The (k, ℓ)-rainbow index of random graphs*

Qingqiong Cai, Xueliang Li, Jiangli Song

Center for Combinatorics and LPMC-TJKLC

Nankai University

Tianjin 300071, China

Email: cqqnjnu620@163.com, lxl@nankai.edu.cn, songjiangli@mail.nankai.edu.cn

Abstract

A tree in an edge-colored graph G is said to be a rainbow tree if no two edges on the tree share the same color. Given two positive integers k, ℓ with $k \geq 3$, the *(k, ℓ)-rainbow index* $rx_{k,\ell}(G)$ of G is the minimum number of colors needed in an edge-coloring of G such that for any set S of k vertices of G, there exist ℓ internally disjoint rainbow trees connecting S. This concept was introduced by Chartrand et. al., and there have been very few known results about it. In this paper, we establish a sharp threshold function for $rx_{k,\ell}(G_{n,p}) \leq k$ and $rx_{k,\ell}(G_{n,M}) \leq k$, respectively, where $G_{n,p}$ and $G_{n,M}$ are the usually defined random graphs.

Keywords: rainbow index, random graphs, threshold function

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1 Introduction

All graphs in this paper are undirected, finite and simple. We follow [3] for graph theoretical notation and terminology not defined here. Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \rightarrow \{1, 2, \cdots, t\},\ t \in \mathbb{N}$, where adjacent edges may be colored the same. A path of G is said to be a *rainbow path* if no two edges on the path have the same color. An edge-colored graph G is called *rainbow connected* if for every pair of distinct vertices of G there exists a rainbow path connecting them. The *rainbow*

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connection number of a graph G, denoted by $rc(G)$, is defined as the minimum number of colors that are needed in order to make G rainbow connected. For any two vertices u and v of G, a rainbow $u-v$ geodesic in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v. The graph G is *strongly rainbow connected* if there exists a rainbow $u-v$ geodesic for any pair of vertices u and v in G. Similarly, we define the *strong rainbow connection number* of a connected graph G, denoted by $src(G)$, as the smallest number of colors that are needed in order to make G strongly rainbow connected. Clearly, we have $diam(G) \leq rc(G) \leq src(G) \leq m$, where $diam(G)$ denotes the diameter of G and m is the number of edges of G. The *rainbow k-connectivity* of G, denoted by $rc_k(G)$, is defined as the minimum number of colors in an edge-coloring of G such that every two distinct vertices of G are connected by k internally disjoint rainbow paths. These concepts were introduced by Chartrand et al. in [7, 8]. Recently, a lot of relevant results have been published; see [5, 6, 10, 11, 12, 18]. The interested readers can see [16, 17] for a survey on this topic.

Here we recall the concept of generalized connectivity. Let G be a connected graph of order n and size m. For $S \subseteq V(G)$, an *S-tree* is a tree connecting the vertices of S. Suppose that $\{T_1, T_2, \cdots, T_\ell\}$ is a set of S-trees. They are called *internally disjoint* if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \bigcap V(T_j) = S$ for every pair of distinct integers i, j with $1 \leq i, j \leq \ell$ (note that the trees are vertex-disjoint in $G \setminus S$). For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number of internally disjoint S-trees in G. The k-connectivity $\kappa_k(G)$ of G is defined by $\kappa_k(G)$=min$\{\kappa(S)\}$, where the minimum is taken over all k-element subsets S of $V(G)$. We refer to [9, 13, 14, 15] for more details about the generalized connectivity.

A tree T in an edge-colored graph G is called a *rainbow tree* if no two edges of T have the same color. Given two positive integers k, ℓ with $2 \leq k \leq n$ and $1 \leq \ell \leq \kappa_k(G)$, the *$(k,\ell)$-rainbow index* $rx_{k,\ell}(G)$ of G is the minimum number of colors needed in an edge-coloring of G such that for any set S of k vertices of G, there exist ℓ internally disjoint rainbow S-trees. In particular, for $\ell = 1$, we often write $rx_k(G)$ rather than $rx_{k,1}(G)$ and call it the *k-rainbow index*. It is easy to see that $rx_{2,\ell}(G) = rc_\ell(G)$. So the (k,ℓ)-rainbow index can be viewed as a generalization of the rainbow connectivity. In the sequel, we always assume $k \geq 3$.

The concept of (k,ℓ)-rainbow index was also introduced by Chartrand et al.; see [9]. They determined the k-rainbow index of all unicyclic graphs and the $(3,\ell)$-rainbow index of complete graphs for $\ell = 1, 2$. In [4], we investigated the (k,ℓ)-rainbow index of complete graphs for every pair of integers k, ℓ. We proved that for every pair of positive integers k, ℓ with $k \geq 3$, there exists a positive integer $N = N(k,\ell)$ such that $rx_{k,\ell}(K_n) = k$ for every integer $n \geq N$, which settled down the two conjectures in [9].

