The 3-rainbow index of a graph*

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Abstract

Let G be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \ldots, q\}$, $q \in \mathbb{N}$, where adjacent edges may be colored the same. A tree T in G is a *rainbow tree* if no two edges of T receive the same color. For a vertex subset $S \subseteq V(G)$, a tree that connects S in G is called an S-tree. The minimum number of colors that are needed in an edge-coloring of G such that there is a rainbow S-tree for each k-subset S of $V(G)$ is called the k-rainbow index of G, denoted by $rx_k(G)$. In this paper, we first determine the graphs of size m whose 3-rainbow index equals m, $m-1$, $m-2$ or 2. We also obtain the exact values of $rx_3(G)$ when G is a regular multipartite complete graph or a wheel. Finally, we give a sharp upper bound for $rx_3(G)$ when G is 2-connected and 2-edge connected. Graphs G for which $rx_3(G)$ attains this upper bound are determined.

Keywords: rainbow tree, S-tree, k-rainbow index.

AMS subject classification 2010: 05C05, 05C15, 05C75.

1 Introduction

We follow the terminology and notation of Bondy and Murty [1]. Let G be a nontrivial connected graph with an edge-coloring $c : E(G) \to \{1, 2, \ldots, q\}$, $q \in \mathbb{N}$, where adjacent edges may be colored the same. A path of G is a *rainbow path* if no two edges of the path are colored the same. The graph G is *rainbow connected* if for every two vertices u and v of G, there is a rainbow path connecting u and v. The minimum number of colors for which there is an edge-coloring of G such that G is rainbow connected is called the

*Supported by NSFC Nos. 11371205 and 11071130.

rainbow connection number of G, denoted by $rc(G)$. Results on the rainbow connections can be found in [2, 3, 5, 6, 7].

These concepts were introduced by Chartrand et al. in [3]. In [4], they generalized the concept of rainbow path to rainbow tree. A tree T in G is a *rainbow tree* if no two edges of T receive the same color. For $S \subseteq V(G)$, a *rainbow* S-*tree* is a rainbow tree that connects the vertices of S. Given a fixed integer k with $2 \leq k \leq n$, the edge-coloring c of G is called a *k-rainbow coloring* if for every k-subset S of $V(G)$, there exists a rainbow S-tree. In this case, G is called *k-rainbow connected*. The minimum number of colors that are needed in a *k-rainbow coloring* of G is called the *k-rainbow index* of G, denoted by $rx_k(G)$. Clearly, when $k = 2$, $rx_2(G)$ is the rainbow connection number $rc(G)$ of G. For every connected graph G of order n, it is easy to see that $rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G)$.

The *Steiner distance* $d(S)$ of a subset S of vertices in G is the minimum size of a tree in G that connects S. Such a tree is called a *Steiner S-tree* or simply a *Steiner tree*. The *k-Steiner diameter* $sdiam_k(G)$ of G is the maximum Steiner distance of S among all k-subsets S of G. Then there is a simple upper bound and a lower bound for $rx_k(G)$.

Observation 1 ([4])**.** For every connected graph G of order $n \geq 3$ and each integer k with $3 \leq k \leq n$, $k - 1 \leq sdiam_k(G) \leq rx_k(G) \leq n - 1$.

They showed that trees are composed of a class of graphs whose k-rainbow index attains the upper bound.

Proposition 1 ([4])**.** Let T be a tree of order $n \geq 3$. For each integer k with $3 \leq k \leq n$, $rx_k(T) = n - 1$.

Before showing Proposition 1, they gave the following observation.

Observation 2 ([4])**.** Let G be a connected graph of order n containing two bridges e and f. For each integer k with $2 \leq k \leq n$, every k-rainbow coloring of G must assign distinct colors to e and f.

For $k = 2$, $rx_2(G) = rc(G)$, which has been studied extensively, see [6, 7]. But for $k \geq 3$, very few results has been obtained. In this paper, we focus on $k = 3$. By Observation 1, we have $rx_3(G) \geq 2$. On the other hand, if G is a nontrivial connected graph of size m, then the coloring that assigns distinct colors to the edges of G is a 3-rainbow coloring, hence $rx_3(G) \leq m$. So we want to determine the graphs whose 3-rainbow index equals the values 2, m, $m - 1$ and $m - 2$, respectively. The following results are needed.

Lemma 1 ([4])**.** *For* $3 \leq n \leq 5$, $rx_3(K_n) = 2$.

Lemma 2 ([4])**.** *Let G be a connected graph of order $n \geq 6$. For each integer k with $3 \leq k \leq n$, $rx_k(G) \geq 3$.*

Theorem 1 ([4]). *For each integer k and n with $3 \leq k \leq n$,*

$$rx_k(C_n) = \begin{cases} n-2, & \text{if } k=3 \text{ and } n \geq 4; \\ n-1, & \text{if } k=n=3 \text{ or } 4 \leq k \leq n. \end{cases} \tag{1}$$

Theorem 2 ([4]). *If G is a unicyclic graph of order $n \geq 3$ and girth $g \geq 3$, then*

$$rx_k(G) = \begin{cases} n-2, & k=3 \text{ and } g \geq 4; \\ n-1, & g=3 \text{ or } 4 \leq k \leq n. \end{cases} \tag{2}$$

The following observation is easy to verify.

Observation 3. Let G be a connected graph and H be a connected spanning subgraph of G. Then $rx_3(G) \leq rx_3(H)$.

In Section 2, we determine the graphs whose 3-rainbow index equals the values m, $m-1$, $m-2$ or 2. In Section 3, we determine the 3-rainbow index for the complete bipartite graphs $K_{r,r}$ and complete t-partite graphs $K_{t\times r}$ as well as the wheel W_n. Finally, we give a sharp upper bound of $rx_3(G)$ for 2-connected graphs and 2-edge connected graphs, and graphs whose 3-rainbow index attains the upper bound are characterized.

2 Graphs with $rx_3(G)$= $\mathbf{m, m-1, m-2 or 2}$

From Lemma 2, if $rx_3(G) = 2$, then the order n of G satisfies $3 \leq n \leq 5$.

