The generalized 3-edge-connectivity of lexicographic product graphs*

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Abstract. The generalized k-edge-connectivity $\lambda_k(G)$ of a graph G is a natural generalization of the concept of edge-connectivity. The generalized edge-connectivity has many applications in networks. The lexicographic product of two graphs G and H, denoted by $G \circ H$, is an important method to construct large graphs from small ones. In this paper, we mainly study the generalized 3-edge-connectivity of $G \circ H$, and get lower and upper bounds of $\lambda_3(G \circ H)$. An example is given to show that all bounds are sharp.

Keywords: edge-disjoint paths, edge-connectivity, Steiner tree, edge-disjoint Steiner trees, generalized edge-connectivity.

1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [3]. For a graph G, the *local edge-connectivity* between two distinct vertices u and v, denoted by $\lambda(u, v)$, is the maximum number of pairwise edge-disjoint uv-paths. A nontrivial graph G is *k-edge-connected* if $\lambda(u, v) \geq k$ for any two distinct vertices u and v of G. The *edge-connectivity* $\lambda(G)$ of a graph G is the maximum value of k for which G is k-edge-connected.

Naturally, the concept of edge-connectivity can be extended to a new concept, the generalized k-edge-connectivity, which was introduced by Li et al. [22]. For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, *a Steiner tree connecting S* (or simply, *an S-tree*) is a such subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Two S-trees T and T' are said to be edge-disjoint if $E(T) \cap E(T') = \emptyset$. The *generalized local edge-connectivity* $\lambda(S)$ is the maximum number of pairwise edge-disjoint Steiner trees connecting S. For an integer k with $2 \leq k \leq n$, the *generalized k-edge-connectivity* $\lambda_k(G)$ of G is defined as $\lambda_k(G) = \min\{\lambda(S) \mid S \subseteq V(G), |S| = k\}$. Obviously, $\lambda_2(G) = \lambda(G)$. Set $\lambda_k(G) = 0$ if G is disconnected. Similarly, the concept of the generalized k-connectivity was

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introduced by Hager in [11] and it is also studied in [5]. We refer to [17–19, 22, 24, 30] for some known results of the generalized connectivity and edge-connectivity.

The generalized edge-connectivity has a close relation to an important problem, the *Steiner tree packing problem*, which asks for finding a set of maximum number of edge-disjoint S-trees in a given graph G where $S \subseteq V(G)$, see [9, 31]. An extreme of Steiner tree packing problem is the *Spanning tree packing problem* where $S = V(G)$. For any graph G, the *spanning tree packing number* or *STP number*, is the maximum number of edge-disjoint spanning trees contained in G. For the STP number, we refer to [1, 25, 26]. The difference between the Steiner tree packing problem and the generalized edge-connectivity is as follows: the former problem studies local properties of graphs since S is given beforehand, while the latter problem focuses on global properties of graphs since S runs over all k-subsets of $V(G)$.

The generalized edge-connectivity and the Steiner tree packing problem have applications in $VLSI$ circuit design, see [9, 27]. In this application, a Steiner tree is needed to share an electronic signal by a set of terminal nodes. A Steiner tree is also used in computer communication networks and optical wireless communication networks, see [6, 7]. Another application arises in the Internet Domain. Suppose that a given graph G represents a network. We select arbitrary k vertices as nodes. Suppose one of the nodes in G is a *broadcaster* and all other nodes are *users*. The broadcaster wants to broadcast as many streams of movies as possible, so that the users have the maximum number of choices. Each stream of movie is broadcasted via a tree connecting all the users and the broadcaster. So, in essence we need to find the maximum number of Steiner trees connecting all the users and the broadcaster, namely, we want to get $\lambda(S)$, where S is the selected k nodes. Clearly, it is a Steiner tree packing problem. Furthermore, if we want to know whether for any k nodes the network G has above properties, then we need to compute $\lambda_k(G) = \min\{\lambda(S)\}$ in order to prescribe the reliability and the security of the network.

From a theoretical perspective, both extremes of the generalized edge-connectivity problem are fundamental theorems in combinatorics. One extreme is when we have two terminals. In this case edge-disjoint trees are just edge-disjoint paths between the two terminals, and so the problem becomes the well-known edge version of Menger theorem. The other extreme is when all the vertices are terminals. In this case edge-disjoint trees are just spanning trees of the graph, and the problem becomes the classical Nash-Williams-Tutte theorem, see [23, 29].

Graph product is an important method to construct large graphs from small ones. So it has many applications in the design and analysis of networks, see [9, 14, 15]. The lexicographic product (or composition), Cartesian product, strong product and the direct product are the main four standard products of graphs. More information about the (edge-) connectivity of these four product graphs can be found in [4, 8, 10, 12, 13, 16, 32]. The generalized 3-edge-connectivity of Cartesian product graphs was studied and the lower bound is given in [28]. In this paper, we study the generalized 3-edge-connectivity of lexicographic product graphs and provide both sharp lower and upper bounds.

Theorem 1. *Let G and H be two non-trivial graphs such that G is connected. Then* $\lambda_3(H) + \lambda_3(G)|V(H)| \leq \lambda_3(G \circ H) \leq \min\Big\{\Big\lfloor \frac{4\lambda_3(G)+2}{3} \Big\rfloor |V(H)|^2, \delta(H) + \delta(G)|V(H)|\Big\}$. *Moreover, the lower and upper bounds are sharp.*

Note that the vertex version, the generalized 3-connectivity of Cartesian product and lexicographic product graphs, was studied in [16, 20]. The results there are quite different from ours.

