Color degree condition for long rainbow paths in edge-colored graphs*

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Abstract

Let G be an edge-colored graph. A rainbow (heterochromatic, or multicolored) path of G is such a path in which no two edges have the same color. Let the color degree of a vertex v to be the number of different colors that are used on edges incident to v, and denote it by $d^c(v)$. In a previous paper, we showed that if $d^c(v) \geq k$ (color degree condition) for every vertex v of G, then G has a rainbow path of length at least $\lceil (k+1)/2 \rceil$. Later, in another paper we first showed that if $k \leq 7$, G has a rainbow path of length at least $k-1$, and then, based on this we used induction on k and showed that if $k \geq 8$, then G has a rainbow path of length at least $\lceil (3k)/5 \rceil + 1$. In 2010, Gyárfás and Mhalla showed that in any proper edge-colored complete graph K_n, there is a rainbow path with no less than $(2n+1)/3$ vertices. In the present paper, by using a simpler approach we further improve the result by showing that if $k \geq 8$, G has a rainbow path of length at least $\lceil (2k)/3 \rceil + 1$.

Keywords: edge-colored graph, color degree, color neighborhood, rainbow (heterochromatic, or multicolored) path.

AMS Subject Classification (2000): 05C38, 05C15

1. Introduction

We use Bondy and Murty [3] for terminology and notation not defined here and consider simple graphs only.

*Supported by NSFC Nos. 10901035, 11371205, PCSIRT and CSC

Let $G = (V, E)$ be a graph. By an *edge-coloring* of G we mean a function $C : E \to \mathbb{N}$, the set of natural numbers. If G is assigned such a coloring, then we say that G is an *edge-colored graph.* Denote the colored graph by (G, C), and call $C(e)$ the *color* of an edge $e \in E$. A subgraph is called *rainbow (heterochromatic, or multicolored)* if any two edges of it have different colors. For a subgraph H of G, we denote $C(H) = \{C(e) \mid e \in E(H)\}$ and $c(H) = |C(H)|$. For a vertex v of G, the *color neighborhood* $CN(v)$ of v is defined as the set $\{C(e) \mid e \text{ is incident with } v\}$ and the *color degree* is $d^c(v) = |CN(v)|$, i.e., the number of different colors that are used on edges incident to v. Given a positive integer k, C is a $k-good$ coloring if $d^c(v) \geq k$ for any vertex v of G. If u and v are two vertices on a path P, uPv denotes the segment of P from u to v, whereas $vP^{-1}u$ denotes the same segment but from v to u.

There are many existing literature dealing with the existence of paths and cycles with special properties in edge-colored graphs. The rainbow Hamiltonian cycle or path problem was studied by Hahn and Thomassen [14], Rödl and Winkler (see [11]), Frieze and Reed [11], and Albert, Frieze and Reed [1]. In [2], Axenovich, Jiang and Tuza gave the range of the maximum k such that there exists a k-good coloring of $E(K_n)$ that contains no properly colored copies of a path with fixed number of edges, no rainbow copies of a path with fixed number of edges, no properly colored copies of a cycle with fixed number of edges and no rainbow copies of a cycle with fixed number of edges, respectively. In [9], Erdös and Tuza studied the rainbow paths in infinite complete graph K_ω. In [10], Erdös and Tuza studied the values of k, such that every k-good coloring of K_n contains a rainbow copy of F where F is a given graph with e edges ($e < n/k$). In [15], Manoussakis, Spyratos and Tuza studied (s, t)-cycle in 2-edge-colored graphs, where (s, t)-cycle is a cycle of length $s + t$ and s consecutive edges are in one color and the remaining t edges are in the other color. In [16], Manoussakis, Spyratos, Tuza and Voigt studied conditions on the minimum number k of colors, sufficient for the existence of given types (such as families of internally pairwise vertex-disjoint paths with common endpoints, hamiltonian paths and hamiltonian cycles, cycles with a given lower bound of their length, spanning trees, stars, and cliques) of properly edge-colored subgraphs in a k-edge-colored complete graph. In [6], Chou, Manoussakis, Megalaki, Spyratos and Tuza showed that for a 2-edge-colored graph G and three specified vertices x, y and z, to decide whether there exists a color-alternating path from x to y passing through z is NP-complete. Many results in these papers were proved by using probabilistic methods.

In [2], Axenovich, Jiang and Tuza considered the local variation of anti-Ramsey problem. Namely, they studied the maximum k such that there exists a k-good edge-coloring of K_n containing no rainbow copies of a given graph H, and denoted by $g(n, H)$. They showed that for a fixed integer $k \geq 2$, $k - 1 \leq g(n, P_{k+1}) \leq 2k - 3$, i.e., if K_n is edge-colored by a $(2k-2)$-good coloring, then there must exist a rainbow path P_{k+1}, there exists a $(k-1)$-good coloring of K_n such that no rainbow path P_{k+1} exists.

In [4], the authors considered the long rainbow paths in general graphs with a k-good coloring and showed that if G is an edge-colored graph with $d^c(v) \geq k$ (color degree condition) for every vertex v of G, then G has a rainbow path of length at least $\lceil (k+1)/2 \rceil$. In [5], we first showed that if $3 \leq k \leq 7$, G has a rainbow path of length at least $k-1$, and then, based on this we used induction on k and showed that if $k \geq 8$, then G has a rainbow path of length at least $\lceil (3k)/5 \rceil + 1$. In the present paper, by using a simpler approach we further improve the result by showing that if $k \geq 8$, G has a rainbow path of length at least $\lceil (2k)/3 \rceil + 1$. This improves the result of [12], in which Gyárfás and Mhalla showed that in any properly edge-colored complete graph K_n, there is a rainbow path with no less than $(2n+1)/3$ vertices. Later, H. Gebauer, and F. Mousset showed in [13] that in any properly edge-colored complete graph K_n, there is a rainbow path with no less than $3n/4 - o(n)$ vertices.

For more references on edge-colorings and cycles, see [7, 8, 17, 18, 19].

2. Some properties of a longest rainbow path

In this section we will give some properties of a longest rainbow path. All these properties will help us to get better lower bounds of the length of a longest rainbow path.

Proposition 2.1 *Let G be an edge-colored graph and suppose that $P = u_1u_2\ldots u_lu_{l+1}$ is a longest rainbow path, v be a vertex not belonging to the path P. For any integer j, $2 \leq j \leq l-1$, if both the two edges u_jv, $u_{j+1}v$ exist, then $|\{C(u_jv), C(u_{j+1}v)\} \setminus C(P)| \leq 1$.*

Proof. By contradiction, if there exists an integer j_0, $2 \leq j_0 \leq l-1$, such that both the two edges $u_{j_0}v$, $u_{j_0+1}v$ exist and $|\{C(u_{j_0}v), C(u_{j_0+1}v)\} \setminus C(P)| = 2$. Then $u_1Pu_{y_{j_0}}vu_{y_{j_0}+1}Pu_{l+1}$ is a rainbow path of length $l+1$, a contradiction. ∎

Proposition 2.2 *Let G be an edge-colored graph and suppose $P = u_1u_2\ldots u_lu_{l+1}$ is a longest rainbow path. If there exists an integer x such that $3 \leq x \leq l$ and $C(u_1u_x) \notin C(P)$, then for any vertex $v \in N(u_{l+1}) \setminus V(P)$, the color of the edge $u_{l+1}v$ is different from $C(u_{x-1}u_x)$.*

Proof. By contradiction. If there exists a vertex $v \in N(u_{l+1}) \setminus V(P)$ such that $C(u_{l+1}v) = C(u_{x-1}u_x)$ (as shown in Figure 2.1), then $u_{x-1}P^{-1}u_1u_xPu_{l+1}v$ is a rainbow path of length $l+1$, a contradiction, which completes the proof. ∎

Proposition 2.3 *Let G be an edge-colored graph and suppose $P = u_1u_2\ldots u_lu_{l+1}$ is a longest rainbow path. If there exists a vertex $v \in N(u_{l+1}) \setminus V(P)$ and an integer x ($2 \leq x \leq l-2$) such that u_xv and $u_{x+2}v$ are edges of G and $|\{C(u_xv), C(u_{x+2}v)\} \setminus C(P)| = 2$, then for any vertex $w \in N(u_{l+1}) \setminus (V(P) \cup \{v\})$, the color of the edge $u_{l+1}w$ is different from $C(u_xu_{x+1})$ and $C(u_{x+1}u_{x+2})$.*

