On extremal graphs with at most ℓ internally disjoint Steiner trees connecting any $n-1$ vertices*

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Abstract

The maximum local connectivity was first introduced by Bollobás. The problem of determining the maximum number of edges in a graph with $\overline{\kappa} \leq \ell$ has been studied extensively. We consider a generalization of the above concept and problem. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local connectivity* $\kappa(S)$ is the maximum number of internally disjoint trees connecting S in G. The parameter $\overline{\kappa}_k(G) = max\{\kappa(S)|S \subseteq V(G), |S| = k\}$ is called the *maximum generalized local connectivity* of G. In this paper the problem of determining the largest number $f(n;\overline{\kappa}_k \leq \ell)$ of edges for graphs of order n that have maximum generalized local connectivity at most ℓ is considered. The exact value of $f(n;\overline{\kappa}_k \leq \ell)$ for $k = n, n-1$ is determined. For a general k, we construct a graph to obtain a sharp lower bound.

Keywords: (edge-)connectivity, Steiner tree, internally (edge-)disjoint trees, generalized local (edge-)connectivity.

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1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to [7] for graph theoretical notation and terminology not described here. For any two distinct vertices x and y in G, the *local connectivity* $\kappa_G(x, y)$ is the maximum number of internally disjoint paths connecting x and y. Then $\kappa(G) = \min\{\kappa_G(x, y)|x, y \in V(G), x \neq y\}$ is defined

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to be the *connectivity* of G. In contrast to this parameter, $\overline{\kappa}(G)=\max\{\kappa_G(x,y)|x,y\in V(G),x\neq y\}$, first introduced by Bollobás (see [4] for example), is called the *maximum local connectivity* of G. As we have seen, the connectivity and maximum local connectivity are two extremes of the local connectivity of a graph. A invariant lying between these two extremes is the *average connectivity* $\widehat{\kappa}(G)$ of a graph, which is defined to be $\widehat{\kappa}(G)=\sum_{x,y\in V(G)}\kappa_G(x,y)/\binom{n}{2}$; see [3]. The problem of determining the smallest number of edges, $h_1(n;\overline{\kappa}\geq r)$, which guarantees that any graph with n vertices and $h_1(n;\overline{\kappa}\geq r)$ edges will contain a pair of vertices joined by r internally disjoint paths was posed by Erdös and Gallai; see [1] for details. Bollobás [4] considered the problem of determining the largest number of edges, $f(n;\overline{\kappa}\leq \ell)$, for graphs with n vertices and local connectivity at most ℓ, that is, $f(n;\overline{\kappa}\leq \ell)=\max\{e(G)||V(G)|=n \ and \ \overline{\kappa}(G)\leq \ell\}$. One can see that $h_1(n;\overline{\kappa}\geq \ell+1)=f(n;\overline{\kappa}\leq \ell)+1$. Similarly, let $\lambda_G(x,y)$ denote the *local edge-connectivity* connecting x and y in G. Then $\lambda(G)=\min\{\lambda_G(x,y)|x,y\in V(G),x\neq y\}$, $\overline{\lambda}(G)=\max\{\lambda_G(x,y)|x,y\in V(G),x\neq y\}$ and $\widehat{\lambda}(G)=\sum_{x,y\in V(G)}\kappa_G(x,y)/\binom{n}{2}$ are the *edge-connectivity, maximum local edge-connectivity* and *average edge-connectivity*, respectively. For the connectivity and edge-connectivity, Oellermann gave a survey paper on this subject; see [34] for details. For more details on the average (edge-)connectivity, we refer to [2]. The edge version of the above problems can be defined similarly. Set $g(n;\overline{\lambda}\leq \ell)=\max\{e(G)||V(G)|=n \ and \ \overline{\lambda}(G)\leq \ell\}$. Let $h_2(n;\overline{\lambda}\geq r)$ denote the smallest number of edges which guarantees that any graph with n vertices and $h_2(n;\overline{\kappa}\geq r)$ edges will contain a pair of vertices joined by r edge-disjoint paths. Similarly, $h_2(n;\overline{\lambda}\geq \ell+1)=g(n;\overline{\lambda}\leq \ell)+1$. The problem of determining the precise value of the parameters $f(n;\overline{\kappa}\leq \ell)$, $g(n;\overline{\lambda}\leq \ell)$, $h_1(n;\overline{\kappa}\geq r)$, or $h_2(n;\overline{\kappa}\geq r)$ has obtained wide attention and many results have been obtained; see [4, 5, 6, 18, 19, 20, 28, 29, 36].

Mader was one of the first authors that considered the 'connectedness' properties of sets of vertices in a graph other than just 2-sets; see [29, 30]. In [30], he studied an extension of Menger's theorem to independent sets of three or more vertices. We know that from Menger's theorem that if $S=\{u,v\}$ is a set of two independent vertices in a graph G, then the maximum number of internally disjoint u-v paths in G equals the minimum number of vertices that separate u and v. For a set $S=\{u_1,u_2,\cdots,u_k\}$ $(k\geq 2)$ in a graph G, an *S-path* is defined as a path between a pair of vertices of S that contains no other vertices of S. Two S-paths P_1 and P_2 are said to be *internally disjoint* if they are vertex-disjoint except possibly for the vertices in S. If S is a set of independent vertices of a graph G, then a vertex set $U\subseteq V(G)$ with $U\cap S=\varnothing$ is said to *totally separate* S if every two vertices of S belong to different components of $G\setminus U$. Let S be a set of at least three independent vertices in a graph G. Let $\mu(G)$ denote the maximum number of internally disjoint S-paths and $\mu'(G)$ the minimum number of vertices that totally separate S. A natural extension of Menger' s theorem may well be suggested, namely: If