Theorem 3. *Let G be a connected graph of order n. Then $rx_3(G) = 2$ if and only if $G = K_5$ or G is a 2-connected graph of order 4 or G is of order 3.*

Proof. If $n = 3$, it is easy to see that $rx_3(G) = 2$.

If $n = 4$, assume that G is not 2-connected, then there is a cut vertex v. It is easy to see that a tree connecting the vertices of $G - v$ has size 3, thus $rx_3(G) \geq 3$, a contradiction.

If $n = 5$, let $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$. Assume that $rx_3(G) = 2$ but G is not K_5. Let $c : E(G) \to \{1, 2\}$ be a rainbow coloring of G. Since every three vertices belong to a rainbow path of length 2, there is no monochromatic triangle. Now we show that the maximum degree $\Delta(G)$ is 4. If $\Delta(G)$ is 2, then G is a cycle or a path, and it is easy to check that $rx_3(G)$ is 3 or 4, a contradiction. Assume that $\Delta(G)$ is 3. Let $deg(v_1) = 3$ and $N(v_1) = \{v_2, v_3, v_4\}$. Then at least two edges incident to v_1 have the same color, say $c(v_1v_2) = c(v_1v_3) = 1$. Consider $\{v_1, v_2, v_5\}$, $\{v_1, v_3, v_5\}$, this forces $c(v_2v_5) = c(v_3v_5) = 2$. Consider $\{v_1, v_2, v_3\}$, it implies that $c(v_2v_3) = 2$, but now $\{v_2, v_3, v_5\}$ forms a monochromatic triangle, a contradiction. Thus $\Delta(G) = 4$. Suppose $d(v_1) = 4$. If there are three edges incident to v_1 colored the same, say $c(v_1v_2) = c(v_1v_3) = c(v_1v_4) = 1$, then consider the three vertices v_2, v_3 and v_4. Since these three vertices must belong to a rainbow path of length 2, without loss of generality, assume that $c(v_2v_3) = 1$ and $c(v_3v_4) = 2$. However then $\{v_1, v_2, v_3\}$ is a monochromatic triangle, which is impossible. Therefore only two edges incident to v_1 are assigned the same color. Since G is not K_5, G is a spanning subgraph of $K_5 - e$. Since $d(v_1) = 4$, we may assume that G is a spanning subgraph of $K_5 - v_3v_4$. Let $G' = K_5 - v_3v_4$. Consider $\{v_1, v_3, v_4\}$, v_1v_3 and v_1v_4 must have different colors, without loss of generality, assume that $c(v_1v_3) = 1$ and $c(v_1v_4) = 2$. By symmetry, suppose $c(v_1v_2) = 1$ and $c(v_1v_5) = 2$. Then $c(v_2v_3) = 2$, $c(v_4v_5) = 1$. Consider $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, $\{v_2, v_3, v_5\}$, then $c(v_2v_4) = 1$, $c(v_3v_5) = 2$, $c(v_2v_5) = 1$, but now $\{v_2, v_4, v_5\}$ forms a monochromatic triangle, which is impossible. Hence, $rx_3(G) \geq rx_3(G') \geq 3$, contradicting to the assumption. □

Theorem 4. *Let G be a connected graph of size $m \geq 3$. Then*

(1) $rx_3(G) = m$ if and only if G is a tree.

(2) $rx_3(G) = m - 1$ if and only if G is a unicyclic graph with girth 3.

(3) $rx_3(G) = m - 2$ if and only if G is a unicyclic graph with girth at least 4.

Proof. (1). By Proposition 1, if G is a tree, then $rx_3(G) = n - 1 = m$. Conversely, if $rx_3(G) = m$ but G is not a tree, then $m \geq n$. By Observation 1, $rx_3(G) \leq n-1 \leq m-1$, a contradiction.

(2). If G is a unicyclic graph with girth 3, by Theorem 2, $rx_3(G) = n - 1 = m - 1$. Conversely, if $rx_3(G) = m - 1$, then by (1), G must contain cycles. If G contains at least two cycles, then $m \geq n + 1$. By Observation 1, $rx_3(G) \leq n - 1 \leq m - 2$, a contradiction. Thus, G contains exactly one cycle. If the cycle of G is of length at least 4, then by Theorem 2, $rx_3(G) = n - 2 = m - 2$, a contradiction. Thus, the cycle of G is of length 3, the result holds.

(3). If G is a unicyclic graph with girth at least 4, by Theorem 2, $rx_3(G) = n-2 = m-2$. Conversely, if $rx_3(G) = m - 2$ and $m \geq n + 2$, then by Observation 1, $rx_3(G) \leq n - 1 \leq m - 3$, a contradiction. Thus, $m \leq n + 1$. If $m = n$, then G is a unicyclic graph. By Theorem 2, the girth of G is at least 4. If $m = n+1$, and there are two edge-disjoint cycles C_1 and C_2 of length g_1 and g_2 such that $g_1 \geq g_2$, then if $g_1 \geq 4$, we assign $g_1 - 2$ colors to C_1, $g_2 - 1$ new colors to C_2 and assign new distinct colors to all the remaining edges, which make G 3-rainbow connected, hence $rx_3(G) \leq m - 3$, a contradiction. Therefore $g_1 = g_2 = 3$. In this case, we assign each cycle with three colors 1, 2, 3, and assign new colors to all the remaining edges, then G is 3-rainbow connected, thus $rx_3(G) \leq m-3$. If these two cycles are not edge-disjoint, we can also use $m - 3$ colors to make G 3-rainbow connected, a contradiction. □

3 The 3-rainbow index of some special graphs

In this section, we determine the 3-rainbow index of some special graphs. First, we consider the regular complete bipartite graphs $K_{r,r}$. It is easy to see that when $r = 2$, $rx_3(K_{2,2}) = 2$ and, logically, we can define $rx_3(K_{1,1}) = 0$.