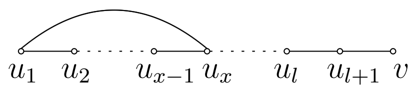


Figure 2.1

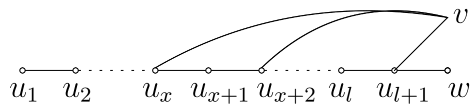


Figure 2.2

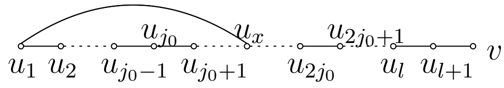


Figure 2.3

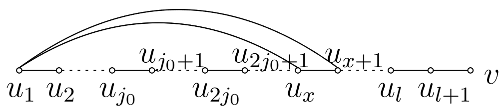


Figure 2.4

Proof. By contradiction. If there exists a vertex $w \in N(u_{l+1}) \setminus (V(P) \cup \{v\})$ such that $C(u_{l+1}w) \in \{C(u_x u_{x+1}), C(u_{x+1}u_{x+2})\}$ (as shown in Figure 2.2), then $u_1 P u_x v u_{x+2} P u_{l+1} w$ is a rainbow path of length $l+1$, a contradiction, which completes the proof. ∎

Proposition 2.4 *Let G be an edge-colored graph and $P = u_1 u_2 \dots u_l u_{l+1} v$ be a path in G such that:*

(a) $u_1 P u_{l+1}$ is a longest rainbow path in G;

(b) $C(u_{l+1}v) = C(u_{j_0} u_{j_0+1})$ for some integer j_0 with $1 \leq j_0 \leq l$.

(c) P was chosen so that j_0 is minimum under the condition (b).

Then we have

(1) for any integer x, $j_0 + 1 \leq x \leq 2j_0$, if the vertex u_x is adjacent to the vertex u_1, then the color of $u_1 u_x$ must appear in P;

(2) for any integer x, $2j_0 \leq x \leq l$, if both the vertices u_x and u_{x+1} are adjacent to the vertex u_1, then $|\{C(u_1 u_x), C(u_1 u_{x+1})\} \setminus C(P)| \leq 1$.

Proof. (1) By contradiction. If there exists an integer x such that $j_0 + 1 \leq x \leq 2j_0$, the vertex u_x is adjacent to the vertex u_1 and the color of the edge $u_1 u_x$ does not appear in C(P) (as shown in Figure 2.3), then $P' = u_{x-1} P^{-1} u_1 u_x P u_{l+1} v$ is a path satisfying that $u_{x-1} P' u_{l+1}$ is a rainbow path of length l and $C(u_{l+1}v) = C(u_{j_0+1} u_{j_0})$ (note that $v \notin V(u_{x-1} P' u_{l+1})$), where $u_{j_0+1} u_{j_0}$ is the $(x - j_0 - 1)$-th edge in this rainbow path $u_{x-1} P' u_{l+1}$. Since $x - j_0 - 1 \leq 2j_0 - j_0 - 1 = j_0 - 1$, this contradicts the choice of P, which completes the proof of (1).

(2) By induction. If there exists an integer x such that $2j_0 + 1 \leq x \leq l$, both the vertices u_x and u_{x+1} are adjacent to the vertex u_1, and the two edges $u_1 u_x$ and $u_1 u_{x+1}$ have distinct colors both of which do not appear in $C(P)$ (see Figure **??**), then $P'' = u_2 P u_x u_1 u_{x+1} P u_{l+1}$ is a path satisfying that $u_2 P'' u_{l+1}$ is a rainbow path of length l and $C(u_{l+1}v) = C(u_{j_0} u_{j_0+1})$ (note that $v \notin V(u_2 P'' u_{l+1})$) is the $(j_0 - 1)$-th edge in the rainbow path $u_2 P'' u_{l+1}$, contradicting the choice of P and completing the proof of (2). ∎

3. New lower bounds for the length of a longest rainbow path

In this section we will give two better lower bound for the length of a longest rainbow path in G when $k \geq 8$. As an induction initial, we need the following result as a lemma.

Lemma 3.1 *[5] Let G be an edge-colored graph and k ($3 \leq k \leq 7$) an integer. Suppose that $d^c(v) \geq k$ for every vertex v of G. Then G has a rainbow path of length at least $k-1$.*

As we showed in [5], $k-1$ is the best lower bound of the length of a longest rainbow path. Therefore, we shall only consider the case when $k \geq 8$ now. We will begin this with an important Lemma.

Lemma 3.2 *Let G be an edge-colored graph and suppose $d^c(v) \geq k \geq 8$ for every vertex $v \in V(G)$. If the length of a longest rainbow path in G is $l \leq \lceil (2k)/3 \rceil$, then there is a path $P = u_1u_2 \ldots u_l u_{l+1} v$ in G such that $u_1 P u_{l+1}$ is a rainbow path of length l and $C(u_{l+1}v) = C(u_1u_2)$.*

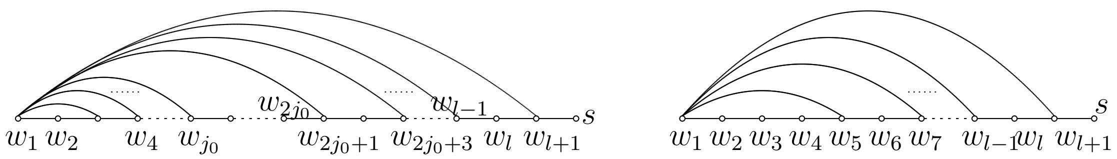
Proof. Let $P' = w_1w_2 \ldots w_l w_{l+1} s$ be a path in G such that
(a) $w_1 P' w_{l+1}$ is a rainbow path of length l;
(b) $C(w_{l+1}s) = C(w_{j_0}w_{j_0+1})$ for some integer j_0 with $1 \leq j_0 \leq l$;
(c) P' was chosen so that j_0 is minimum under the condition (b).

Denote $c_j = C(w_j w_{j+1})$, $j = 1, 2, \ldots, l$. Now we will show that $j_0 = 1$ by contradiction, and then P' is a path we want.

Suppose that $j_0 > 1$. First, we can easily get that $j_0 \leq \lceil (l+1)/2 \rceil$, this is because $CN(w_{l+1}) \subseteq \{C(w_j w_{l+1}) : 1 \leq j \leq l-1,\ w_j \in N(w_{l+1})\} \cup \{c_{j_0}, c_{j_0+1}, \ldots, c_l\}$, and then $k \leq |CN(w_{l+1})| \leq (l-1) + (l - j_0 + 1) = 2l - j_0$.

Since $w_1 P' w_{l+1}$ is a longest rainbow path in G, for any vertex $t \in N(w_{l+1}) \setminus \{w_1, \ldots, w_{l+1}\}$ and any vertex $t' \in N(w_1) \setminus \{w_1, \ldots, w_{l+1}\}$, the color of the edge $w_{l+1}t$ or the edge $w_1 t'$ must appear in P'. This implies that there are at least $k - l$ different colors not in $C(P')$ appearing on some edges in the edge set $\{w_1 t' : t' \in N(w_1) \cap \{w_1, \ldots, w_{l+1}\}\}$. In another words, there are $k - l$ different integers $x_1, x_2, \ldots, x_{k-l}$, such that $3 \leq x_1 < x_2 < \ldots < x_{k-l} \leq l+1$, $w_{x_i} \in N(w_1)$, $1 \leq i \leq k - l$, and the subgraph induced by the edge set $\{w_1w_2, w_2w_3, \ldots, w_l w_{l+1}, w_1 w_{x_1}, w_1 w_{x_2}, \ldots, w_1 w_{x_{k-l}}\}$ is rainbow.

Now we consider the integer set $\{x_1, x_2, \ldots, x_{k-l}\}$. By Proposition 2.4, we can easily get that $\{j_0 + 1, j_0 + 2, \ldots, 2j_0\} \cap \{x_1, x_2, \ldots, x_{k-l}\} = \emptyset$ and if $2j_0 + 1 \leq l$, then for any integer x, $2j_0 + 1 \leq x \leq l$, at most one of $\{x, x+1\}$ belongs to $\{x_1, \ldots, x_{k-l}\}$. Using these two facts, we can get that $k - 1 \leq \lceil (l+1)/2 \rceil - 2$. We will show this in the following three cases:



$j_0 \geq 3$ $\qquad$ $j_0 = 2$

Figure 3.1

Case 1. $2j_0 + 1 \leq l$.

In this case, $k - l \leq (j_0 - 2) + \lceil (l - 2j_0 + 1)/2 \rceil = \lceil (l+1)/2 \rceil - 2$.