In this paper, we study the (k, ℓ)-rainbow index of random graphs and establish a sharp threshold function for the property $rx_{k,\ell}(G_{n,p}) \leq k$ and $rx_{k,\ell}(G_{n,M}) \leq k$, respectively, where $G_{n,p}$ and $G_{n,M}$ are defined as usual; see [2].

2 Basic notation on random graphs

The two most frequently occurring probability models of random graphs are $\mathcal{G}(n,p)$ and $\mathcal{G}(n,M)$. The model $\mathcal{G}(n,p)$ consists of all graphs on n vertices, in which the edges are chosen independently and randomly with probability p; whereas the model $\mathcal{G}(n,M)$ consists of all graphs on n vertices and M edges, in which each graph has the same probability. Let $G_{n,p}$, $G_{n,M}$ stand for random graphs from the models $\mathcal{G}(n,p)$ and $\mathcal{G}(n,M)$. We say that an event $E = E(n)$ happens *almost surely* (or *a.s.* for short) if $\lim_{n\to\infty} Pr[E(n)] = 1$. Let K, G, H be three graphs on n vertices. A property Q is said to be *monotone* if whenever $G \subseteq H$ and G satisfies Q, then H also satisfies Q. Moreover, We call a property Q *convex* if whenever $K \subset G \subset H$, and K satisfies Q and H satisfies Q, then G also satisfies Q. For a graph property Q, a function $p(n)$ is called a *threshold function* of Q if

- $\frac{p^{'}(n)}{p(n)} \to 0$, then $G_{n,p^{'}(n)}$ almost surely does not satisfy Q; and
- $\frac{p^{''}(n)}{p(n)} \to \infty$, then $G_{n,p^{''}(n)}$ almost surely satisfies Q.

Furthermore, $p(n)$ is called a *sharp threshold function* of Q if there are two positive constants c and C such that

- for every $p^{'}(n) \leq cp(n)$, $G_{n,p^{'}(n)}$ almost surely does not satisfy Q; and
- for every $p^{''}(n) \geq Cp(n)$, $G_{n,p^{''}(n)}$ almost surely satisfies Q.

Similarly, we can define $M(n)$ as a threshold function of Q in the model $\mathcal{G}(n,M)$; see [2].

It is well known that all monotone graph properties have a threshold function [2]. Obviously, for every pair of positive integers k, ℓ, the property that the (k, ℓ)-rainbow index is at most k is monotone, and thus has a threshold.

3 Main results

As Caro et al. pointed out, the random graph setting poses several intriguing questions. In [5], Caro et al. proved that $p = \sqrt{logn/n}$ is a sharp threshold for the property $rc(G_{n,p}) \leq 2$. This was generalized by Fujita et al. [10], who obtained that $p = \sqrt{logn/n}$

is a sharp threshold for the property $rc_k(G_{n,p}) \leq 2$ and $M = \sqrt{n^3 logn}$ is a sharp threshold for the property $rc_k(G_{n,M}) \leq 2$ for all integer $k \geq 1$. In this section, we employ similar methods to study the (k, ℓ)-rainbow index of random graphs $G_{n,p}$ and $G_{n,M}$.

Theorem 1. *For every pair of positive integers k, ℓ with $k \geq 3$, $\sqrt[k]{\frac{log_a n}{n}}$ is a sharp threshold function for the property $rx_{k,\ell}(G_{n,p}) \leq k$, where $a = \frac{k^k}{k^k - k!}$.*

Proof. The proof will be two-fold. For the first part, we show that there exists a positive constant c_1 such that for every $p \geq c_1 \sqrt[k]{\frac{log_a n}{n}}$, almost surely $rx_{k,\ell}(G_{n,p}) \leq k$, which can be derived from the following two claims.

Claim 1: For any $c_1 \geq 3$, if $p \geq c_1 \sqrt[k]{\frac{log_a n}{n}}$, then almost surely any k vertices in $G_{n,p}$ have at least $2k log_a n$ common neighbors.