Theorem 5. *For each integer r with $r \geq 3$, $rx_3(K_{r,r}) = 3$.*

Proof. Let U and W be the partite sets of $K_{r,r}$, where then $|U| = |W| = r$. Suppose that $U = \{u_1, \cdots, u_r\}$ and $W = \{w_1, \cdots, w_r\}$. If $S \subseteq U$ and $|S| = 3$, then every S-tree has size at least 3; hence $rx_3(K_{r,r}) \geq 3$.

Next we show that $rx_3(K_{r,r}) \leq 3$. We define a coloring c: $E(K_{r,r}) \to \{1, 2, 3\}$ as follows.

$$c(u_i w_j) = \begin{cases} 1, & 1 \leq i = j \leq r; \\ 2, & 1 \leq i < j \leq r; \\ 3, & 1 \leq j < i \leq r. \end{cases} \tag{3}$$

Now we show that c is a 3-rainbow coloring of $K_{r,r}$. Let S be a set of three vertices of $K_{r,r}$. We consider two cases.

Case 1. The vertices of S belong to the same partite set of $K_{r,r}$. Without loss of generality, let $S = \{u_i, u_j, u_k\}$, where $i < j < k$. Then $T = \{u_i w_j, u_j w_j, u_k w_j\}$ is a rainbow S-tree.

Case 2. The vertices of S belong to different partite sets of $K_{r,r}$. Without loss of generality, let $S = \{u_i, u_j, w_k\}$, where $i < j$.

Subcase 2.1. $k < i < j$. Then $T = \{u_i w_k, u_i w_j, u_j w_j\}$ is a rainbow S-tree.

Subcase 2.2. $i \leq k \leq j$. Then $T = \{u_i w_k, u_j w_k\}$ is a rainbow S-tree.

Subcase 2.3. $i < j < k$. Then $T = \{u_i w_i, u_j w_i, u_j w_k\}$ is a rainbow S-tree. □

With the aid of Theorem 5, we are now able to determine the 3-rainbow index of complete t-partite graph $K_{t\times r}$. Note that we always have $t \geq 3$. When $r = 1$, $rx_3(K_{t\times 1}) = rx_3(K_t)$, which was given in [4].

Theorem 6. *Let $K_{t\times r}$ be a complete t-partite graph, where $r \geq 2$ and $t \geq 3$. Then $rx_3(K_{t\times r}) = 3$.*

Proof. Let $U_1, U_2, \cdots, U_t$ be the t partite sets of $K_{t\times r}$, where $|U_i| = r$. Suppose that $U_i = \{u_{i1}, \cdots, u_{ir}\}$. If $S \subseteq U_i$ and $|S| = 3$, then every S-tree has size at least 3, hence $rx_3(K_{r,r}) \geq 3$.

Next we show that $rx_3(K_{t\times r}) \leq 3$. We define a coloring c: $E(K_{t\times r}) \to \{1, 2, 3\}$ as follows.

$$c(u_{ai} u_{bj}) = \begin{cases} 1, & 1 \leq i = j \leq r; \\ 2, & 1 \leq i < j \leq r; \\ 3, & 1 \leq j < i \leq r, \end{cases} \tag{4}$$

where $1 \leq a < b \leq t$.

We now show that c is a 3-rainbow coloring of $K_{t\times r}$. Let S be a set of three vertices of $K_{t\times r}$.

Case 1. The vertices of S belong to the same partite set. Without loss of generality, let $S = \{u_{a1}, u_{a2}, u_{a3}\}$. Then $T = \{u_{a1}u_{b2}, u_{a2}u_{b2}, u_{a3}u_{b2}\}$ is a rainbow S-tree.

Case 2. Two vertices of S belong to the same partite set. Without loss of generality, let $S = \{u_{ai}, u_{aj}, u_{bk}\}$. If $k < i < j$, then $T = \{u_{ai}u_{bk}, u_{ai}u_{bj}, u_{aj}u_{bj}\}$ is a rainbow S-tree. If $i \leq k \leq j$, then $T = \{u_{ai}u_{bk}, u_{aj}u_{bk}\}$ is a rainbow S-tree. If $i < j < k$, then $T = \{u_{ai}u_{bi}, u_{aj}u_{bi}, u_{aj}u_{bk}\}$ is a rainbow S-tree.

Case 3. Each vertex of S belongs to a distinct partite set. Let $S = \{u_{ai}, u_{bj}, u_{ck}\}$, $a < b < c$.

Subcase 3.1 $i = j = k$. Without loss of generality, let $S = \{u_{a1}, u_{b1}, u_{c1}\}$, then $T = \{u_{a1}u_{b1}, u_{a1}u_{b2}, u_{b2}u_{c1}\}$ is a rainbow S-tree.

Subcase 3.2 $i = j \neq k$. Without loss of generality, let $S = \{u_{a1}, u_{b1}, u_{c2}\}$, then $T = \{u_{a1}u_{b1}, u_{b1}u_{c2}\}$ is a rainbow S-tree.

Subcase 3.3 $i \neq j \neq k$. Without loss of generality, let $S = \{u_{a1}, u_{b2}, u_{c3}\}$, then $T = \{u_{a1}u_{c1}, u_{c1}u_{b2}, u_{b2}u_{c3}\}$ is a rainbow S-tree. □

Another well-known class of graphs are the wheels. For $n \geq 3$, the *wheel* W_n is a graph constructed by joining a vertex v to every vertex of a cycle $C_n : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$. Given an edge-coloring c of W_n, for two adjacent vertices v_i and v_{i+1}, we define an edge-coloring of the graph by identifying v_i and v_{i+1} to a new vertex v' as follows: set $c(vv') = c(vv_{i+1})$, $c(v_{i-1}v') = c(v_{i-1}v_i)$, $c(v'v_{i+2}) = c(v_{i+1}v_{i+2})$, and keep the coloring for the remaining edges. We call this coloring the *identified-coloring* at v_i and v_{i+1}. Next we determine the 3-rainbow index of wheels.