Case 2. $2j_0 + 1 = l + 1$, i.e. $l = 2j_0$.

In this case, $\{x_1, x_2, \ldots, x_{k-l}\} \subseteq \{3, 4, \cdots, j_0, l+1\}$, so we have $k - l \leq j_0 - 2 + 1 = 1/2 - 1 = \lceil (l+1)/2 \rceil - 2$ (the last equation holds because l is even).

Case 3. $2j_0 + 1 > l + 1$, i.e. $j_0 > l/2$.

In this case, $\{x_1, x_2, \ldots, x_{k-l}\} \subseteq \{3, 4, \cdots, j_0\}$, so we have $k - l \leq j_0 - 2 \leq \lceil (l+1)/2 \rceil - 2$.

Therefore we shall only consider the case when $k \equiv 2 \pmod 3$ (note that in this case l is even) and $\{x_1, x_2, \ldots, x_{k-l}\}$ is equal to $\{3, \ldots, j_0, 2j_0 + 1, 2j_0 + 3, \ldots, l-1, l+1\}$ if $j_0 \geq 3$, or $\{2j_0+1, 2j_0+3, \ldots, l-1, l+1\}$ if $j_0 = 2$ (as shown in Figure 3.1).

By the fact that $w_{2j}(P')^{-1}w_1w_{2j+1}P'w_{l+1}$ is a rainbow path of length l for any integer j, $j \in \{j_0, j_0+1, \ldots, l/2\}$, and the choice of P', we have that $\{C(w_{l+1}t) : t \in N(w_{l+1}) \backslash P'\} = \{c_{j_0}\}$. Now $CN(w_{l+1}) = \{C(w_{l+1}t) : t \in N(w_{l+1}) \cap P'\} \cup \{c_{j_0}\}$, so $d^c(w_{l+1}) = |CN(w_{l+1})| \leq l + 1 < k$, a contradiction, which concludes that $j_0 = 1$, and P' is the path we want. ∎

By using this lemma, we can easily get a better lower bound of the length of a longest rainbow path.

Theorem 3.3 *Let G be an edge-colored graph. If $d^c(v) \geq k \geq 7$ for any vertex $v \in V(G)$, then G has a rainbow path of length at least $\lceil (2k)/3 \rceil$.*

Proof. By contradiction. Suppose a longest rainbow path in G has a length $l \leq \lceil (2k)/3 \rceil - 1$.

Since $l \leq \lceil (2k)/3 \rceil - 1 < \lceil (2k)/3 \rceil$, we can get by Lemma 3.2 that there exists a longest rainbow path $P = u_1u_2 \cdots u_lu_{l+1}$ and a vertex $v \notin V(P)$ such that $C(u_{l+1}v) = C(u_1u_2)$.

Notice that $u_2Pu_{l+1}v$ is also a rainbow path of length l, i.e., a longest rainbow path. Therefore, for any vertex $u \notin \{u_2, u_3, \cdots, u_l\}$, $C(vu) \in C(P)$. Without loss of generality, suppose that $|\{C(u_{x_1}v), C(u_{x_2}v), \cdots, C(u_{x_t}v)\} \setminus C(P)| = |CN(v) \setminus C(P)| = t$ where $2 \leq x_1 < x_2 < \cdots < x_t \leq l$.

By Lemma 2.1, we have that $x_{j+1} - x_j > 1$ for any $1 \leq j \leq t-1$. Then

$$t \leq \lceil \frac{l-1}{2} \rceil \leq \frac{l}{2}.$$

On the other hand, $CN(v) \subseteq C(P) \cup \{C(u_{x_1}v), C(u_{x_2}v), \cdots, C(u_{x_t}v)\}$. Therefore, $k \leq d^c(v) \leq l + t$. This implies that

$$t \geq k - l.$$

From the two inequations above, we can get that $k - l \leq t \leq l/2$. So $l \geq (2k)/3$, a contradiction. Therefore, G has a rainbow path of length at least $\lceil (2k)/3 \rceil$. ∎

In the remaining part of this section, we will show that under the color degree condition, the length of a longest rainbow path is at least $\lceil (2k)/3 \rceil + 1$.

Theorem 3.4 *Let G be an edge-colored graph. If $d^c(v) \geq k \geq 7$ for any vertex $v \in V(G)$, then G has a rainbow path of length at least $\lceil (2k)/3 \rceil + 1$.*

Proof. We will prove the theorem by induction on k.

If $k = 7$, our Lemma 2.1 guarantees that G has a rainbow path of length at least 6, where $6 = \lceil (2 \times 7)/3 \rceil + 1$.

So we may assume that $k \geq 8$ and that the result holds for all smaller values of k.

Now we need only to show that if $d^c(v) \geq k$ for any $v \in V(G)$, G has a rainbow path of length $\lceil (2k)/3 \rceil + 1$. By the assumption, we know that G has a rainbow path of length $\lceil (2(k-1))/3 \rceil + 1$, which is equal to $\lceil 2k/3) \rceil + 1$ when $k \equiv 0 \pmod 3$, and $\lceil (2k)/3 \rceil$ otherwise. So if $k \equiv 0 \pmod 3$, we are done. Therefore, the rest is only to show that if $k \equiv 1, 2 \pmod 3$, G has a rainbow path of length $\lceil (2k)/3 \rceil + 1$. We will show this by contradiction.

Assume that a longest rainbow path in G is of length $l = \lceil (2k)/3 \rceil$. Then we have that $k - l \geq 2$, and we can get by Lemma 3.2 that G has a rainbow path $P = u_1u_2 \ldots u_lu_{l+1}$ and there exists a vertex $v \in N(u_{l+1}) \setminus V(P)$ such that $C(u_{l+1}v) = C(u_1u_2)$. Denote $c_j = C(u_ju_{j+1})$, $j = 1, 2, \ldots, l$.

Since $d^c(v) \geq k$, $d^c(u_1) \geq k$, and the two paths P and $u_2Pu_{l+1}v$ are both rainbow paths of length l, we have that there are at least $k - l$ different colors not belonging to the color set $C(P)$ appearing in the edge set $\{C(u_1u_j) : 3 \leq j \leq l+1, \text{ and } u_j \in N(u_1)\}$, and there are also at least $k - l$ different colors not belonging to the color set $C(P)$ appearing in the color set $\{C(u_jv) : 2 \leq j \leq l, \text{ and } u_j \in N(v)\}$. So we can conclude that there exist two integer sets

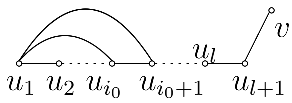


Figure 3.2

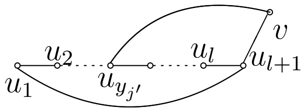


Figure 3.3

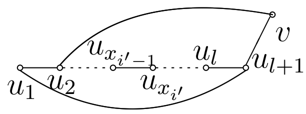


Figure 3.4

$\{x_1, x_2, \ldots, x_{k-l}\}$ and $\{y_1, y_2, \ldots, y_{k-l}\}$, such that $3 \leq x_1 < x_2 < \ldots < x_{k-l} \leq l+1$, $2 \leq y_1 < y_2 < y_3 < \ldots < y_{k-l} \leq l$, $u_{x_i} \in N(u_1)$ $(i = 1, 2, \ldots, k-l)$, $u_{y_j} \in N(v)$ $(j = 1, 2, \ldots, k-l)$, and $|\{C(u_1u_{x_1}), C(u_1u_{x_2}), \ldots, C(u_1u_{x_{k-l}})\} \setminus C(P)| = k-l$, $|\{C(u_{y_1}v), C(u_{y_2}v), \ldots, C(u_{y_{k-l}}v)\} \setminus C(P)| = k-l$.

Note that we can easily get the following three claims:

Claim 1 For any integer i, $3 \leq i \leq l$, if both the two edges u_1u_i, u_1u_{i+1} exist, then $|\{C(u_1u_i), C(u_1u_{i+1})\} \setminus C(P)| \leq 1$.

Otherwise, there exists an integer i_0, $3 \leq i_0 \leq l$, such that both the two edges $u_1u_{i_0}$, $u_1u_{i_0+1}$ exist and $|\{C(u_1u_{i_0}), C(u_1u_{i_0+1})\} \setminus C(P)| = 2$ (see Figure 3.2). Then $u_2Pu_{i_0}u_1u_{i_0+1}Pu_{l+1}v$ is a rainbow path of length $l+1$, a contradiction.

Claim 2 If the edge u_1u_{l+1} exists, then the color of the edge must appear in P.

