} = \sum_{i=1}^{n} \frac{q^i(q;q)_{i-1}}{(yq;q)_i}.$$

In the special case $y = q^m$, we obtain (1.2).

We take now $\mathbb{A} = \{q^M, q^{M-1}, q^{M-2}, \cdots, q, q^{-1}, \cdots q^{M-n}\}$, with $M \in \mathbb{N}$, and $y = 1$. Eq(1.8) becomes:

$$R.H.S = \sum_{i=0}^{n-1} \frac{R(1, \mathbb{A}_i)}{R(1, \mathbb{A}_{i+1})} = - \sum_{i=0, i \neq M}^{n} \frac{q^{i-M}}{1 - q^{i-M}},$$

and

$$L.H.S = \sum_{i=0, i \neq M}^{n} \frac{1}{1 - q^{M-i}} \frac{\prod_{j \neq i, M}(1 - q^{M-j})}{\prod_{j \neq i, M}(q^{M-i} - q^{M-j})}$$

$$= \sum_{i=0 i \neq M}^{n} \frac{(-1)^{M+i} q^{\binom{i+1}{2} - \binom{M+1}{2}} (q;q)_M (q;q)_{n-M}}{(q;q)_i (q;q)_{n-i}},$$

which proves (1.4).

As a final comment, we would like to stress that we have just used simple alphabets in Newton and Lagrange interpolation; it is easy to generalize the above formulas by taking more sophisticated alphabets.

References

[1] G. E. Andrews and K. Uchimura, *Identities in combinatorics IV: differentiation and harmonic numbers*, Utilitas Mathematica, 28(1985), 265-269.

[2] K. Dilcher, *Some q-series identities related to divisor functions*, Discrete Math 145(1995), 83-93.

[3] P. Kirchenhofer, *A note on alternating sums*, Electronic J. Combinatorics 3(2) (1996), R7, 10 p.

[4] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon, Oxford,1979.

[5] H. Prodinger, *Some applications of the q-Rice formula*, Random Structures and Algo 19(2001), 552-557.

[6] K. Uchimura, *A generalization of identities for the divisor generating function*, Utilitas Mathematica 25(1984), 377-379.

[7] L. Van Hamme, *Advanced problem 6407*, American Math. Monthly 40(1982), 703-704.