2 Preliminary and notation

Let $G = (V, E)$ be a graph and S be an s-subset of V. $G[S]$ denotes the induced subgraph of G on S and $\mathcal{E}^{|S|}$ denotes the empty graph on S, that is, the union of s isolated vertices. *Connect x to S* is to join x to each vertex of S for a vertex x outside S. Given two sets X, Y of vertices, we call a path P an *XY-path* if the end-vertices of P are in X and Y, respectively, and all inner vertices are in neither X nor Y. If u and v are two vertices on a path P, uPv will denote the segment of P from u to v. Two distinct paths are *edge-disjoint* if they have no edges in common; *internally disjoint* if they have no internal vertices in common; *vertex-disjoint* if they have no vertices in common. For $X = \{x_1, x_2, \cdots, x_k\}$ and $Y = \{y_1, y_2, \cdots, y_k\}$, an *$XY$-linkage* is defined as a set Q of k vertex-disjoint XY-paths $x_iP_iy_i$, $1 \leq i \leq k$.

Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$. The *lexicographic product* (*or composition*) $G \circ H$ of G and H is defined as follows: $V(G \circ H) = V_1 \times V_2$, two vertices (u, v) and (u', v') are adjacent if and only if either $uu' \in E_1$ or $u = u'$, $vv' \in E_2$. In other words, $G \circ H$ is obtained by substituting a copy $H(u)$ of H for every vertex u of G and joining each vertex of $H(u)$ with every vertex of $H(u')$ if $uu' \in E_1$. The vertex set $G(v) = \{(u, v)|u \in V_1\}$ for some fixed vertex v of H is called a layer of graph G or simply a *G-layer*. Analogously, we define the *H-layer* with respect to a vertex u of G and denote it by $H(u)$. It is not hard to see that any *G-layer* induces a subgraph of $G \circ H$ that is isomorphic to G and any *H-layer* induces a subgraph of $G \circ H$ that is isomorphic to H. For any $u, u' \in V(G)$ and $v, v' \in V(H)$, $(u, v), (u, v') \in V(H(u))$, $(u', v), (u', v') \in V(H(u'))$, $(u, v), (u', v) \in V(G(v))$, $(u, v'), (u', v') \in V(G(v'))$. We view (u, v') and (u', v) as *the vertices corresponding to* (u, v) in $G(v')$ and $H(u')$, respectively. Similarly, we can define the path and tree corresponding to some path and tree, respectively. The edge $(u, v)(u', v')$ is called a *first-type* edge if $uu' \in E_1$ and $v = v'$; a *second-type* edge if $vv' \in E_2$ and $u = u'$; a *third-type* edge if $uu' \in E_1$ and $v \neq v'$. For a subset W of $V(G)$ with $W = \{u_1, \cdots, u_t\}$, we denote $H(W) = H(u_1) \cup \cdots \cup H(u_t)$. We use K_W to denote a subgraph of $G \circ H$, where $V(K_W) = V(G[W] \circ H)$, $E(K_W) = E(G[W] \circ H) \setminus E(H(W))$, namely, the end-vertices of an edge of K_W are in different H-layers.

Unlike the other products, the lexicographic product does not satisfy the commutative law, that is, $G \circ H$ could not be isomorphic to $H \circ G$. By a simple observation, $G \circ H$ is connected if and only if G is connected. Moreover, $\delta(G \circ H) = \delta(G)|V(H)| + \delta(H)$.

Let $G = (V, E)$ be a connected graph, $S = \{x, y, z\} \subseteq V$, and T be an S-tree. We call T a *type I S-tree* if it is just a path whose end-vertices belong to S; a *type II S-tree* if it has exactly three leaves x, y, z. Note that each vertex in a type I S-tree has degree two except the two end-vertices in S. If T is of type II, every vertex in $T \setminus S$ has degree two except one vertex of degree three. By deleting some vertices and edges of an S-tree T, it is easy to check that T is of type I or II. Because our aim is to get as many S-trees as possible, in this paper, each S-tree is of type I or II. Therefore, we get the following proposition.

Proposition 1. *Let $G = (V, E)$ be a graph with $\lambda_3(G) = k \geq 2$, $S = \{x, y, z\} \subseteq V$. Then there exist $k-2$ edge-disjoint S-trees $T_1, \cdots, T_{k-2}$ such that $E(T_i) \cap E(G[S]) = \emptyset$ where $1 \leq i \leq k-2$.*

Proof. By the definition of an S-tree, we know that $|E(T_i) \cap E(G[S])| \leq 2$ and $|\{T_i \,|\, E(T_i) \cap E(G[S]) \neq \emptyset\}| \leq 3$. Let $T_1, \cdots, T_k$ be k edge-disjoint S-trees. If $|\{T_i \,|\, E(T_i) \cap E(G[S]) \neq \emptyset\}| \leq 2$, we are done. Thus, it remains to consider the case when $G[S]$ is a triangle. Without loss of generality, assume that $|\{T_i \,|\, E(T_i) \cap E(G[S]) \neq \emptyset\}| = 3$ and $E(T_i) \cap E(G[S]) \neq \emptyset$, where $i = 1, 2, 3$. Then T_1, T_2, T_3 have the structures F_1 or F_2 shown in Figure 1. Furthermore, we can obtain T_1', T_2', T_3' from T_1, T_2, T_3 such that $E(T_1') \cap E(G[S]) = \emptyset$. See figures F_1' and F_2' in Figure 1, where the S-tree T_1' is shown by gray lines. Thus $T_1', T_4, \cdots, T_k$ are our desired $k-2$ edge-disjoint S-trees.

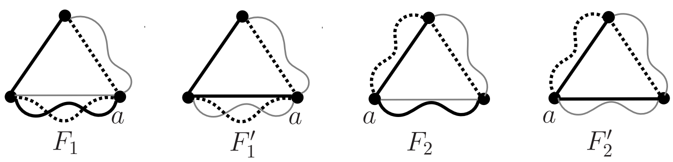

Fig. 1. Three S-trees of type I.

Li et al. [22, 21] got the following results which will be useful for our proof.

Observation 1 *([22]) For any graph G of order n, $\lambda_k(G) \leq \lambda(G)$. Moreover, the upper bound is tight.*

Observation 2 *([22]) If G is a connected graph, then $\lambda_k(G) \leq \delta(G)$. Moreover, the upper bound is tight.*

Proposition 2. *([21]) Let G be a connected graph of order n with minimum degree δ. If there are two adjacent vertices of degree δ, then $\lambda_k(G) \leq \delta - 1$ for $3 \leq k \leq n$. Moreover, the upper bound is sharp.*

From Proposition 2, it is easy to get the following observation.