Theorem 7. *For $n \geq 3$, the 3-rainbow index of the wheel W_n is*

$$rx_3(W_n) = \begin{cases} 2, & n = 3; \\ 3, & 4 \leq n \leq 6; \\ 4, & 7 \leq n \leq 16; \\ 5, & n \geq 17. \end{cases} \tag{5}$$

Proof. Suppose that W_n consists of a cycle $C_n : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ and another vertex v joined to every vertex of C_n.

Since $W_3 = K_4$, it follows by Lemma 1 that $rx_3(W_3) = 2$.

If $n = 6$, let $S = \{v_1, v_2, v_4\}$. Since every S-tree has size at least 3, $rx_3(W_6) \geq 3$. Next we show that $rx_3(W_6) \leq 3$ by providing a rainbow 3-coloring of W_6 as follows:

$$c(e) = \begin{cases} 1, & \text{if e} \in \{vv_1, vv_4, v_2v_3, v_5v_6\}; \\ 2, & \text{if e} \in \{vv_2, vv_5, v_3v_4, v_1v_6\}; \\ 3, & \text{if e} \in \{vv_3, vv_6, v_4v_5, v_1v_2\} \end{cases} \tag{6}$$

If $n = 5$, since $|W_5| = 6$, by Lemma 2, $rx_3(W_5) \geq 3$. Then we show that $rx_3(W_5) \leq 3$. We provide a rainbow 3-coloring of W_5 obtained from the rainbow 3-coloring of W_6 by the identified-coloring at v_5 and v_6.

If $n = 4$, by Theorem 3, $rx_3(W_4) \geq 3$. Then we show that $rx_3(W_4) \leq 3$. We provide a rainbow 3-coloring of W_4 obtained from the rainbow 3-coloring of W_6 by the identified-coloring at v_5 and v_6, v_4 and v_5, respectively.

Claim 1. If $7 \leq n \leq 16$, $rx_3(W_n) = 4$.

First we show that $rx_3(W_7) \geq 4$. Assume, to the contrary, that $rx_3(W_7) \leq 3$. Let $c : E(W_7) \to \{1, 2, 3\}$ be a rainbow 3-coloring of W_7. Since $d(v) = 7 > 2 \times 3$, there exists $A \subseteq V(C_n)$ such that $|A| = 3$ and all edges in $\{uv : u \in A\}$ are colored the same. Thus, there must exist at least two vertices $v_i, v_j \in A$ such that $d_{C_7}(v_i, v_j) \geq 2$ and a vertex $v_k \in C_7$ such that $v_k \notin \{v_{i-1}, v_{i+1}, v_{j-1}, v_{j-1}\}$. Let $S = \{v_i, v_j, v_k\}$. Note that the only S-tree of size 3 is $T = vv_i \cup vv_j \cup vv_k$, but $c(vv_i) = c(vv_j)$, it follows that there is no rainbow S-tree, which is a contradiction. Similarly, we have $rx_3(W_n) \geq 4$ for all $n \geq 8$.

Second, we show that $rx_3(W_{16}) \leq 4$, which we establish by defining a rainbow 4-coloring c of W_{16} shown in Figure 1. It is easy to check that c is a rainbow 4-coloring of W_{16}. Therefore, $rx_3(W_{16}) = 4$.

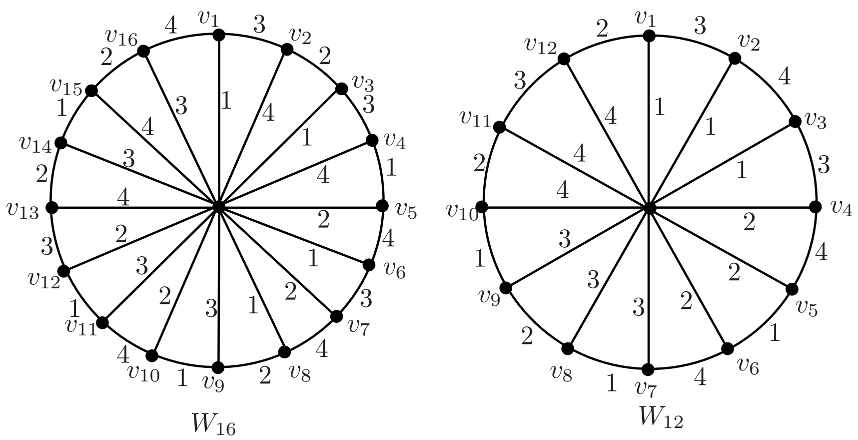

Figure 1. 3-rainbow coloring of W_{16} and W_{12}

When $13 \leq n \leq 15$, we provide a rainbow 4-coloring of W_{15}, W_{14}, W_{13} from the rainbow 4-coloring c of W_{16} by consecutively using the identified-colorings at v_1 and v_{16}, v_{12} and v_{13}, v_8 and v_9.

When $n = 12$, we define a rainbow 4-coloring of W_{12} shown in Figure 1.

When $7 \leq n \leq 11$, we provide a rainbow 4-coloring of W_{11}, W_{10}, W_9, W_8, W_7 from the rainbow 4-coloring c of W_{12} by consecutively identified-colorings at v_1 and v_2, v_4 and v_5, v_7 and v_8, v_{10} and v_{11}, v_{11} and v_{12}.

Claim 2. If $n \geq 17$, $rx_3(W_n) = 5$.