Otherwise, we have that the edge u_1u_{l+1} exists and the color of it is not contained in $C(P)$. Since $k-l \geq 2$, there exists an integer j', $1 \leq j' \leq k-l$ such that $C(u_{y_{j'}}v) \neq C(u_1u_{l+1})$ (see Figure 3.3). Then $vu_{y_{j'}}Pu_{l+1}u_1Pu_{y_{j'}-1}$ is a rainbow path of length $l+1$, a contradiction.

Claim 3 If the edge u_2v exists, then the color of the edge must appear in P.

Otherwise, we have that the edge u_2v exists and the color of it is not contained in the color set $C(P)$. Since $k-l \geq 2$, there exists an integer i', $1 \leq i' \leq k-l$, such that $C(u_1u_{x_{i'}}) \neq C(u_2v)$ (see Figure 3.4). Then $u_{x_{i'}-1}P^{-1}u_2vu_{l+1}P^{-1}u_{x_{i'}}u_1$ is a rainbow path of length $l+1$, a contradiction.

From the four claims above, we can get that $3 \leq x_1 < x_1+1 < x_2 < x_2+1 < \ldots < x_{k-l} \leq l$ and $3 \leq y_1 < y_1+1 < y_2 < y_2+1 < \ldots < y_{k-l} \leq l$.

Now we distinguish the following two cases:

Case 1. $k \equiv 1 \pmod 3$. (Then $l = \lceil (2k)/3 \rceil = (2k+1)/3$ must be odd.)

Since $3 \leq y_1 < y_1+1 < y_2 < y_2+1 < \ldots < y_{k-l} \leq l$, we have $2(k-l-1) \leq y_{k-l} - y_1 \leq l-3$. On the other hand, we have $2(k-l-1) = l-3$. This implies that $\{y_1, y_2, \ldots, y_{k-l}\} = \{3, 5, \ldots, l-2, l\}$. Then by Proposition 2.3, we can conclude that $\{c_3, c_4, \ldots, c_{l-2}, c_l\} \cap \{C(u_{l+1}v') : v' \in N(u_{l+1}) \setminus V(P)\} = \emptyset$. So $CN(u_{l+1}) \subseteq \{C(u_{l+1}v') : v' \in N(u_{l+1}) \cap V(P)\} \cup \{c_1, c_2, c_l\} = \{C(u_{l+1}v') : v' \in N(u_{l+1}) \cap V(P)\} \cup \{c_1, c_2\}$, and hence we get that $d^c(u_{l+1}) = |CN(u_{l+1})| \leq l+2 < k$ (the last inequality holds because $k \geq 8$, $k \equiv 1 \pmod 3$ and $l = (2k+1)/3$) a contradiction.

Case 2. $k \equiv 2 \pmod 3$. (Then $l = \lceil (2k)/3 \rceil = (2k+1)/3$ must be even.)

Since $3 \le y_1 < y_1 + 1 < y_2 < y_2 + 1 < \ldots < y_{k-l} \le l$, we have $2(k-l-1) \le y_{k-l} - y_1 \le l - 3$. On the other hand, we have $2(k-l-1) = (l-3) - 1$. Then we can conclude that $y_{j+1} = y_j + 2$ for $j = 1, 2, \ldots, k-l-1$ or there exists an integer j_0 such that $1 \le j_0 \le k-l-1$, $y_{j_0+1} = y_{j_0} + 3$, and $y_{j+1} = y_j + 2$ for any $1 \le j \le k-l-1$ and $j \ne j_0$.

Case 2.1 $y_{j+1} = y_j + 2$ for $j = 1, 2, \ldots, k-l-1$.

Now we have $\{C(u_{l+1}v') : v' \in N(u_{l+1}) \setminus V(P)\} \cap \{c_{y_1}, c_{y_1+1}, c_{y_2}, c_{y_2+1}, \ldots, c_{y_{k-l}-1}\} = \emptyset$ by Proposition 2.3, and $\{C(u_{l+1}v') : v' \in N(u_{l+1}) \setminus V(P)\} \cap \{c_{x_1-1}, c_{x_2-1}, \ldots, c_{x_{k-l}-1}\} = \emptyset$ by Proposition 2.2. Therefore, $\{C(u_{l+1}v') : v' \in N(u_{l+1}) \setminus V(P)\} \subseteq C(P) \setminus (\{c_{y_1}, c_{y_1+1}, c_{y_2}, \ldots, c_{y_{k-l}-1}\} \cup \{c_{x_1-1}, c_{x_2-1}, \ldots, c_{x_{k-l}-1}\})$, since P is a longest rainbow path.

Notice that $y_{j+1} = y_j + 2$ for $j = 1, 2, \ldots, k-l-1$ and $3 \le x_1 < x_1 + 1 < x_2 < \ldots < x_{k-l} \le l$. Then we have $(y_{k-l} - 1) - y_1 = 2(k-l-1) - 1 < 2(k-l-1) \le (x_{k-l} - 1) - (x_1 - 1)$. This implies that $\{c_{x_1-1}, c_{x_2-1}, \ldots, c_{x_{k-l}-1}\} \setminus \{c_{y_1}, c_{y_1+1}, c_{y_2}, \ldots, c_{y_{k-l}-1}\} \ne \emptyset$. So we get

$$\begin{aligned} k &\le d^c(u_{l+1}) = |CN(u_{l+1})| \\ &\le |\{c_1, c_2, \ldots, c_l\} \setminus (\{c_{y_1}, c_{y_1+1}, \ldots, c_{y_{k-l}-1}\} \cup \{c_{x_1-1}, c_{x_2-1}, \ldots, c_{x_{k-l}-1}\})| \\ &\quad + |\{C(u_{l+1}u_j) : 1 \le j \le l-1 \text{ and } u_j \in N(u_{l+1})\}| \\ &\le (l - 2(k-l-1) - 1) + (l-1) = 4l - 2k \qquad (3.1) \end{aligned}$$

Since if $k \equiv 2 \pmod 3$ and $k > 8$, then $4l - 2k < k$, a contradiction. So we shall only consider the case when $k = 8$.

If $k = 8$, we have $l = (2k+1)/3 = 6$, then we have $y_1 = 3$, $y_2 = 5$ or $y_1 = 4$, $y_2 = 6$. Denote $c_7 = C(u_{y_1}v_1)$, $c_8 = C(u_{y_2}v_1)$. On the other hand, since $4l - 2k = 8 = k$, from equation (3.1) the only case we need to consider is the case when all the edges $u_1u_{l+1}, u_2u_{l+1}, \ldots, u_{l-1}u_{l+1}$ exist and

$$\begin{aligned} &CN(u_{l+1}) = (\{c_1, c_2, \ldots, c_l\} \setminus (\{c_{y_1}, c_{y_1+1}\} \cup \{c_{x_1-1}, c_{x_2-1}\})) \\ &\cup \{C(u_{l+1}u_j) : 1 \le j \le l-1 \text{ and } u_j \in N(u_{l+1})\}, \qquad (3.2) \\ &|\{c_{x_1-1}, c_{x_2-1}\} \setminus \{c_{y_1}, c_{y_1+1}\}| = 1, \qquad (3.3) \\ &|\{C(u_{l+1}u_j) : 1 \le j \le l-1 \text{ and } u_j \in N(u_{l+1})\}| = l-1, \qquad (3.4) \\ &\{C(u_1u_{l+1}), C(u_2u_{l+1}), \ldots, C(u_{l-1}u_{l+1})\} \\ &\cap (C(P) \setminus (\{c_{y_1}, c_{y_1+1}\} \cup \{c_{x_1-1}, c_{x_2-1}\})) = \emptyset. \qquad (3.5) \end{aligned}$$

Case 2.1.1 $y_1 = 3$ and $y_2 = 5$ (see Figure 3.5).

Then by equation (3.3), we need only to consider the cases when $x_1 = 3$ and $x_2 = 5$, or $x_1 = 4$ and $x_2 = 6$. Now we can conclude by Claim 2 and equations (3.4), (3.5) that $C(u_1u_7) \in \{c_2, c_3, c_4\}$ if $x_1 = 3$, $x_2 = 5$, and $C(u_1u_7) \in \{c_3, c_4, c_5\}$ if $x_1 = 4$, $x_2 = 6$. It is easy to check from Figure 3.5 that if $C(u_1u_7) = c_3$ or c_5, then $u_4u_5vu_3u_2u_1u_7u_6$ is a rainbow path of length 7; if $C(u_1u_7) = c_2$ or c_4, then

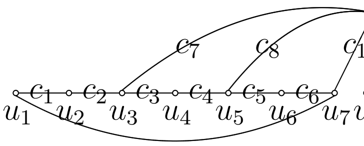


Figure 3.5

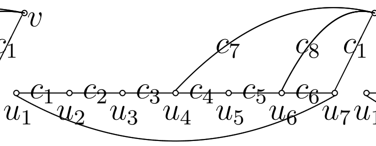


Figure 3.6

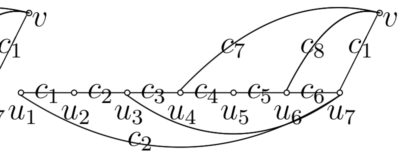


Figure 3.7

$u_4u_3vu_5u_6u_7u_1u_2$ is a rainbow path of length 7. In another words, there always is a rainbow path of length 7 in all these cases, a contradiction.