Observation 3 *Let G be a connected graph with $\lambda_3(G) = k$, and x, y be two adjacent vertices of G. Then $d_G(x) \geq k+1$ or $d_G(y) \geq k+1$.*

Example 1. Let G be a path of length two and H be a complete graph of order four, and T_1, T_2 be two edge-disjoint S-trees in H, where $S = \{x, y, z\} \subseteq V(H)$. The structure of $G \circ (T_1 \cup T_2)$ is shown as F_a in Figure 2, where the edges of a complete bipartite graph is simplified by bold black crossing edges. Note that $E(G \circ T_1) \cap E(G \circ T_2) = E(G \circ \mathcal{E}^{|S|})$.

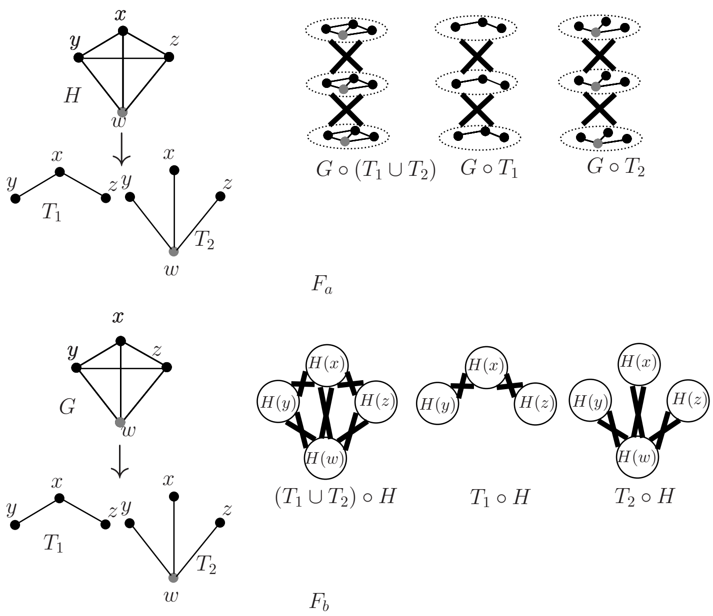

Fig. 2. The structures of $G \circ (T_1 \cup T_2)$ and $(T_1 \cup T_2) \circ H$.

Remark 1. Two edge-disjoint S-trees T_1, T_2 in H may have other vertices in common except S. If $V(T_1) \cap V(T_2) = W$, then $E(G \circ T_1) \cap E(G \circ T_2) = E(G \circ \mathcal{E}^{|W|})$.

Example 2. Let G be a complete graph of order four and H be an arbitrary graph, and T_1, T_2 be two edge-disjoint S-trees in G, where $S = \{x, y, z\} \subseteq V(G)$. The structure of $(T_1 \cup T_2) \circ H$ is shown as F_b in Figure 2 and $E(T_1 \circ H) \cap E(T_2 \circ H) = E(H(S))$.

Remark 2. Two edge-disjoint S-trees T_1, T_2 in G may have other vertices in common except S. If $V(T_1) \cap V(T_2) = W$, then $E(T_1 \circ H) \cap E(T_2 \circ H) = E(H(W))$.

3 Lower bound of $\lambda_3(G \circ H)$

In this section, we mainly prove the following theorem.

Theorem 2. *Let G and H be two non-trivial graphs such that G is connected. Then $\lambda_3(G \circ H) \geq \lambda_3(H) + \lambda_3(G)|V(H)|$. Moreover, the lower bound is sharp.*

By the following corollary, we know that the bound of the above theorem is sharp.

Corollary 1. $\lambda_3(P_s \circ P_t) = t + 1$.

Proof. By Theorem 2, $\lambda_3(P_s \circ P_t) \geq t + 1$. On the other hand, by Observation 2, $\lambda_3(P_s \circ P_t) \leq \delta(P_s \circ P_t) = t + 1$. Thus $\lambda_3(P_s \circ P_t) = t + 1$.

Let G be a graph with $V(G) = \{u_1, u_2, \cdots, u_{n_1}\}$ and $\lambda_3(G) = r_1$, and let H be a graph with $V(H) = \{v_1, v_2, \cdots, v_{n_2}\}$ and $\lambda_3(H) = r_2$. Set $S = \{x, y, z\} \subseteq V(G \circ H)$. Firstly, we give the sketch of the proof of Theorem 2. In total, the desired $r_2 + r_1 n_2$ S-trees are obtained on two stages: r_2 edge-disjoint S-trees by first-type and second-type edges on Stage I and $r_1 n_2$ edge-disjoint S-trees by the remaining first-type edges and the third-type edges on Stage II. Note that if H is disconnected, then $\lambda_3(H) = 0$ as defined, thus we omit Stage I immediately. Next we shall prove Theorem 2 by a series of lemmas according to the position of x, y, z in $G \circ H$.

Lemma 1. *If x, y, z belong to the same H-layer, then there exist $r_2 + r_1 n_2$ edge-disjoint S-trees.*

Proof. Without loss of generality, assume that $x, y, z \in H(u_1)$, $x = (u_1, v_1)$, $y = (u_1, v_2)$ and $z = (u_1, v_3)$. On Stage I, since $\lambda_3(H) = r_2$, there are r_2 edge-disjoint S-trees in $H(u_1)$. On Stage II, by Observation 2, u_1 has r_1 neighbors in G, say $\beta_1, \beta_2, \cdots, \beta_{r_1}$. Thus $T^*_{ij} = x(\beta_i, v_j) \cup y(\beta_i, v_j) \cup z(\beta_i, v_j)$ ($1 \leq i \leq r_1$ and $1 \leq j \leq n_2$) are $r_1 n_2$ S-trees. These $r_2 + r_1 n_2$ S-trees are obviously edge-disjoint, as desired.