First we show that $rx_3(W_{17}) \geq 5$. Assume, to the contrary, that $rx_3(W_{17}) \leq 4$. Let $c : E(W_{17}) \to \{1, 2, 3, 4\}$ be a rainbow 4-coloring of W_{17}. Since $d(v) = 17 > 4 \times 4$, there exists $A \subseteq V(C_n)$ such that $|A| = 5$ and all edges in $\{uv : u \in A\}$ are colored the same, say 1. Suppose that $A = \{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5}\}$, where $i_1 \leq i_2 \leq i_3 \leq i_4 \leq i_5$. There exists k such that $d_{C_{17}}(v_{i_k}, v_{i_{k+1}}) \geq 3$, where $1 \leq k \leq 4$. Let $S = \{v_{i_k}, v_{i_{k+1}}, v_{i_{k+3}}\}$. Since $d_{C_{17}}(v_{i_k}, v_{i_{k+3}}) \geq 2$ and $d_{C_{17}}(v_{i_{k+1}}, v_{i_{k+3}}) \geq 2$, the only possible S-tree is the path $P = v_{i_{k+1}}v_{i_{k+2}}v_{i_{k+3}}v_{i_{k+4}}v_{i_{k+5}}$, where addition is performed modulo 5. Thus color 1 must appear in P and every edge of the path must have a distinct color. By symmetry, we consider two cases. If $c(v_{i_{k+1}}v_{i_{k+2}}) = 1$. Suppose $c(v_{i_{k+2}}v_{i_{k+3}}) = 2$, $c(v_{i_{k+3}}v_{i_{k+4}}) = 3$. There exists a vertex v_0, where $c(vv_0) = 2$ or 3, such that $d(v_0, A) \geq 3$. It is easy to see that there is no rainbow $\{v_0, v_{i_{k+2}}, v_{i_{k+4}}\}$-tree. If $c(v_{i_{k+2}}v_{i_{k+3}}) = 1$. We can also find such a vertex v_0 such that there exists no $\{v_0, v_{i_{k+2}}, v_{i_{k+3}}\}$-tree, which is a contradiction.

To show that $rx_3(W_n) \leq 5$ for $n \geq 17$, define a rainbow 5-coloring of W_n as follows:

$$c(e) = \begin{cases} j, & e = vv_i \text{ and } i \equiv j \ (mod\ 5),\ 1 \leq j \leq 5; \\ i+3, & e = v_iv_{i+1}. \end{cases} \tag{7}$$

It is easy to see that c is a 5-rainbow coloring of W_n. Therefore, $rx_3(W_n) = 5$ for $n \geq 17$. □

4 The 3-rainbow index of 2-connected and 2-edge-connected graphs

In this section, we give a sharp upper bound of the 3-rainbow index for 2-connected and 2-edge-connected graphs. We start with some lemmas that will be used in the sequel.

Lemma 3. *Let G be a connected graph and $\{V_1, V_2, \cdots, V_k\}$ a partition of $V(G)$. If each V_i induces a connected subgraph H_i of G, then $rx_3(G) \leq k - 1 + \sum_{i=1}^{k} rx_3(H_i)$.*

Proof. Let G' be a graph obtained from G by contracting each H_i to a single vertex. Then G' is a graph of order k, so $rx_3(G') \leq k - 1$. Given an edge-coloring of G' with $k - 1$ colors such that G' is 3-rainbow connected. Now go back to G, and color each edge connecting vertices in distinct H_i with the color of the corresponding edge in G'. For each $i = 1, 2, \cdots, k$, we use $rx_3(H_i)$ new colors to assign the edges of H_i such that H_i is 3-rainbow connected. The resulting edge-coloring makes G 3-rainbow connected. Therefore, $rx_3(G) \leq k - 1 + \sum_{i=1}^{k} rx_3(H_i)$. □

To *subdivide* an edge e is to delete e, add a new vertex x, and join x to the ends of e. Any graph derived from a graph G by a sequence of edge subdivisions is called a *subdivision* of G. Given a rainbow coloring of G, if we subdivide an edge $e = uv$ of G by xu and xv, then we assign xu the same color as e and assign xv a new color, which also make the subdivision of G 3-rainbow connected. Hence, the following lemma holds.

Lemma 4. *Let G be a connected graph, and H be a subdivision of G. Then $rx_3(H) \leq rx_3(G) + |V(H)| - |V(G)|$.*

The Θ-*graph* is a graph consisting of three internally disjoint paths with common end vertices and of lengths a, b, and c, respectively, such that $a \leq b \leq c$. Then $a+b+c = n+1$.

Lemma 5. *Let G be a Θ-graph of order n. If $n \geq 7$, then $rx_3(G) \leq n - 3$.*

Proof. Let the three internally disjoint paths be P_1, P_2, P_3 with the common end vertices u and v, and the lengths of P_1, P_2, P_3 are a, b, c, respectively, where $a \leq b \leq c$.

(1). $b \geq 3$. Then $c \geq b \geq 3$, $a \geq 1$. First, we consider the graph Θ_1 with $a = 1$, $b = 3$ and $c = 3$. We color uP_1v with color 3, uP_2v with colors $2, 3, 1$, and uP_3v with colors $1, 3, 2$. The resulting coloring makes Θ_1 rainbow connected. Thus, $rx_3(\Theta_1) \leq 3 = |V(\Theta_1)| - 3$. For a general Θ-graph G with $b \geq 3$ and $n \geq 7$, it is a subdivision of Θ_1, hence by Lemma 4, $rx_3(G) \leq rx_3(\Theta_1) + |V(G)| - |V(\Theta_1)| \leq |V(G)| - 3$.

(2). $a = 1$, $b = 2$. Then since $a + b + c = n + 1 \geq 8$, $c \geq 5$. Consider the graph Θ_2 with $a = 1$, $b = 2$ and $c = 5$. We rainbow color uP_1v with color 4, uP_2v with colors $1, 3$, and uP_3v with colors $2, 3, 4, 2, 1$. Thus, $rx_3(\Theta_2) \leq 4 = |V(\Theta_2)| - 3$. For a general Θ-graph G with $a = 1$, $b = 2$, $c \geq 5$, it is a subdivision of Θ_2, hence by Lemma 4, $rx_3(G) \leq rx_3(\Theta_2) + |V(G)| - |V(\Theta_2)| \leq |V(G)| - 3$.

(3). $a = 2$, $b = 2$, Then since $a + b + c = n + 1 \geq 8$, $c \geq 4$. Consider the graph Θ_3 with $a = 2$, $b = 2$ and $c = 3$. We rainbow color uP_1v with colors $3, 2$, uP_2v with colors $2, 1$, and uP_3v with colors $1, 2, 3$. Thus, $rx_3(\Theta_3) \leq 3 = |V(\Theta_3)| - 3$. For a general Θ-graph G with $a = 2$, $b = 2$, $c \geq 4$, it is a subdivision of Θ_3, hence by Lemma 4, $rx_3(G) \leq rx_3(\Theta_3) + |V(G)| - |V(\Theta_3)| \leq |V(G)| - 3$.