Case 2.1.2 $y_1 = 4$ and $y_2 = 6$ (see Figure 3.6).

Then by equation (3.3), we need only to consider the cases when $x_1 = 3$ and $x_2 = 5$, or $x_1 = 3$ and $x_2 = 6$, or $x_1 = 4$ and $x_2 = 6$.

Now we can conclude by Claim 2 and equations (3.4), (3.5) that $C(u_1u_7) \in \{c_2, c_4, c_5\}$ if $x_1 = 3$, $x_2 = 5$ or $x_1 = 3$, $x_2 = 6$, and $C(u_1u_7) \in \{c_3, c_4, c_5\}$ if $x_1 = 4$, $x_2 = 6$.

It is easy to check from Figure 3.6 that if $C(u_1u_7) = c_3$ or c_5, then $u_5u_4vu_6u_7u_1u_2u_3$ is a rainbow path of length 7; if $C(u_1u_7) = c_4$, then $u_5u_6vu_4u_3u_2u_1u_7$ is a rainbow path of length 7, a contradiction. It remains us to consider the case when $C(u_1u_7) = c_2$ (see Figure 3.7), and $3 \in \{x_1, x_2\}$ only.

Then we have $x_1 = 3$, $x_2 = 5$, or $x_1 = 3$, $x_2 = 6$. So $|\{C(u_2u_7), C(u_3u_7), C(u_4u_7), C(u_5u_7)\}| = 4$ and $\{C(u_2u_7), C(u_3u_7), C(u_4u_7), C(u_5u_7)\} \cap \{c_1, c_3, c_4, c_5, c_6\} \subseteq \{c_4, c_5\}$, since $C(u_1u_7) = c_2$ and because of the equations (3.2), (3.4), (3.5). So the edge u_3u_7 is in color c_4, or color c_5, or some color not appearing in P. It is easy to check from Figure 3.7 that if the edge u_3u_7 is in color c_4, then $u_2u_1u_7u_3u_4vu_6u_5$ is a rainbow path of length 7; if $C(u_3u_7) = c_5$, then $u_5u_4vu_6u_7u_3u_2u_1$ is a rainbow path of length 7; if $C(u_3u_7) = c_7$, then $u_2u_1u_7u_3u_4u_5u_6v$ is a rainbow path of length 7; if the edge u_3u_7 is in a color not belonging to the color set $\{c_1, c_2, \dots, c_7\}$, then $vu_4u_5u_6u_7u_3u_2u_1$ is a rainbow path of length 7. So there always is a rainbow path of length 7 in all these cases, a contradiction.

Case 2.2 There exists an integer j_0 such that $1 \le j_0 \le k-l-1, y_{j_0+1} = y_{j_0}+3$, and $y_{j+1} = y_j + 2$ for any $1 \le j \le k-l-1$ and $j \ne j_0$.

Then we have $\{C(u_{l+1}v') : v' \in N(u_{l+1}) \setminus V(P)\} \cap \{c_{y_1}, c_{y_1+1}, \dots, c_{y_{j_0}-1}, c_{y_{j_0+1}}, c_{y_{j_0+1}+1}, \dots, c_{y_{k-l}-1}\} = \emptyset$ by Lemma 2.3, and $\{C(u_{l+1}v') : v' \in N(u_{l+1}) \setminus V(P)\} \cap \{c_{x_1-1}, c_{x_2-1}, \dots, c_{x_{k-l}-1}\} = \emptyset$ by Lemma 2.2. Therefore,

$$\begin{aligned} CN(u_{l+1}) &= \{C(u_{l+1}v') : v' \in N(u_{l+1}) \cap V(P)\} \cup \\ &\quad \{C(u_{l+1}v') : v' \in N(u_{l+1}) \setminus V(P)\} \\ &\subseteq \{C(u_{l+1}u_j) : 1 \le j \le l-1 \text{ and } u_j \in N(u_{l+1})\} \cup \\ &\quad (C(P) \setminus (\{c_{x_1-1}, c_{x_2-1}, \dots, c_{x_{k-l}-1}\} \cup \\ &\quad \{c_{y_1}, c_{y_1+1}, \dots, c_{y_{j_0}-1}, c_{y_{j_0+1}}, c_{y_{j_0+1}+1}, \dots, c_{y_{k-l}-1}\})). \quad (3.6) \end{aligned}$$

Since $3 \le x_1 < x_1 + 1 < x_2 < x_2 + 1 < \ldots < x_{k-l-1} < x_{k-l-1} + 1 < x_{k-l} \le l$, we can easily get that there are at most $(j_0 - 1)$ different integers $i(1 \le i \le k-l)$

such that x_i appears in the set $\{y_1, y_1+1, \ldots, y_{j_0}-1\}$ and at most $(k-l-j_0-1)$ different integers $i(1 \leq i \leq k-l)$ such that x_i appears in the set $\{y_{j_0+1}, y_{j_0+1}+1, \ldots, y_{k-l}-1\}$. This implies that

$$\begin{aligned} & |\{c_{x_1-1}, c_{x_2-1}, \ldots, c_{x_{k-l}-1}\} \setminus \{c_{y_1}, c_{y_1+1}, \ldots, c_{y_{j_0}-1}, c_{y_{j_0+1}}, c_{y_{j_0+1}+1}, \ldots, c_{y_{k-l}-1}\}| \\ \geq & \ 2. \end{aligned} \tag{3.7}$$

Consequently, we have $k \leq |CN(u_{l+1})| \leq (l-1) + (l - 2(j_0-1) - 2(k-l-j_0-1) - 2) = 4l - 2k + 1$. So we shall only consider the case when $k = 8$ and the case when $k = 11$, since if $k \equiv 2 \pmod 3$ and $k > 11$, then $4l - 2k + 1 < k$, a contradiction.

Case 2.2.1 $k = 8$. In this case, $l = 6$ and $y_1 = 3$, $y_2 = 6$. Denote $c_7 = C(u_3v)$ and $c_8 = C(u_6v)$. We distinguish the following cases according to x_1 and x_2:

Case 2.2.1.1 $x_1 = 3$ and $x_2 = 5$ (see Figure 3.8).

Then we can get from Proposition 2.2 that $\{C(u_7v') : v' \in N(u_7) \setminus V(P)\} \cap \{c_2, c_4\} = \emptyset$. So $CN(u_7) \subseteq \{C(u_ju_7) : 1 \leq j \leq 6 \text{ and } u_j \in N(u_7)\} \cup \{C(u_7v') : v' \in N(u_7) \setminus V(P)\} \subseteq \{C(u_ju_7) : 1 \leq j \leq 5 \text{ and } u_j \in N(u_7)\} \cup \{c_1, c_3, c_5, c_6\}$. Since $|CN(u_7)| \geq 8$, this implies that

$$|\{C(u_ju_7) : 1 \leq j \leq 5 \text{ and } u_j \in N(u_7)\} \setminus \{c_1, c_3, c_5, c_6\}| \geq 4. \tag{3.8}$$

Now, we will consider the existence of the edge u_1u_7 in G and the color of it if it does exist.

Subcase 1 The edge u_1u_7 exists and $C(u_1u_7) = c_2$.

It is obvious that $vu_3u_4u_5u_6u_7u_1u_2$ is a rainbow path of length 7 in this subcase, a contradiction.

Subcase 2 The edge u_1u_7 exists and $C(u_1u_7) = c_4$ (see Figure 3.9).

In this case, $\{C(u_ju_7) : 2 \leq j \leq 5 \text{ and } u_j \in N(u_7)\} \setminus \{c_1, c_3, c_4, c_5, c_6, c_7, c_8\} \neq \emptyset$, because of the inequality (3.8).

Subcase 2.1 There exists some $i \in \{2, 4, 5\}$ such that the edge u_iu_7 exists and the color of it does not belong to the color set $\{c_1, c_3, c_4, c_5, c_6, c_7, c_8\}$.