Lemma 2. *If exactly two of x, y and z belong to the same H-layer, then there exist $r_2 + r_1 n_2$ edge-disjoint S-trees.*

Proof. Assume that $x, y \in H(u_1)$, $z \in H(u_2)$. Let x'' and y'' be the vertices in $H(u_2)$ corresponding to x and y, and z' be the vertex in $H(u_1)$ corresponding to z, respectively. Consider the following two cases.

Case 1. $z' \in \{x, y\}$.

Without loss of generality, assume that $z' = x$, $x = (u_1, v_1)$, $y = (u_1, v_2)$ and $z = (u_2, v_1)$.

By Observation 1, there are r_2 edge-disjoint v_1v_2-paths $P_1, P_2, \cdots, P_{r_2}$ in H such that $\ell(P_1) \leq \ell(P_2) \leq \cdots \leq \ell(P_{r_2})$. Denote the neighbor of v_1 in P_i by α_i ($1 \leq i \leq r_2$). Set $D = \{\alpha_1, \alpha_2, \cdots, \alpha_{r_2}\}$. Notice that $\alpha_p \neq \alpha_q$ if $p \neq q$. Similarly, there are r_1 edge-disjoint u_1u_2-paths $Q_1, Q_2, \cdots, Q_{r_1}$ in G such that $\ell(Q_1) \leq$

$\ell(Q_2) \leq \cdots \leq \ell(Q_{r_1})$. For each i with $1 \leq i \leq r_1$, set $Q_i = u_1\beta_{i,1}\beta_{i,2}\cdots\beta_{i,t_i-1}u_2$ and $\ell(Q_i) = t_i$. Also, note that $\beta_{p,1} \neq \beta_{q,1}$ if $p \neq q$.

On Stage I, the desired r_2 S-trees are obtained associated with the longest u_1u_2-path Q_{r_1}. If v_1 and v_2 are nonadjacent in H, then $T_i^* = P_i(u_1) \cup Q_{r_1}(\alpha_i) \cup z(u_2, \alpha_i)$ $(1 \leq i \leq r_2)$ are r_2 S-trees as shown in Figure 3(a), where $P_i(u_1)$ is the path in $H(u_1)$ corresponding to P_i in H, and $Q_{r_1}(\alpha_i)$ is the path in $G(\alpha_i)$ corresponding to Q_{r_1} in G. Now v_1 and v_2 are adjacent in H, that is, $P_1 = v_1v_2$ and $(u_1, \alpha_1) = y$. It follows from Observation 3 that $d_H(v_1) \geq r_2 + 1$ or $d_H(v_2) \geq r_2 + 1$, without loss of generality, say $d_H(v_1) \geq r_2 + 1$. For P_1, $T_1^* = xy \cup x(u_1, \alpha_{r_2+1}) \cup Q_{r_1}(\alpha_{r_2+1}) \cup z(u_2, \alpha_{r_2+1})$ is an S-tree, where $\alpha_{r_2+1} \notin D$, α_{r_2+1} is a neighbor of v_1 in H, and $Q_{r_1}(\alpha_{r_2+1})$ is the path in $G(\alpha_{r_2+1})$ corresponding to Q_{r_1}, see Figure 3(b). For P_i $(2 \leq i \leq r_2)$, set $T_i^* = P_i(u_1) \cup Q_{r_1}(\alpha_i) \cup z(u_2, \alpha_i)$. It is easy to see that these r_2 S-trees are edge-disjoint. The case that $d_H(v_2) \geq r_2 + 1$ can be proved similarly.

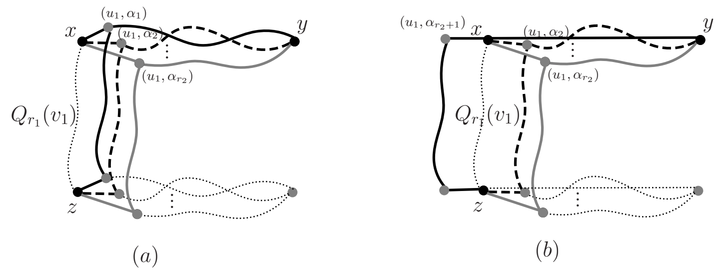

Fig. 3. The r_2 edge-disjoint S-trees where the edges of an S-tree are shown by the same type of lines.

Up to now, we should remark that the first-type edges incident with x and y in $G \circ H$ are not used whether or not v_1 and v_2 are adjacent in H. Since a vertex in $V(H) \setminus \{v_1, v_2\}$ may belong to more than one v_1v_2-path, we make use of either the r_2 neighbors of v_1 or the r_2 neighbors of v_2 to get our desired r_2 edge-disjoint S-trees.

Define a new graph $(G \circ H)^*$ from $G \circ H$ by deleting the edges of r_2 S-trees on Stage I. On Stage II, with the aid of Q_i $(1 \leq i \leq r_1)$, we successively construct r_1n_2 S-trees in $(G \circ H)^*$ in non-decreasing order of the length of Q_i. We distinguish two subcases by the length t_1 of Q_1.

Subcase 1.1. $t_1 \geq 2$.

Recall that $Q_1 = u_1\beta_{1,1}\beta_{1,2}\cdots\beta_{1,t_1-1}u_2$. We will obtain n_2 internally disjoint xy-paths $A_1, A_2, \cdots, A_{n_2}$ in $K_{u_1,\beta_{1,1}}$, and a $V(H(\beta_{1,1}))V(H(\beta_{1,t_1-1}))$-linkage $B_1, B_2, \cdots, B_{n_2}$ by third-type edges associated with $\beta_{1,1}Q_1\beta_{1,t_1-1}$. Thus, $T_i^* = A_i \cup B_i \cup (\beta_{1,t_1-1}, v_i)z$ are n_2 edge-disjoint S-trees, where the subscript i $(1 \leq i \leq n_2)$ of v_i is expressed module n_2 as one of $1, 2, \cdots, n_2$. Indeed, this can be done. Set $A_i = x(\beta_{1,1}, v_i)y$ for $1 \leq i \leq n_2$. If $t_1 = 2$, then $B_i = \emptyset$. If $t_1 \geq 3$, then $B_i = (\beta_{1,1}, v_i)(\beta_{1,2}, v_{i+1})(\beta_{1,3}, v_i)(\beta_{1,4}, v_{i+1})\cdots(\beta_{1,t_1-1}, v_i)$ and $T_i^* = A_i \cup B_i \cup$