For any $S \subseteq V(G_{n,p})$ with $|S| = k$, let $D(S)$ denote the event that the vertices in S have at least $2k log_a n$ common neighbors. Then it suffices to prove that, for $p = c_1 \sqrt[k]{\frac{log_a n}{n}}$, $Pr[\bigcap_S D(S)\ ] \to 1$, as $n \to \infty$. Define X as the number of common neighbors of all the vertices in S. Then $X \sim Bin\left(n-k, (c_1 \sqrt[k]{\frac{log_a n}{n}})^k\right)$ and $E(X) = \frac{n-k}{n} c_1^k log_a n$. Assume that $n > \frac{c_1^k k}{c_1^k - 2k}$. Using the Chernoff Bound [1], we get that

$$
\begin{aligned}
Pr[\overline{D(S)}] &= Pr[X < 2k log_a n] \\
&= Pr[X < \frac{c_1^k(n-k)}{n} log_a n \left(1 - \frac{(c_1^k - 2k)n - c_1^k k}{c_1^k(n-k)}\right)] \\
&\leq e^{-\frac{c_1^k(n-k)}{2n} log_a n \left(\frac{(c_1^k-2k)n - c_1^k k}{c_1^k(n-k)}\right)^2} \\
&< n^{-\frac{c_1^k(n-k)}{2n} \left(\frac{(c_1^k-2k)n - c_1^k k}{c_1^k(n-k)}\right)^2}.
\end{aligned}
$$

Note that the assumption $n > \frac{c_1^k k}{c_1^k - 2k}$ ensures $\frac{(c_1^k-2k)n - c_1^k k}{c_1^k(n-k)} > 0$. So we can apply the Chernoff Bound to scaling the above inequalities. The last inequality holds, since $1 < a = \frac{k^k}{k^k - k!} < e$ and then $log_a n > \ln n$.

It follows from the union bound that

$$
\begin{aligned}
Pr[\bigcap_S D(S)\ ] &= 1 - Pr[\bigcup_S \overline{D(S)}\ ] \\
&\geq 1 - \sum_S Pr[\ \overline{D(S)}\ ]
\end{aligned}
$$

$$> 1-\binom{n}{k} n^{-\frac{c_1^k(n-k)}{2n}\left(\frac{(c_1^k-2k)n-c_1^k k}{c_1^k(n-k)}\right)^2}$$
$$> 1-n^{k-\frac{c_1^k(n-k)}{2n}\left(\frac{(c_1^k-2k)n-c_1^k k}{c_1^k(n-k)}\right)^2}.$$

It is not hard to see that $c_1>3$ can guarantee $k-\frac{c_1^k(n-k)}{2n}\left(\frac{(c_1^k-2k)n-c_1^k k}{c_1^k(n-k)}\right)^2<0$ for sufficiently large n. Then $\lim\limits_{n\to\infty} 1-n^{k-\frac{c_1^k(n-k)}{2n}\left(\frac{(c_1^k-2k)n-c_1^k k}{c_1^k(n-k)}\right)^2}=1$, which implies that $\lim\limits_{n\to\infty} Pr[\bigcap\limits_S D(S)\,]=1$ as desired.

Claim 2: If any k vertices in $G_{n,p}$ have at least $2klog_a n$ common neighbors, then there exists a positive integer $N=N(k)$ such that $rx_{k,\ell}(G_{n,p})\leq k$ for every integer $n\geq N$.

Let $C=\{1,2,\cdots,k\}$ be a set of k different colors. We color the edges of $G_{n,p}$ with the colors from C randomly and independently. For $S\subseteq V(G_{n,p})$ with $|S|=k$, define $F(S)$ as the event that there exist at least ℓ internally disjoint rainbow S-trees. It suffices to prove that $\Pr[\bigcap\limits_S F(S)\,]>0$.

Suppose $S=\{v_1,v_2,\cdots,v_k\}$. For any common neighbor u of the vertices in S, let $T(u)$ denote the star with $V(T(u))=\{u,v_1,v_2,\cdots,v_k\}$ and $E(T(u))=\{uv_1,uv_2,\cdots,uv_k\}$. Set $\mathcal{T}=\{T(u)|u$ is a common neighbor of the vertices in $S\}$. Then $\mathcal{T}$ is a set of at least $2klog_a n$ internally disjoint S-trees. It is easy to see that q:=Pr[T$\in\mathcal{T}$ is a rainbow tree]$=\frac{k!}{k^k}<\frac{1}{2}$. So $1-q>q$. Define Y as the number of rainbow S-trees in $\mathcal{T}$. Then we have

$$\begin{aligned}
Pr[\overline{F(S)}] &\leq Pr[Y\leq \ell-1]\\
&\leq \sum_{i=0}^{\ell-1}\binom{2klog_a n}{i} q^i(1-q)^{2klog_a n-i}\\
&\leq (1-q)^{2klog_a n}\sum_{i=0}^{\ell-1}\binom{2klog_a n}{i}\\
&\leq (1-q)^{2klog_a n}(1+2klog_a n)^{\ell-1}\\
&= \frac{(1+2klog_a n)^{\ell-1}}{n^{2k}}.
\end{aligned}$$