It is easy to check in Figure 3.9 that if $i = 2$, then $vu_3u_4u_5u_6u_7u_2u_1$ is a rainbow path of length 7; if $i = 4$, then $u_5u_6vu_3u_4u_7u_1u_2$ is a rainbow path of length 7; if $i = 5$, then $u_4u_3vu_6u_5u_7u_1u_2$ is a rainbow path of length 7, a contradiction.

Subcase 2.2 The edge u_3u_7 exists and the color of it does not belong to $\{c_1, c_3, c_4, c_5, c_6, c_7, c_8\}$.

In this subcase, $u_2u_1u_5u_4u_3u_7u_6v$ is a rainbow path of length 7 if $C(u_1u_5) = c_7$. On the other hand, since $5 = x_2$, so we may assume that $C(u_1u_5) = c_8$ (see Figure 3.10).

Since $u_1u_2u_3vu_6u_5u_4$ is a rainbow path of length 6 with the color set $\{c_1, c_2, c_4, c_5, c_7, c_8\}$, $\{C(u_1w) : w \in N(u_1) \cap \{u_3, u_4, u_6, v\}\} \setminus \{c_1, c_2, c_4, c_5, c_6, c_7, c_8\} \neq \emptyset$. It is easy to check from Figure 3.10 that if the edge u_1u_4 exists and the color of it does not belong to the color set $\{c_1, c_2, c_4, c_5, c_6, c_7, c_8\}$, then $u_5u_4u_1u_2u_3vu_6u_7$ is a rainbow path of length 7; if the edge u_1u_6 exists and the color of it does not belong to the color set $\{c_1, c_2, c_4, c_5, c_6, c_7, c_8\}$, then $u_2u_3vu_7u_6u_1u_5u_4$ is a rainbow path of length 7; if the edge u_1v exists and the color of it does not belong to the color set $\{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$, then $vu_1u_2u_3u_4u_5u_6u_7$ is a rainbow path of length 7; if the edge u_1v exists and the color of it is c_3, then $u_2u_3u_1vu_7u_6u_5u_4$ is a rainbow path of length 7, a contradiction. So the edge u_1u_3 exists and is in a color not belonging to the color set $\{c_1, c_2, c_4, c_5, c_6, c_7, c_8\}$, which implies that the edge u_1u_3 is in a color not belonging to the color set $\{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$ since $3 = x_1$. Denote $c_9 = C(u_1u_3)$ (as shown in Figure 3.11).

From the analysis above, we now have $\{C(u_1w) : w \in N(u_1)\cap\{u_3, u_4, u_6, v\}\} \subseteq \{c_1, c_2, c_4, c_5, c_6, c_7, c_8, c_9\}$. On the other hand, because of the fact that P is a rainbow path of length 6 and Claim 1, we have $\{C(u_1w) : w \in N(u_1) \cap \{u_3, u_4, u_6, v\}\}\cap\{c_7\} = \emptyset$. So $\{C(u_1w) : w \in N(u_1)\cap\{u_3, u_4, u_6, v\}\} \subseteq \{c_1, c_2, c_4, c_5, c_6, c_8, c_9\}$, and then $\{C(u_1w) : w \in N(u_1)\cap\{u_3, u_4, \ldots, u_6, u_7, v\}\} \subseteq \{c_1, c_2, c_4, c_5, c_6, c_8, c_9\}$. Since $d^c(u_1) \geq 8$ and P is a rainbow path of length 6, there exists a vertex $v' \notin \{u_1, u_2, \ldots, u_7, v\}$ such that $C(u_1v') = c_3$. Then, $v'u_1u_2u_3vu_6u_5u_4$ is a rainbow path of length 7, a contradiction.

Subcase 3 The edge u_1u_7 exists and the color of it is other than c_2 and c_4, or the edge u_1u_7 does not exist.

We can conclude from Claim 2 and the inequality (3.8) that the edges u_2u_7, u_3u_7, u_4u_7, u_5u_7 all exist and $|\{C(u_2u_7), C(u_3u_7), C(u_4u_7), C(u_5u_7)\}\setminus\{c_1, c_3, c_5, c_6\}| = 4$. Then

$$CN(u_7) = \{C(u_2u_7), C(u_3u_7), C(u_4u_7), C(u_5u_7)\} \cup \{c_1, c_3, c_5, c_6\}. \tag{3.9}$$

Note that if the edge u_1u_7 exists and $C(u_1u_7) = c_3$, then $u_4u_5u_6vu_3u_2u_1u_7$ is a rainbow path of length 7; if $C(u_1u_7) = c_5$, then $u_5u_4u_3v_1u_6u_7u_1u_2$ is a rainbow path of length 7. So we can conclude from the equation (3.9) that there exist two vertices $v', v'' \notin V(P)$ such that $C(u_7v') = c_3$, $C(u_7v'') = c_5$. On the other hand, since $x_2 = 5$, the edge u_1u_5 exists and has a color not belonging to the color set $C(P)$. If $C(u_1u_5) \neq c_7$, then $vu_3u_2u_1u_5u_6u_7v'$ is a rainbow path of length 7, and so we assume that $C(u_1u_5) = c_7$ (as shown in Figure 3.12). Now $vu_6u_5u_1u_2u_3u_4$ is a rainbow path of length 6 with color set $\{c_1, c_2, c_3, c_5, c_7, c_8\}$, and so we get that there exists an integer j, $1 \leq j \leq 5$, such that the edge u_jv exists and $C(u_jv) \notin \{c_1, c_2, c_3, c_5, c_6, c_7, c_8\}$, since $d^c(v) \geq 8$. It is easy to check from Figure 3.12 that if the edge u_1v exists and the color of it does not belong to the color set $\{c_1, c_2, c_3, c_5, c_6, c_7, c_8\}$, then $u_4u_3u_2u_1vu_6u_7v''$ is a rainbow path of length 7; if the edge u_2v exists and the color of it does not belong to the color set $\{c_1, c_2, c_3, c_5, c_6, c_7, c_8\}$, then $u_4u_3u_2vu_7u_6u_5u_1$ is a rainbow path of length 7; if the edge u_4v exists and the color of it does not belong to the color set $\{c_1, c_2, c_3, c_5, c_6, c_7, c_8\}$, then $vu_4u_3u_2u_1u_5u_6u_7$ is a rainbow path of

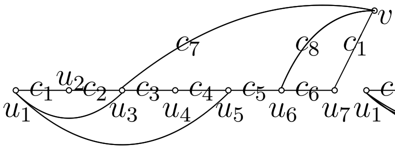


Figure 3.8

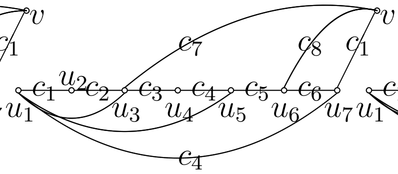


Figure 3.9

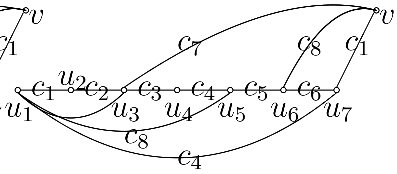


Figure 3.10

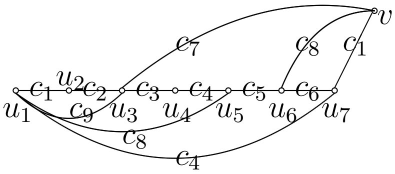


Figure 3.11

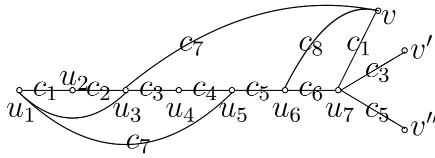


Figure 3.12

length 7; if the edge u_5v exists and the color of it does not belong to the color set $\{c_1, c_2, c_3, c_5, c_6, c_7, c_8\}$, then $u_4u_3u_2u_1u_5vu_6u_7$ is a rainbow path of length 7, a contradiction.

Case 2.2.1.2 $x_1 = 3$ and $x_2 = 6$ (see Figure 3.13).