$(\beta_{1,t_1-1}, v_i)z$ when t_1 is even; $B_i = (\beta_{1,1}, v_i)(\beta_{1,2}, v_{i+1})(\beta_{1,3}, v_i)(\beta_{1,4}, v_{i+1}) \cdots (\beta_{1,t_1-1}, v_{i+1})$ and $T_i^* = A_i \cup B_i \cup (\beta_{1,t_1-1}, v_{i+1})z$ when t_1 is odd. For example, let $n_2 = 4$. Then 4 edge-disjoint S-trees are shown in Figure 4 when $t_1 = 2$, $t_1 = 3$ and $t_1 = 4$, respectively.

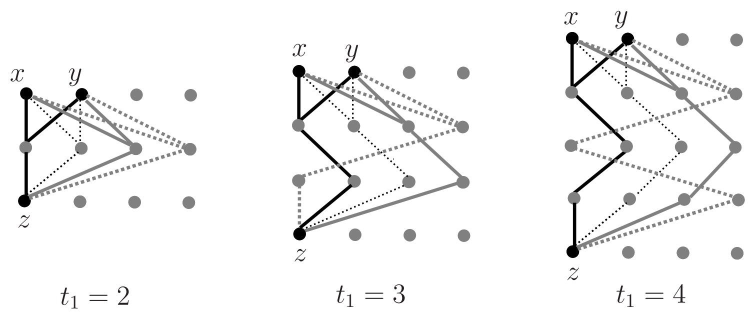

Fig. 4. The 4 edge-disjoint S-trees where the edges of an S-tree are shown by the same type of lines.

Subcase 1.2. $t_1 = 1$ and $Q_1 = u_1u_2$.

Since $\lambda_3(G) = r_1$, it follows from Observation 3 that $d_G(u_1) \geq r_1 + 1$ or $d_G(u_2) \geq r_1 + 1$.

If $d_G(u_1) \geq r_1 + 1$, then denote another neighbor of u_1 in G by $\beta_{r_1+1,1}$ except u_2 and $\beta_{i,1}$ $(2 \leq i \leq r_1)$. We obtain n_2 edge-disjoint S-trees associated with Q_1 as follows. Let $T_1^* = (\beta_{r_1+1,1}, v_1)x \cup (\beta_{r_1+1,1}, v_1)y \cup xz$, $T_2^* = (\beta_{r_1+1,1}, v_2)x \cup (\beta_{r_1+1,1}, v_2)y \cup yz$, $T_i^* = (u_2, v_i)x \cup (u_2, v_i)y \cup (u_2, v_i)(u_1, v_{i+1}) \cup (u_1, v_{i+1})z$ for $3 \leq i \leq n_2 - 1$, $T_{n_2}^* = (u_2, v_{n_2})x \cup (u_2, v_{n_2})y \cup (u_2, v_{n_2})(u_1, v_3) \cup (u_1, v_3)z$; see Figure 5$(a)$.

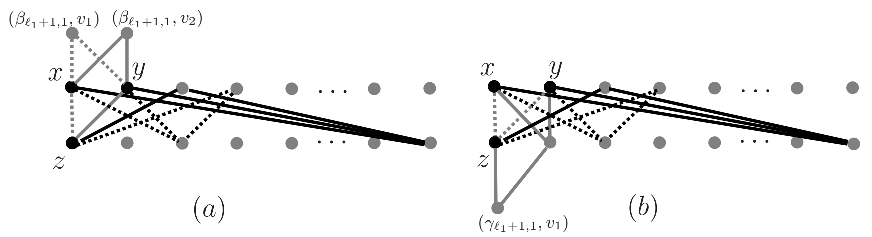

Fig. 5. The n_2 edge-disjoint S-trees where the edges of an S-tree are shown by the same type of lines.

If $d_G(u_2) \geq r_1 + 1$, then denote another neighbor of u_2 in G by γ_{r_1+1} except u_1 and β_{i,t_i-1} $(2 \leq i \leq r_1)$. For Q_1, set $T_1^* = xz \cup zy$, $T_2^* = xy'' \cup y''y \cup (\gamma_{r_1+1}, v_1)y'' \cup (\gamma_{r_1+1}, v_1)z$, $T_i^* = (u_2, v_i)x \cup (u_2, v_i)y \cup (u_2, v_i)(u_1, v_{i+1}) \cup (u_1, v_{i+1})z$ for $3 \leq$

$i \leq n_2 - 1$, and $T^*_{n_2} = (u_2, v_{n_2})x \cup (u_2, v_{n_2})y \cup (u_2, v_{n_2})(u_1, v_3) \cup (u_1, v_3)z$; see Figure 5$(b)$.

In both subcases, similar to Subcase 1.1, we are able to get n_2 edge-disjoint S-trees associated with Q_i $(2 \leq i \leq r_1)$, it follows that n_2r_1 edge-disjoint S-trees are obtained, as desired.

Case 2. $z' \notin \{x, y\}$.

Assume that $x = (u_1, v_1)$, $y = (u_1, v_2)$ and $z = (u_2, v_3)$. Let $S' = \{v_1, v_2, v_3\}$ and $S'' = \{x, y, z'\}$.

By Observation 1, there are r_1 edge-disjoint u_1u_2-paths $Q_1, Q_2, \cdots, Q_{r_1}$ in G such that $\ell(Q_1) \leq \ell(Q_2) \leq \cdots \leq \ell(Q_{r_1})$. By Proposition 1, r_2 edge-disjoint S'-trees $T_1, T_2, \cdots, T_{r_2}$ exist in H such that $0 \leq |\{T_i | E(T_i) \cap E(H[S'])\}| \leq 2$. Suppose $E(T_i) \cap E(H[S']) = \emptyset$ for $3 \leq i \leq r_2$. According to whether T_1 and T_2 share edges with $E(H[S'])$ or not, we get the desired S-trees in the following subcases.