Every Θ-graph with $n \geq 7$ is one of the above cases, therefore $rx_3(G) \leq n - 3$. □

A 3-*sun* is a graph G which is defined from $C_6 = v_1, v_2, \cdots, v_6, v_1$ by adding three edges v_2v_4, v_2v_6 and v_4v_6.

Lemma 6. *Let G be a 2-connected graph of order 6. If G is a spanning subgraph of a 3-sun, then $rx_3(G) = 4$. Otherwise, $rx_3(G) = 3$.*

Proof. Since G is a 2-connected graph of order 6, G is a graph with a cycle $C_6 = v_1v_2 \cdots v_6v_1$ and some additional edges.

If G is a subgraph of a 3-sun, then since every tree connecting the three vertices $\{v_1, v_3, v_5\}$ must have size at least 4, which implies that $rx_3(G) \geq 4$. On the other hand, $rx_3(G) \leq rx_3(C_6) \leq 4$. Therefore, $rx_3(G) = 4$.

If there is an edge between the two antipodal vertices of C_6, then by Lemma 5, $rx_3(G) = 3$.

If G contains the edges v_1v_3 and v_2v_6, then it contains Θ_3, defined in Lemma 5, as a spanning subgraph, thus $rx_3(G) = 3$.

If G contains the edges v_1v_5 and v_2v_4, we give a rainbow 3-coloring c of G: $c(v_1v_2) = c(v_4v_5) = 1$, $c(v_2v_3) = c(v_2v_4) = c(v_1v_5) = c(v_5v_6) = 2$, $c(v_3v_4) = c(v_1v_6) = 3$. □

Let H be a subgraph of a graph G. An *ear* of H in G is a nontrivial path in G whose ends are in H but whose internal vertices are not. A nested sequence of graphs is a sequence $\{G_0, G_1, \cdots, G_k\}$ of graphs such that $G_i \subset G_{i+1}$, $0 \leq i < k$. An *ear decomposition* of a 2-connected graph G is a nested sequence $\{G_0, G_1, \cdots, G_k\}$ of 2-connected subgraphs of G such that: (1) G_0 is a cycle; (2) $G_i = G_{i-1} \cup P_i$, where P_i is an ear of G_{i-1} in G, $1 \leq i \leq k$; (3) $G_k = G$. We call an ear decomposition *nonincreasing* if $\ell(P_1) \geq \ell(P_2) \geq \cdots \geq \ell(P_k)$, where $\ell(P_i)$ denotes the length of P_i.

Theorem 8. *Let G be a 2-connected graph of order $n \geq 4$. Then $rx_3(G) \leq n - 2$, with equality if and only if $G = C_n$ or G is a spanning subgraph of 3-sun or G is a spanning subgraph of $K_5 - e$ or G is a spanning subgraph of K_4.*

Proof. Since G is 2-connected, G contains a cycle. Let C be the largest cycle of G, then $|C| \geq 4$, $rx_3(C) \leq |V(C)| - 2$. Let $H_1 = C$, $H_2, H_3, \cdots, H_{n-|V(C)|+1}$ be subgraphs of G, each is a single vertex, then by Lemma 3, $rx_3(G) \leq n - |V(C)| + rx_3(H_1) \leq n - 2$.

If $G = C$, then by Theorem 1, $rx_3(G) = n - 2$.

If $G \neq C$, then G contains a nonincreasing ear decomposition $\{G_0, G_1, \cdots, G_k\}$. Let $H_1 = C \cup P_1$, then H_1 is a Θ-graph. We choose $H_2, H_3, \cdots, H_{n-|V(H_1)|+1}$ as subgraphs of G with a single vertex each, then by Lemma 3, $rx_3(G) \leq n - |V(H_1)| + rx_3(H_1)$.

If $|V(H_1)| \geq 7$, then by Lemma 5, $rx_3(H_1) \leq |V(H_1)| - 3$, hence $rx_3(G) \leq n - 3$.

If $|V(H_1)| = 6$, we consider three cases.

Case 1. $|V(C)| = 6$. Then $\ell(P_1) = 1$. Hence $\ell(P_1) = \ell(P_2) = \cdots = \ell(P_k) = 1$, G is a graph of order 6. By Lemma 6, $rx_3(G) = 4$ if and only if G is a spanning subgraph of a 3-sun.

Case 2. $|V(C)| = 5$. Then $\ell(P_1) = 2$. Let u and v be the end vertices of P_1. If $d_C(u, v) = 1$, then we can find a cycle larger than C, contradicting the choice of C. Otherwise, $d_C(u, v) = 2$, it is the graph Θ_3 defined in Lemma 5, then $rx_3(H_1) = rx_3(\Theta_3) \leq 3 = |V(H_1)| - 3$, thus $rx_3(G) \leq n - 3$.

Case 3. $|V(C)| = 4$. Then $\ell(P_1) = 3$. Let u and v be the end vertices of P_1. Either $d_C(u, v) = 1$ or $d_C(u, v) = 2$, we can always find a cycle larger than C, a contradiction.

If $|V(H_1)| = 5$, there are two cases to be considered. If $|V(C)| = 5$, then $\ell(P_1) = 1$, hence G is a graph of order 5. By Theorem 3, $rx_3(G) = 3 = n - 2$ except for K_5, whose 3-rainbow index is 2. If $|V(C)| = 4$, then $\ell(P_1) = 2$. Let u and v be the end vertices of P_1. Note that $d_C(u, v) = 2$. If $\ell(P_2) = 1$, then G is a graph of order 5. If $\ell(P_2) \geq 2$, let u' and v' be the end vertices of P_2, then $\{u', v'\} = \{u, v\}$, otherwise, we can find a cycle larger than C. Let $H'_1 = H_1 \cup P_2$, then H'_1 is a graph consisting of 4 internally disjoint paths of length 2 with common vertices u and v. We assign the edges of the four paths with colors 12, 21, 31, 13, the resulting coloring makes H'_1 rainbow connected, thus, $rx_3(H'_1) \leq 3 = |V(H'_1)| - 3$. Let $H'_2, H'_3, \cdots, H'_{n-|V(H'_1)|+1}$ be subgraphs of G, each is a single vertex, then by Lemma 3, $rx_3(G) \leq n - |V(H'_1)| + rx_3(H'_1) \leq n - 3$.