Then we can get from Proposition 2.2 that $\{C(u_{l+1}v') : v' \in N(u_{l+1})\setminus V(P)\} \cap \{c_2, c_5\} = \emptyset$. So $CN(u_{l+1}) \subseteq \{C(u_ju_7) : 1 \leq j \leq 6 \text{ and } u_j \in N(u_7)\} \cup \{C(u_{l+1}v') : v' \in N(u_{l+1}) \setminus V(P)\} \subseteq \{C(u_ju_7) : 1 \leq j \leq 5 \text{ and } u_j \in N(u_7)\} \cup \{c_1, c_3, c_4, c_6\}$. Since $|CN(u_7)| \geq 8$, this implies that

$$|\{C(u_ju_7) : 1 \leq j \leq 5 \text{ and } u_j \in N(u_7)\} \setminus \{c_1, c_3, c_4, c_6\}| \geq 4. \tag{3.10}$$

Note that if the edge u_1u_7 exists and $C(u_1u_7) = c_2$, then $vu_3u_4u_5u_6u_7u_1u_2$ is a rainbow path of length 7; if the edge u_1u_7 exists and $C(u_1u_7) = c_5$, then $u_5u_4u_3vu_6u_7u_1u_2$ is a rainbow path of length 7, a contradiction. Then we can get from Claim 2 and the inequality (3.10) that the four edges u_2u_7, u_3u_7, u_4u_7, u_5u_7 all exist, and

$$|\{C(u_2u_7), C(u_3u_7), C(u_4u_7), C(u_5u_7)\} \setminus \{c_1, c_3, c_4, c_6\}| = 4. \tag{3.11}$$

Now we consider the color of the edge u_2u_7. If $C(u_2u_7) \notin \{c_1, c_3, c_4, c_5, c_6, c_7\}$, then $vu_3u_4u_5u_6u_7u_2u_1$ is a rainbow path of length 7; if $C(u_2u_7) = c_5$, then $u_5u_4u_3vu_6u_7u_2u_1$ is a rainbow path of length 7, a contradiction. So we get from the inequality(3.11) that $C(u_2u_7) = c_7$ (see Figure 3.14).

On the other hand, $u_1u_2u_3vu_6u_5u_4$ is a rainbow path of length 6 with the color set $\{c_1, c_2, c_4, c_5, c_7, c_8\}$, so we have that there exists a vertex $w \in \{u_3, u_4, u_5, u_6, v\}$ such that the edge u_1w exists and $C(u_1w) \notin \{c_1, c_2, c_3, c_4, c_5, c_7, c_8\}$ since $d^c(u_1) \geq 8$. It is easy to check from Figure 3.14 that if the edge u_1u_3 has a color not belonging to the color set $\{c_1, c_2, c_3, c_4, c_5, c_7, c_8\}$, then $u_1u_3u_2u_7vu_6u_5u_4$ is a rainbow path of length 7; if the edge u_1u_4 exists and has a color not belonging to

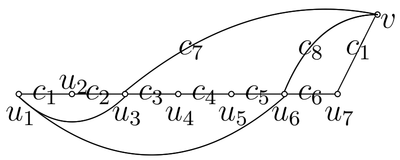


Figure 3.13

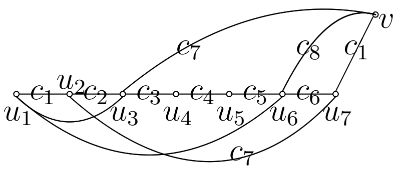


Figure 3.14

the color set $\{c_1, c_2, c_3, c_4, c_5, c_7, c_8\}$, then $u_1u_4u_3u_2u_7vu_6u_5$ is a rainbow path of length 7; if the edge u_1u_5 exists and has a color not belonging to the color set $\{c_1, c_2, c_3, c_4, c_5, c_7, c_8\}$, then $u_1u_5u_4u_3u_2u_7vu_6$ is a rainbow path of length 7; if the edge u_1u_6 has a color not belonging to the color set $\{c_1, c_2, c_3, c_4, c_5, c_7, c_8\}$, then $u_1u_6u_5u_4u_3u_2u_7v$ is a rainbow path of length 7; if the edge u_1v exists and has a color not belonging to the color set $\{c_1, c_2, c_3, c_4, c_5, c_7, c_8\}$, then $u_1vu_7u_2u_3u_4u_5u_6$ is a rainbow path of length 7, a contradiction.

Case 2.2.1.3 $x_1 = 4$ and $x_2 = 6$ (see Figure 3.15).

Then we can get from Proposition 2.2 that $\{C(u_{l+1}v') : v' \in N(u_{l+1})\setminus V(P)\} \cap \{c_3, c_5\} = \emptyset$. So $CN(u_{l+1}) \subseteq \{C(u_ju_7) : 1 \le j \le 6 \text{ and } u_j \in N(u_7)\} \cup \{C(u_{l+1}v') : v' \in N(u_{l+1}) \setminus V(P)\} \subseteq \{C(u_ju_7) : 1 \le j \le 5 \text{ and } u_j \in N(u_7)\} \cup \{c_1, c_2, c_4, c_6\}$. Since $|CN(u_7)| \ge 8$, this implies that

$$|\{C(u_ju_7) : 1 \le j \le 5 \text{ and } u_j \in N(u_7)\} \setminus \{c_1, c_2, c_4, c_6\}| \ge 4. \qquad (3.12)$$

Note that if the edge u_1u_7 exists and $C(u_1u_7) = c_3$, then $vu_3u_2u_1u_7u_6u_5u_4$ is a rainbow path of length 7; if $C(u_1u_7) = c_5$, then $vu_6u_7u_1u_2u_3u_4u_5$ is a rainbow path of length 7, a contradiction. So we can get from Claim 2 and the inequality (3.12) that all the four edges u_2u_7, u_3u_7, u_4u_7, u_5u_7 exist and

$$|\{C(u_2u_7), C(u_3u_7), C(u_4u_7), C(u_5u_7)\} \setminus \{c_1, c_2, c_4, c_6\}| = 4, \qquad (3.13)$$

$$CN(u_7) = \{C(u_2u_7), C(u_3u_7), C(u_4u_7), C(u_5u_7)\} \cup \{c_1, c_2, c_4, c_6\}. \qquad (3.14)$$

If $C(u_1u_4) \ne c_7$, then $vu_3u_2u_1u_4u_5u_6u_7$ is a rainbow path of length 7, a contradiction. So we have $C(u_1u_4) = c_7$ and then $C(u_1u_6) \notin \{c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$ because $x_1 = 4$, $x_2 = 6$ and from the way we choose x_1, x_2.

If the edge u_1u_7 exists and $C(u_1u_7) = c_2$, then $vu_3u_4u_5u_6u_7u_1u_2$ is a rainbow path of length 7, a contradiction. So we can conclude from the equations (3.13) and (3.14) that there exists a vertex $v' \notin \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ such that $C(u_7v') = c_2$ (see Figure 3.16). Then, $u_1u_6u_5u_4u_3vu_7v'$ is a rainbow path of length 7, a contradiction.

So, in the case $k = 8$, there always is a rainbow path of length 7 in G.

Case 2.2.2 $k = 11$, then $l = 8$.

Denote $c_9 = C(u_{y_1}v_1)$, $c_{10} = C(u_{y_2}v_1)$ and $c_{11} = C(u_{y_3}v_1)$.

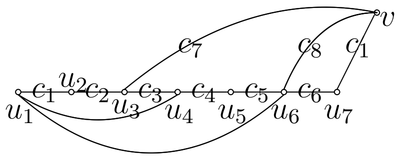


Figure 3.15

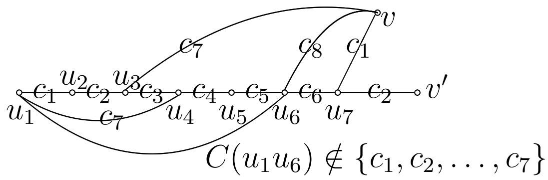


Figure 3.16

By the two equations (3.6) and (3.7), we have $11 = k \leq |CN(u_{l+1})| \leq (l-1) + (l - 2(j_0 - 1) - 2(k - l - j_0 - 1) - 2) = 4l - 2k + 1 = 11$. So we shall only consider the case when all the edges $u_1u_9, u_2u_9, \ldots, u_7u_9$ exist and

$$|\{C(u_1u_9), C(u_2u_9), \ldots, C(u_7u_9)\}| = 7, \tag{3.15}$$

$$\{C(u_1u_9), C(u_2u_9), \ldots, C(u_7u_9)\} \cap (C(P) \setminus (\{c_{x_1-1}, c_{x_2-1}, c_{x_3-1}\} \cup \{c_{y_1}, c_{y_1+1}, \ldots, c_{y_{j_0}-1}, c_{y_{j_0+1}}, c_{y_{j_0+1}+1}, \ldots, c_{y_3-1}\})) = \emptyset, \tag{3.16}$$

$$CN(u_9) = \{C(u_1u_9), C(u_2u_9), \ldots, C(u_7u_9)\} \cup (C(P) \setminus (\{c_{x_1-1}, c_{x_2-1}, c_{x_3-1}\} \cup \{c_{y_1}, c_{y_1+1}, \ldots, c_{y_{j_0}-1}, c_{y_{j_0+1}}, c_{y_{j_0+1}+1}, \ldots, c_{y_3-1}\})), \tag{3.17}$$

$$|\{c_{x_1-1}, c_{x_2-1}, c_{x_3-1}\} \setminus \{c_{y_1}, c_{y_1+1}, \ldots, c_{y_{j_0}-1}, c_{y_{j_0+1}}, c_{y_{j_0+1}+1}, \ldots, c_{y_3-1}\}| = 2. \tag{3.18}$$