Subcase 2.1. $E(T_1) \cap E(H[S']) = \emptyset$ and $E(T_2) \cap E(H[S']) = \emptyset$.

Denote the neighbor of v_3 in T_i by α_i where $1 \leq i \leq r_2$. On Stage I, let $T^*_i = T_i(u_1) \cup Q_{r_1}(\alpha_i) \cup z(u_2, \alpha_i)$, where $T_i(u_1)$ is the tree in $H(u_1)$ corresponding to T_i, $Q_{r_1}(\alpha_i)$ is the path in $G(\alpha_i)$ corresponding to Q_{r_1} for $1 \leq i \leq r_2$. On Stage II, if $\ell(Q_1) \geq 2$, then construct n_2 S-trees similar to Case 1; if $\ell(Q_1) = 1$, then either u_1 or u_2 has a neighbor which is not in each u_1u_2-path Q_i in G. Thus n_2 S-trees associated with Q_1 are shown in Figure 6 (u_1 has another neighbor in G in Figure 6(a) and u_2 has another neighbor in G in Figure 6(b)). Similar to Case 1, we obtain n_2 S-trees associated with Q_i for $2 \leq i \leq r_1$, thus there exist $r_2 + r_1n_2$ edge-disjoint S-trees, as desired.

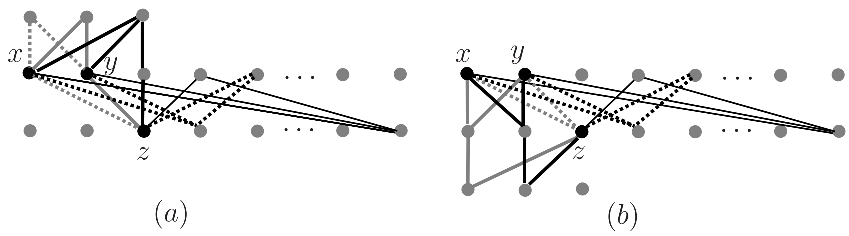

Fig. 6. The n_2 edge-disjoint S-trees where the edges of an S-tree are shown by the same type of lines.

Subcase 2.2. $E(T_1) \cap E(H[S']) \neq \emptyset$ and $E(T_2) \cap E(H[S']) = \emptyset$.

Suppose $|E(T_1) \cap E(H[S'])| = 1$ and $E(T_2) \cap E(H[S']) = \emptyset$. Furthermore, suppose $E(T_1) \cap E(H[S']) = v_1v_2$ and $d_{T_1}(v_2) = 2$ (the other possibilities can be proved similarly). For $1 \leq i \leq r_2$, denote the neighbor of v_3 in T_i by α_i. Then we are able to obtain r_2 S-trees with the aid of α_i on Stage I and r_1n_2 S-trees on Stage II similar to Subcase 2.1. It remains to consider the case that $|E(T_1) \cap E(H[S'])| = 2$ and $E(T_2) \cap E(H[S']) = \emptyset$. On Stage I, if $d_{Q_{r_1}}(u_1, u_2) \geq 2$ and $d_{T_1}(v_2) = 2$ or $d_{Q_{r_1}}(u_1, u_2) \geq 2$ and $d_{T_1}(v_3) = 2$, then an S-tree T^*_1

associated with T_1 has the structure as shown in Figure 7, where $\bar{x}$ is the neighbor of x'' in $Q_{r_1}(v_1)$; if $d_{Q_{r_1}}(u_1, u_2) = 1$, then $T_1^* = xyy''z$ (when u_1 has another neighbor outside Q_i) or $T_1^* = xyz'z$ (when u_2 has another neighbor outside Q_i). We obtain other $r_2 + r_1n_2 - 1$ S-trees similar to Subcase 2.1. Thus there exist $r_2 + r_1n_2$ edge-disjoint S-trees, as desired.

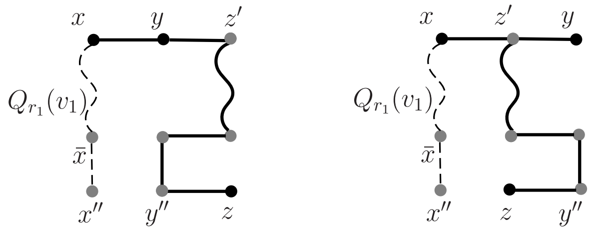

Fig. 7. The solid lines stand for the edges of the S-tree.

Subcase 2.3. $E(T_1) \cap E(H[S']) \neq \emptyset$ and $E(T_2) \cap E(H[S']) \neq \emptyset$.

Without loss of generality, suppose $|E(T_2) \cap E(H[S'])| = 1$. If $|E(T_1) \cap E(H[S'])| = 1$, then assume that the two S'-trees T_1 and T_2 have the structure as one of F_3, F_4, F_5, F_6 in Figure 8, where v_2 is marked. For $1 \leq i \leq r_2$, denote the neighbor of v_2 in $T_i \setminus \{v_1, v_3\}$ by α_i and construct $r_2 + r_1n_2$ S-trees similar to Subcase 2.1. So $|E(T_1) \cap E(H[S'])| = 2$, and then T_1 and T_2 have the structure F_7, where T_1 is shown in Figure 8 by dotted lines. For $2 \leq i \leq r_2$, denote the neighbor of v_2 in $T_i \setminus \{v_1, v_3\}$ by α_i. Construct an S-tree T_1^* similar to Subcase 2.2 and other $r_2 + r_1n_2 - 1$ S-trees similar to Subcase 2.1. Thus, there exist $r_2 + r_1n_2$ edge-disjoint S-trees, as desired.