It yields that

$$\begin{aligned} Pr[\bigcap_S F(S)\ ] &= 1 - Pr[\bigcup_S \overline{F(S)}\ ] \\ &\geq 1 - \sum_S Pr[\ \overline{F(S)}\ ] \\ &\geq 1 - \binom{n}{k}\frac{(1+2klog_a n)^{\ell-1}}{n^{2k}} \\ &> 1 - \frac{(1+2klog_a n)^{\ell-1}}{n^k}. \end{aligned}$$

Obviously, $\lim\limits_{n\to\infty} 1 - \frac{(1+2klog_a n)^{\ell-1}}{n^k} = 1$, and then $\lim\limits_{n\to\infty} Pr[\bigcap\limits_S F(S)\ ] = 1$. Thus there exists a positive integer $N = N(k)$ such that $Pr[\bigcap\limits_S F(S)\ ] > 0$ for every integer $n \geq N$.

For the other direction, we show that there exists a positive constant c_2 such that for every $p \leq c_2\sqrt[k]{\frac{log_a n}{n}}$, almost surely $rx_{k,\ell}(G_{n,p}) \geq k+1$.

It suffices to prove that for a sufficiently small constant c_2, the random graph $G_{n,p}$ with $p = c_2\sqrt[k]{\frac{log_a n}{n}}$ almost surely contains a set S of k vertices satisfying

(i) S is an independent set;

(ii) the vertices in S have no common neighbors.

Clearly, for such S there exists no rainbow S-trees in any k-edge-coloring, which implies that $rx_{k,\ell}(G_{n,p}) \geq k+1$.

Fix a set H of $n^{1/(2k+1)}$ vertices in $G_{n,p}$ (we may and will assume that $n^{1/(2k+1)}/k$ is an integer). Let E_1 be the event that H is an independent set. Then

$$\Pr[E_1] = (1 - c_2\sqrt[k]{\frac{log_a n}{n}})^{\binom{n^{1/(2k+1)}}{2}} = 1 - o(1),$$

where $o(1)$ denotes a function tending to 0 as n tends to infinity.

Partition H into t subsets $H_1, H_2 \ldots, H_t$ arbitrarily, where $t = n^{1/(2k+1)}/k$ and $|H_1| = |H_2| = \ldots = |H_t| = k$. Let E_2 be the event that there exists some H_i without common neighbors in $V(G_{n,p})\backslash H$. Then, for sufficiently small c_2,

$$\Pr[E_2] = 1 - \left(1 - (1 - c_2^k\frac{log_a n}{n})^{n-n^{1/(2k+1)}}\right)^{n^{1/(2k+1)}/k} = 1 - o(1).$$

So, almost surely there exists some set H_i of k vertices satisfying properties (i) and (ii). Thus, for sufficiently small c_2 and every $p \leq c_2\sqrt[k]{\frac{log_a n}{n}}$, almost surely $rx_{k,\ell}(G_{n,p}) \geq k+1$. The proof is thus complete. $\square$

Next we will turn to another well-known random graph model $\mathcal{G}(n, M)$. We start with a useful lemma which reveals the relationship between $\mathcal{G}(n,p)$ and $\mathcal{G}(n,M)$. Set $N = \binom{n}{2}$.

Lemma 2. *[2] If Q is a convex property and $p(1-p)N \to \infty$, then $G_{n,p}$ almost surely has Q if and only if for every fixed x, $G_{n,M}$ almost surely has Q, where $M = \lfloor pN + x\,(p(1-p)N)^{1/2} \rfloor$.*

Clearly, the property that the (k,ℓ)-rainbow index of a given graph is at most k, is a convex property. By Theorem 1 and Lemma 2, we get the following result.

Corollary 3. *For every pair of positive integers k, ℓ with $k \geq 3$, $M(n) = \sqrt[k]{n^{2k-1} log_a n}$ is a sharp threshold function for the property $rx_{k,\ell}(G_{n,M}) \leq k$, where $a = \frac{k^k}{k^k - k!}$.*

Remark: If p is a threshold function for a given property Q, then so is λp for any positive constant λ. It follows that $p(n) = \sqrt[k]{\frac{logn}{n}}$ $\left(M(n) = \sqrt[k]{n^{2k-1} logn}\right)$ is also a sharp threshold function for the property $rx_{k,\ell}(G_{n,p}) \leq k$ $(rx_{k,\ell}(G_{n,M}) \leq k)$, which corresponds to the results in [10].

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