If $|V(H_1)| = 4$, then $|V(C)| = 4$, $\ell(P_1) = 1$, G is a graph of order 4, by Theorem 3, $rx_3(G) = 2 = n - 2$.

Therefore, $rx_3(G) = n - 2$ if and only if $G = C_n$ or G is a spanning subgraph of 3-sun or G is a spanning subgraph of $K_5 - e$ or G is a spanning subgraph of K_4. □

Now we turn to 2-edge-connected graphs. We call an ear is *closed* if its endvertices are identical, otherwise, it is *open*. An open or closed ear is called a *handle*. For a 2-edge-connected graph G, there is a handle-decomposition, that is a sequence $\{G_0, G_1, \cdots, G_k\}$ of graphs such that: (1) G_0 is a cycle; (2) $G_i = G_{i-1} \cup P_i$, where P_i is a handle of G_{i-1} in G, $1 \leq i \leq k$; (3) $G_k = G$. Similar to Theorem 8, we give an upper bound of 2-edge-connected graphs.

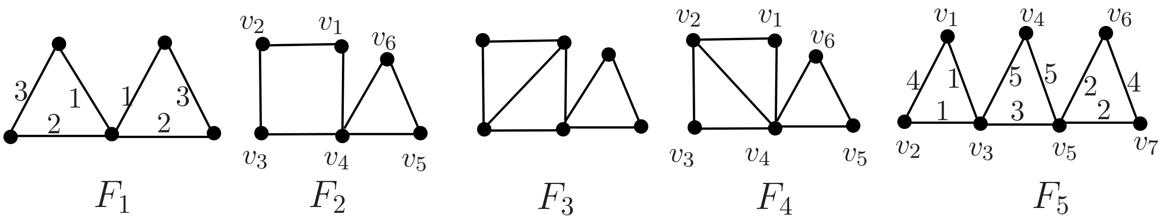

Figure 2. Graphs with $rx_3(G) = n - 2$

Theorem 9. *Let G be a 2-edge-connected graph of order $n \geq 4$. Then $rx_3(G) \leq n - 2$, with equality if and only if G is a graph attaining the upper bound in Theorem 8 or a graph in Figure 2.*

Proof. Let C be the largest cycle of G. If $|V(C)| \geq 4$, then $rx_3(C) \leq |V(C)| - 2$. Otherwise, all cycles of G are of length 3. Since $n \geq 4$, there are at least two triangles C_1 and C_2 with a common vertex v. Let $F_1 = C_1 \cup C_2$, we rainbow color F_1 with three colors, see Figure 2(1), thus $rx_3(F_1) \leq 3 = |V(F_1)| - 2$. Let $H_1 = C$ or F_1, $H_2, H_3, \cdots, H_{n-|V(H_1)|+1}$ be subgraphs of G with a single vertex each, then by Lemma 3, $rx_3(G) \leq n - |V(H_1)| + rx_3(H_1) \leq n - 2$.

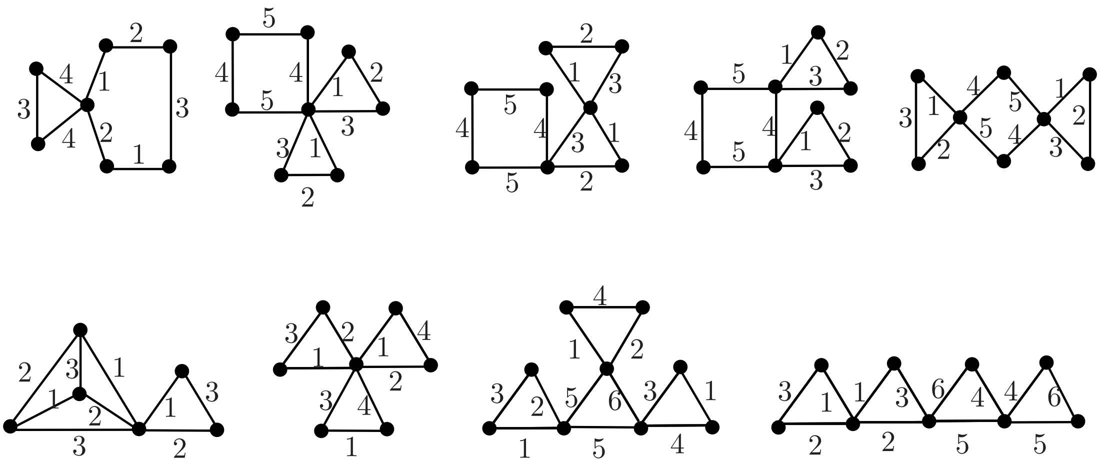

Figure 3. Graphs with $rx_3(G) \leq n-3$

Now we determine the graphs that obtain the upper bound $n-2$.

If $G = C$, then by Theorem 1, $rx_3(G) = n-2$.

If $G \neq C$, then G contains a handle-decomposition $\{G_0, G_1, \cdots, G_k\}$. Let $H_1 \subseteq G$, $H_2, H_3, \cdots, H_{n-|V(H_1)|+1}$ be subgraphs of G with a single vertex each, then by Lemma 3, if we show that $rx_3(H_1) \leq |V(H_1)| - 3$, then we have $rx_3(G) \leq n-3$.

If $|V(C)| \geq 4$ and P_1 is an open ear, we come back to Theorem 8. If $|V(C)| = 3$ and P_1 is an open ear, then a cycle is of length larger than C, a contradiction.