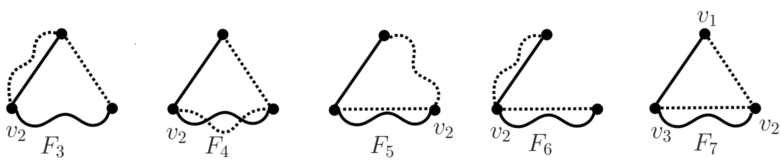

Fig. 8. Two S'-trees of type I.

Lemma 3. *If x, y, z belong to distinct H-layers, then there exist $r_2 + r_1n_2$ edge-disjoint S-trees.*

Proof. Assume that $x \in H(u_1)$, $y \in H(u_2)$ and $z \in H(u_3)$. Let y', z' be the vertex corresponding to y, z in $H(u_1)$, x'', z'' be the vertex corresponding to x, z in $H(u_2)$, and x''', y''' be the vertex corresponding to x, y in $H(u_3)$, respectively. We distinguish the following three cases.

Case 1. x, y, z belong to the same G-layer.

We may assume that $x = (u_1, v_1)$, $y = (u_2, v_1)$, $z = (u_3, v_1)$. It is easily seen that there are r_2 neighbors of v_1 in H, say $\alpha_1, \alpha_2, \cdots, \alpha_{r_2}$, and r_1 edge-disjoint $\{u_1, u_2, u_3\}$-trees $T_1, T_2, \cdots, T_{r_1}$ in G. For a tree T_i in G, set by $T_i(\alpha_j)$ the corresponding tree in $G(\alpha_j)$ for $1 \le i \le r_1$, $1 \le j \le r_2$.

On Stage I, $T_j^* = T_1(\alpha_j) \cup x(u_1, \alpha_j) \cup y(u_2, \alpha_j) \cup z(u_3, \alpha_j)$ $(1 \le j \le r_2)$ are r_2 edge-disjoint S-trees.

On Stage II, if T_j is of type I for some j with $1 \le j \le r_1$, then we may assume that $d_{T_j}(u_2) = 2$. Denote the neighbor of u_1, u_3 in T_j by η_j, γ_j and the neighbors of u_2 by β_j, $\bar{\beta}_j$ (β_j is nearer to u_1 than $\bar{\beta}_j$), where β_j, η_j and $\bar{\beta}_j$, γ_j may be the same vertex. Associated with $u_1T_ju_2$ and $u_2T_ju_3$, there are n_2 edge-disjoint xy-paths $A = \{A_1, \cdots, A_{n_2}\}$ and edge-disjoint yz-paths $B = \{B_1, \cdots, B_{n_2}\}$, respectively. Then $T_{ij}^* = A_i \cup B_i$ $(1 \le i \le n_2)$ are n_2 edge-disjoint S-trees. Indeed, this can be done. We will only provide the construction of A according to $d_{T_j}(u_1, u_2)$, since the construction of B is similar to that of A. If $d_{T_j}(u_1, u_2) = 1$, then set $A_1 = xy$, $A_i = x(u_2, v_i)(u_1, v_{i+1})y$ for $2 \le i \le n_2 - 1$, and $A_{n_2} = x(u_2, v_{n_2})(u_1, v_2)y$; if $d_{T_j}(u_1, u_2) = 2$, then set $A_i = x(\eta_j, v_i)y$ for $1 \le i \le n_2$. It remains to consider the case that $d_{T_j}(u_1, u_2) \ge 3$. Since there is a $V(H(\eta_j))V(H(\beta_j))$-linkage $D_1, D_2, \cdots, D_{n_2}$ by third-type edges of $G \circ H$ associated with $\eta_jT_j\beta_j$, it follows that $A_i = x(\eta_j, v_i) \cup D_i \cup (\beta_j, v_i)y$, where the subscript i $(1 \le i \le n_2)$ of v_i is expressed module n_2 as one of $1, 2, \cdots, n_2$. It remains to consider the case that T_j is of type II. Denote the neighbor of u_1, u_2, u_3 in T_j by η_j, β_j, γ_j and the only one three-degree vertex in T_j by w_j (η_j, β_j, γ_j and w_j may be the same vertex). We find a $V(H(\eta_j))V(H(\beta_j))$-linkage and a $V(H(\gamma_j))V(H(w_j))$-linkage respectively by third-type edges of $G \circ H$, and connect x, y, z respectively to $V(H(\eta_j))$, $V(H(\beta_j))$ and $V(H(\gamma_j))$. Thus, n_2 edge-disjoint S-trees are obtained associated with T_j. Since $1 \le j \le r_1$, it follows that r_1n_2 edge-disjoint S-trees are obtained on Stage II, as desired.

Case 2. Exactly two of x, y, z belong to the same G-layer.

We only consider the case $x = y'$ (the other cases $x = z'$ or $y' = z'$ can be proved by similar arguments). Assume that $x = (u_1, v_1)$, $y = (u_2, v_1)$ and $z = (u_3, v_2)$. Since $\lambda_3(H) = r_2$, there exist r_2 edge-disjoint v_1v_2-paths $P_1, P_2, \cdots, P_{r_2}$ in H such that $\ell(P_1) \le \ell(P_2) \le \cdots \le \ell(P_{r_2})$. For $1 \le i \le r_2$, denote the neighbor of v_1 and v_2 in P_i by α_i and β_i, respectively, and denote by $P_i(u_3)$ in $H(u_3)$ corresponding to P_i. Since $\lambda_3(G) = r_1$, there are r_1 edge-disjoint $\{u_1, u_2, u_3\}$-trees $T_1, T_2, \cdots, T_{r_1}$ in G.