Now we distinguish the following two cases according to j_0:

Case 2.2.2.1 $j_0 = 1$, then $y_1 = 3$, $y_2 = 6$ and $y_3 = 8$ (see Figure 3.17)

In this case $\{c_{y_1}, c_{y_1+1}, \ldots, c_{y_{j_0}-1}, c_{y_{j_0+1}}, c_{y_{j_0+1}+1}, \ldots, c_{y_3-1}\} = \{c_6, c_7\}$. So we can easily get that $\{c_{x_1-1}, c_{x_2-1}\} \cap \{c_6, c_7\} = \emptyset$ and $c_{x_3-1} \in \{c_6, c_7\}$ since $3 \leq x_1 < x_1 + 1 < x_2 < x_2 + 1 < x_3 \leq 8$ and from equation (3.18). Then we have $\{C(u_1u_9), C(u_2u_9), \ldots, C(u_7u_9)\} \cap (\{c_1, c_2, c_3, c_4, c_5, c_8\} \setminus \{c_{x_1-1}, c_{x_2-1}\}) = \emptyset$ from the equation (3.16). So, $C(u_1u_9) \in \{c_{x_1-1}, c_{x_2-1}, c_6, c_7\} \subseteq \{c_2, c_3, c_4, c_5, c_6, c_7\}$ because of Claim 2 and the equations (3.15) and (3.16). It is easy to check in Figure 3.17 that if the edge u_1u_9 has color c_2, then $vu_3u_4u_5u_6u_7u_8u_9u_1u_2$ is a rainbow path of length 9; if the edge u_1u_9 has color c_3, then $u_4u_5u_6u_7u_8u_9u_1u_2u_3v$ is a rainbow path of length 9; if the edge u_1u_9 has color c_5, then $vu_6u_7u_8u_9u_1u_2u_3u_4u_5$ is a rainbow path of length 9; if the edge u_1u_9 has color c_6, then $u_7u_8u_9u_1u_2u_3u_4u_5u_6v$ is a rainbow path of length 9; if the edge u_1u_9 has color c_7, then $vu_8u_9u_1u_2u_3u_4u_5u_6u_7$ is a rainbow path of length 9, a contradiction. So it remains us to consider the case when $C(u_1u_9) = c_4$ and $5 \in \{x_1, x_2\}$.

Then $4 \notin \{x_1, x_2\}$ since $C(u_1u_9) \in \{c_{x_1-1}, c_{x_2-1}, c_6, c_7\}$ and $3 \leq x_1 < x_1 + 1 < x_2 < x_2 + 1 < x_3 \leq 8$. Therefore, we can get from the equations (3.16) and (3.17) that there exists a vertex $v' \notin V(P)$ such that $C(u_9v') = c_3$ (see Figure 3.18). Now $u_5u_6u_7u_8v_1u_3u_2u_1u_9v'$ is a rainbow path of length 9, a contradiction.

Case 2.2.2.2 $j_0 = 2$, then $y_1 = 3$, $y_2 = 5$ and $y_3 = 8$ (see Figure 3.20).

In this case $\{c_{y_1}, c_{y_1+1}, \ldots, c_{y_{j_0}-1}, c_{y_{j_0+1}}, c_{y_{j_0+1}+1}, \ldots, c_{y_3-1}\} = \{c_3, c_4\}$. So we can easily get that $x_1 = 3$, $x_2 = 5$ and $x_3 = 7$, or $x_1 = 3$, $x_2 = 5$ and

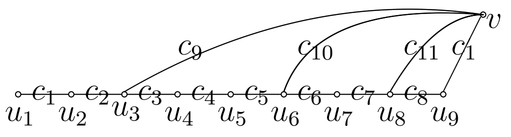


Figure 3.17

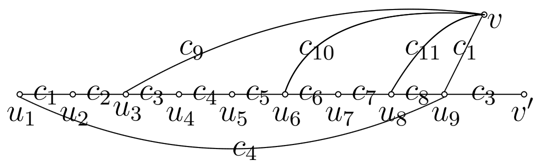


Figure 3.18

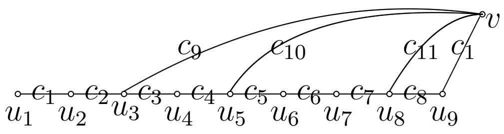


Figure 3.19

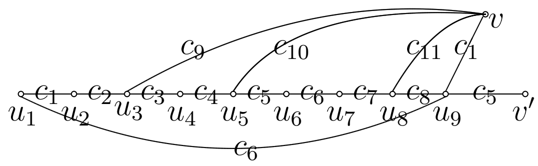


Figure 3.20

$x_3 = 8$, or $x_1 = 4$, $x_2 = 6$ and $x_3 = 8$, since $3 \leq x_1 < x_1 + 1 < x_2 < x_2 + 1 < x_3 \leq 8$ and form equation (3.18). On the other hand, we have that $C(u_1u_9) \in \{c_3, c_4, c_{x_1-1}, c_{x_2-1}, c_{x_3-1}\} \subseteq \{c_2, c_3, c_4, c_5, c_6, c_7\}$ because of Claim 2 and the equations (3.15) and (3.16). It is easy to check in Figure 3.20 that if the edge u_1u_9 has color c_2, then $vu_3u_4u_5u_6u_7u_8u_9u_1u_2$ is a rainbow path of length 9; if the edge u_1u_9 has color c_3, then $u_4u_5u_6u_7u_8u_9u_1u_2u_3v$ is a rainbow path of length 9; if the edge u_1u_9 has color c_4, then $vu_5u_6u_7u_8u_9u_1u_2u_3u_4$ is a rainbow path of length 9; if the edge u_1u_9 has color c_5, then $u_6u_7u_8u_9u_1u_2u_3u_4u_5v$ is a rainbow path of length 9; if the edge u_1u_9 has color c_7, then $vu_8u_9u_1u_2u_3\ u_4u_5u_6u_7$ is a rainbow path of length 9, a contradiction. So it remains us to consider the case when $C(u_1u_9) = c_6$ and $c_6 \in \{c_{x_1-1}, c_{x_2-1}, c_{x_3-1}\}$.

Then we can conclude that $x_1 = 3$, $x_2 = 5$, and $x_3 = 7$ since $C(u_1u_9) \in \{c_3, c_4, c_{x_1-1}, c_{x_2-1}, c_{x_3-1}\}$ and $3 \leq x_1 < x_1+1 < x_2 < x_2+1 < x_3 \leq 8$. Therefore, we can get from the equations (3.16) and (3.17) that there exists a vertex $v' \notin V(P)$ such that $C(u_9v_2) = c_5$ (see Figure **??**). Now $u_7u_8vu_5u_4u_3u_2u_1u_9v'$ is a rainbow path of length 9, a contradiction.

So, in the case $k = 11$, there always is a rainbow path of length 9 in G, a contradiction.

Up to now, from all the above contradictions we can conclude that if $d^c(v) \geq k \geq 7$ for any vertex $v \in V(G)$, then G has a rainbow path of length at least $\lceil (2k)/3 \rceil + 1$ in G. ∎

4. Remarks

In this paper, we consider long rainbow paths in edge-colored general graphs. However, if we restrict graphs to properly edge-colored complete graphs, this is an important topic in combinatorial design [12].

If G is a properly edge-colored complete graph with n vertices, then any vertex v in G has color degree $k = n - 1$. Therefore, by Theorem 3.3, we can get the following conclusion.

Corollary 4.1 *In every proper edge-coloring of K_n, there exists a rainbow path of length at least $\lceil (2n+1)/3 \rceil$.*

This improves the result of [12], since in [12], Gyárfás and Mhalla claimed that there exists a rainbow path with at least $\lceil (2n+1)/3 \rceil$ vertices, i.e., a rainbow path of length at least $\lceil (2n+1)/3 \rceil - 1$.

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