If $|V(C)| \geq 4$ and P_1 is a closed ear, then G_1 is a union of two cycles $C_1 = C$ and $C_2 = P_1$. If both of the cycles are of length at least 4, we rainbow color each cycle C_i with $|V(C_i)| - 2$ colors, which makes G_1 3-rainbow connected. So we assume that C_2 is of length 3. If C_1 is of length 5, we rainbow color G_1 by 4 colors, see Figure 3(1). If C_1 is of length greater than 5, then it is the subdivision of the graph in the case of $|V(C_1)| = 5$. For all the above three cases, we have $rx_3(G_1) \leq |V(G_1)| - 3$. Let $H_1 = G_1$, it follows that $rx_3(G) \leq n-3$.

So it remains the case that $|V(C_1)| = 4$, $|V(C_2)| = 3$, we denote this graph by F_2, see Figure 2(2). Then F_2 is a subdivision of F_1, so $rx_3(F_2) \leq 4$. On the other hand, consider $S = \{v_2, v_5, v_6\}$, every S-tree has size at least 4, hence $rx_3(F_2) = 4 = |V(F_2)| - 2$. Observe that P_2 is a closed ear of length at most 4, then $G_2 = F_2 \cup P_2$. If $\ell(P_2) = 4$, then G_2 contains two cycles of length 4. If $\ell(P_2) = 3$, we rainbow colors G_2 with $|V(G_2)| - 3$ colors, see Figure 3(2-5). For the above two cases, $rx_3(G_2) \leq |V(G_2)| - 3$. Let $H_1 = G_2$, it implies that $rx_3(G) \leq n-3$. If $\ell(P_2) = 1$, then P_2 must be an edge joining the vertices of C_1, there are two graphs, denoted by F_3 and F_4. Similar to F_2, we have $rx_3(F_3) = |V(F_3)| - 2$. For F_4, $rx_3(F_4) \leq rx_3(F_2) \leq 4$. On the other hand, suppose $rx_3(F_4) \leq 3$. Consider $\{v_1, v_3, v_5\}$, $\{v_1, v_3, v_6\}$, we have that $c(v_4v_6) = c(v_4v_5)$, which implies that there is no rainbow $\{v_1, v_5, v_6\}$-tree or $\{v_1, v_5, v_6\}$-tree, a contradiction. Hence

$rx_3(F_4) = 4 = |V(F_4)| - 2$. Observe that P_3 is of length 1, $G_3 = F_3 \cup P_3$ or $F_4 \cup P_3$, we can rainbow color G_3 by 3 colors, see Figure 3(6). Let $H_1 = G_3$, then $rx_3(G) \leq n - 3$.

If $|V(C)| = 3$ and P_1 is a closed ear, then $\ell(P_1) = 3$. Thus $G_1 = F_1$, it is easy to get $rx_3(G_1) = |V(G_1)| - 2$. If P_2 exists, then it must be a closed ear of length 3, there are two cases for the graph G_2. If G_2 is as Figure 3(7), then $rx_3(G_2) \leq |V(G_2)| - 3$, let $H_1 = G_2$, thus $rx_3(G) \leq n - 3$. If G_2 is as Figure 2(5), we prove that its 3-rainbow index is $|V(G_2)| - 2$. From Figure 2(5), we have that $rx_3(G_2) \leq 5$. If $rx_3(G_2) \leq 4$, let $c : E(G) \to \{1, 2, 3, 4\}$ be the rainbow 4-coloring of G_2. Consider $\{v_1, v_4, v_6\}$ and $\{v_1, v_4, v_7\}$, we have $c(v_1v_3) \neq c(v_5v_6)$, $c(v_1v_3) \neq c(v_5v_7)$. If $c(v_5v_6) = c(v_5v_7)$, suppose that $c(v_5v_6) = 1$, $c(v_1v_3) = 2$, consider $\{v_1, v_6, v_7\}$, we may assume $c(v_3v_5) = 3$, $c(v_6v_7) = 4$. Consider $\{v_2, v_6, v_7\}$, $\{v_1, v_2, v_6\}$, $\{v_1, v_2, v_4\}$, $\{v_1, v_4, v_6\}$, we have $c(v_2v_3) = 2$, $c(v_1v_2) = 4$, $c(v_3v_4) = 4, c(v_4v_5) = 1$, but then there is no rainbow tree connecting $\{v_4, v_6, v_7\}$. If $c(v_5v_6) \neq c(v_5v_7)$, then $c(v_1v_3) \neq c(v_2v_3)$, let $c(v_1v_3) = 1$, $c(v_2v_3) = 2$, $c(v_5v_6) = 3$, $c(v_5v_6) = 4$. Consider $\{v_1, v_4, v_6\}$, then the colors 2 and 4 must appear in the triangle $v_3v_4v_5$. Consider $\{v_2, v_4, v_7\}$, then the colors 1 and 3 must appear in the triangle $v_3v_4v_5$, which is impossible. So we consider P_3, if it exists, then it must be a close ear, there are two cases, no matter which case presents, we can give a rainbow coloring with $|V(G_3)| - 3$ colors, see Figure 3(8-9). Let $H_1 = G_3$, then $rx_3(G) \leq n - 3$.

Combining all the above cases, $rx_3(G) = n - 2$ if and only if G is a graph attaining the upper bound in Theorem 8 or a graph in Figure 2. □

References

[1] J.A. Bondy, U.S.R. Murty, *Graph Theory*, GTM 244, Springer, 2008.

[2] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, R. Yuster, On rainbow connection, Electron. J. Combin. 15(1)(2008), R57.

[3] G. Chartrand, G. Johns, K. McKeon, P. Zhang, *Rainbow connection in graphs*, Math. Bohem. 133(2008), 85-98.

[4] G. Chartrand, F. Okamoto, P. Zhang, *Rainbow trees in graphs and generalized connectivity*, Networks 55(2010), 360-367.

[5] G. Chartrand, G. Johns, K. McKeon, P. Zhang, *The rainbow connectivity of a graph*, Networks 54(2)(2009), 75-81.

[6] X. Li, Y. Sun, *Rainbow Connections of Graphs*, Springer Briefs in Math., Springer, 2012.

[7] X. Li, Y. Shi, Y. Sun, *Rainbow connections of graphs*: A survey, Graphs & Combin 29(2013), 1-38.