On Stage I, if $\ell(P_1) \ge 2$, then set $T_i^* = x(u_1, \alpha_i) \cup y(u_2, \alpha_i) \cup zP_i(u_3)(u_3, \alpha_i) \cup T_1(\alpha_i)$ for $1 \le i \le r_2$. Otherwise, $\ell(P_1) = 1$, that is, v_1 is adjacent to v_2. Then $d_H(v_1) \ge r_2 + 1$ or $d_H(v_2) \ge r_2 + 1$. If $d_H(v_1) \ge r_2 + 1$, then $T_1^* = \{x(u_1, \alpha_{r_2+1}), y(u_2, \alpha_{r_2+1}), zx''', x'''(u_3, \alpha_{r_2+1})\}$ $\cup\, T_1(\alpha_{r_2+1})$, where α_{r_2+1} is another neighbor of v_1 except α_i $(1 \le i \le r_2)$. If $d_H(v_2) \ge r_2 + 1$, then $T_1^* = \{xz', z'(u_1, \beta_{r_2+1}), yz'', z''(u_2, \beta_{r_2+1}), z(u_3, \beta_{r_2+1})\} \cup T_1(\beta_{r_2+1})$, where β_{r_2+1} is another neighbor of v_1 except β_i $(1 \le i \le r_2)$.

By similar arguments as in Case 1 of Lemma 3, r_1n_2 edge-disjoint S-trees can be obtained on Stage II.

Case 3. x, y, z belong to different G-layers.

Assume that $x = (u_1, v_1)$, $y = (u_2, v_2)$ and $z = (u_3, v_3)$. Let $S' = \{v_1, v_2, v_3\}$ and $S'' = \{u_1, u_2, u_3\}$.

Since $\lambda_3(H) = r_2$, there are r_2 edge-disjoint S'-trees $T_1, T_2, \cdots, T_{r_2}$ in H. For $1 \leq i \leq r_2$, denote by α_i the vertex in T_i adjacent to a vertex in S', say v_1, and $\ell(T_i)$ denotes the size of T_i. Similarly, there are r_1 edge-disjoint S''-trees $T'_1, T'_2, \cdots, T'_{r_1}$ in G.

On Stage I, if $\ell(T_i) \geq 3$ for each i with $1 \leq i \leq r_2$, then let $T_i^* = x(u_1, \alpha_i) \cup yT_i(u_2)(u_2, \alpha_i) \cup zT_i(u_3)(u_3, \alpha_i) \cup T'_1(\alpha_i)$. Otherwise, similar to Case 2 of Lemma 2, the most difficult case is that there is an S'-tree of size two. Suppose $\ell(T_1) = 2$ and $d_{T_1}(v_2) = 2$. Thus T_1^* has three structures as shown in Figure 9 where T'_1 is of type II in Figure 9(a), T'_1 is of type I and $d_{T'_1}(u_1) = 2$ in Figure 9(b) and T'_1 is of type I and $d_{T'_1}(u_1) = 1$ in Figure 9(c).

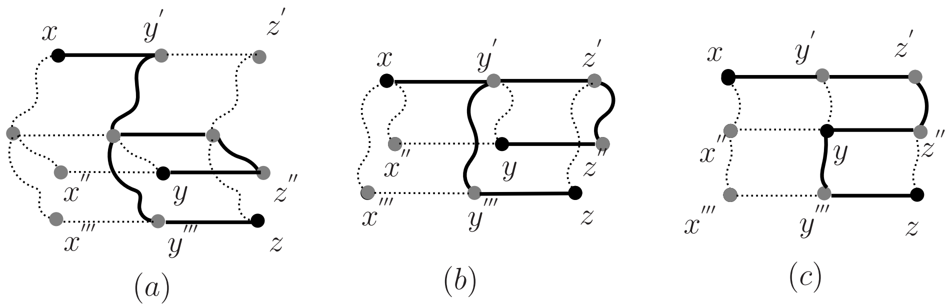

Fig. 9. The S-tree with the aid of T'_1 shown by the solid lines.

On Stage II, $r_1 n_2$ edge-disjoint S-trees are obtained by similar arguments as in Case 1 of Lemma 3.

In each case, we obtain $r_2 + r_1 n_2$ S-trees, and it is easily seen that these S-trees are edge-disjoint, as desired.

From the above three lemmas, Theorem 2 follows immediately.

4 Upper bound of $\lambda_3(G \circ H)$

In this section, we give an upper bound of the generalized 3-edge-connectivity of the lexicographic product of two graphs.

Yang and Xu [33] investigated the classical edge-connectivity of the lexicographic product of two graphs.

Theorem 3. [33] *Let G and H be two non-trivial graphs such that G is connected. Then*

$$\lambda(G \circ H) = \min\{\lambda(G)|V(H)|^2, \delta(H) + \delta(G)|V(H)|\}.$$

In [22], the sharp lower bound of the generalized 3-edge-connectivity of a graph is given as follows.

Proposition 3. *[22] Let G be a connected graph with n vertices. For every two integers s and r with $s \geq 0$ and $r \in \{0, 1, 2, 3\}$, if $\lambda(G) = 4s + r$, then $\lambda_3(G) \geq 3s + \lceil \frac{r}{2} \rceil$. Moreover, the lower bound is sharp. We simply write $\lambda_3(G) \geq \frac{3\lambda(G)-2}{4}$.*

From the above two results, we get the following upper bound of $\lambda_3(G \circ H)$.

Theorem 4. *Let G and H be two non-trivial graphs such that G is connected. Then*

$$\lambda_3(G \circ H) \leq \min\left\{ \left\lfloor \frac{4\lambda_3(G) + 2}{3} \right\rfloor |V(H)|^2, \delta(H) + \delta(G)|V(H)| \right\}.$$

Moreover, the upper bound is sharp.

Proof. By Proposition 3, $\lambda(G) \leq \lfloor \frac{4\lambda_3(G)+2}{3} \rfloor$. By Proposition 1 and Theorem 3, we have $\lambda_3(G \circ H) \leq \lambda(G \circ H) = \min\{\lambda(G)|V(H)|^2, \delta(H) + \delta(G)|V(H)|\}$. It follows that $\lambda_3(G \circ H) \leq \min\left\{ \left\lfloor \frac{4\lambda_3(G)+2}{3} \right\rfloor |V(H)|^2, \delta(H) + \delta(G)|V(H)| \right\}$. Moreover, the example in Corollary 1 shows that the upper bound is sharp.

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