S is a set of independent vertices of a graph G and $|S| \geq 3$, then $\mu(S) = \mu'(S)$. However, the statement is not true in general. Take for example, the graph H_1 obtained from a triangle with vertex set $\{v_1, v_2, v_3\}$ by adding three new vertices u_1, u_2, u_3 and joining v_i to u_i by an edge for $1 \leq i \leq 3$. For $S = \{v_1, v_2, v_3\}$, $\mu(S) = 1$ but $\mu'(S) = 2$. Mader [31] proved that $\mu(S) \geq \frac{1}{2}\mu'(S)$. Moreover, the bound is sharp. Lovász conjectured an edge analogue of this result and Mader proved this conjecture and established its sharpness. For more details, we refer to [30, 31, 33].

For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an *S-Steiner tree* or a *Steiner tree connecting S* (a *Steiner tree* for short) is a such subgraph $T(V', E')$ of G that is a tree with $S \subseteq V'$. Note that when $|S| = 2$ a Steiner tree connecting S is just a path connecting S. Two Steiner trees T and T' connecting S are *internally disjoint* if $E(T) \cap E(T') = \varnothing$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local connectivity* $\kappa(S)$ is the maximum number of internally disjoint trees connecting S in G. For an integer k with $2 \leq k \leq n$, the *k-tree-connectivity* or *generalized k-connectivity* is defined as $\kappa_k(G) = \min\{\kappa(S) | S \subseteq V(G), |S| = k\}$. Thus, $\kappa_2(G) = \kappa(G)$. We knew this concept from [9] for the first time. There the authors obtained the exact value of the generalized k-connectivity of complete graphs. From [12], we know that the concept was introduced actually by Hager in his another paper, but we do not know whether his this paper has been published, yet. Except for the concept of tree-connectivity, Hager also introduced another tree-connectivity parameter, called the *pendant tree-connectivity* of a graph in [12]. For the tree-connectivity, we only search for edge-disjoint trees which include S and are vertex-disjoint with the exception of the vertices in S. But pendant tree-connectivity further requests the degree of each vertex of S in a Steiner tree connecting S is equal to one. Note that it is a specialization of the tree-connectivity. For results on the generalized connectivity or tree-connectivity, see [11, 13, 22, 23, 24, 25, 26].

Chartrand et al. [8] introduced the concept of the k-connectivity of a graph, which is another generalization of the concept of the classical connectivity. Recall that there is another equivalent definition of the connectivity. The *connectivity* of G, written $\kappa(G)$, is the minimum size of a vertex set $S \subseteq V(G)$ such that $G \setminus S$ is disconnected or has only one vertex. Note that we find the above minimum vertex set without regard the number of components of $G \setminus S$. Two graphs with the same connectivity may have different degrees of vulnerability in the sense that the deletion of a vertex cut-set of minimum cardinality from one graph may produce a graph with considerably more components than in the case of the other graph. For example, the star $K_{1,n}$ and the path P_{n+1} ($n \geq 3$) are both trees of order $n+1$ and therefore connectivity 1, but the deletion of a cut-vertex from $K_{1,n}$ produces a graph with n components while the deletion of a cut-vertex from P_{n+1} produces only two components. For an integer k ($k \geq 2$) and a graph G of order n ($n \geq k$), the *k-connectivity* $\kappa'_k(G)$ is the smallest number of vertices whose removal from G of order n ($n \geq k$) produces a graph with at least k components or a graph with fewer than k vertices. Thus, for $k = 2$, $\kappa'_2(G) = \kappa(G)$. For more details about the k-connectivity, we refer to [8, 10, 34, 35]. Note that the generalized k-connectivity (or k-tree-connectivity) and k-connectivity of a graph are indeed different. Take the above graph H_1 for an example. Clearly, $\kappa_3(H_1) = 1$ but $\kappa'_3(H_1) = 2$.

In [21], we generalized the above classical problems. Similar to the classical maximum local connectivity, we introduced the parameter $\overline{\kappa}_k(G) = \max\{\kappa(S)|S \subseteq V(G), |S| = k\}$, which is called the *maximum generalized local connectivity* of G. There we considered the problem of determining the largest number of edges, $f(n; \overline{\kappa}_k \leq \ell)$, for graphs with n vertices and maximum generalized local connectivity at most ℓ, that is, $f(n; \overline{\kappa}_k \leq \ell) = \max\{e(G)||V(G)| = n \textit{ and } \overline{\kappa}_k(G) \leq \ell\}$. We also considered the smallest number of edges, $h_1(n; \overline{\kappa}_k \geq r)$, which guarantees that any graph with n vertices and $h_1(n; \overline{\kappa}_k \geq r)$ edges will contain a set S of k vertices such that there are r internally disjoint S-trees. It is easy to check that $h_1(n; \overline{\kappa}_k \geq \ell + 1) = f(n; \overline{\kappa}_k \leq \ell) + 1$ for $0 \leq \ell \leq n - \lceil k/2 \rceil - 1$. In [21], we determined that $f(n; \overline{\kappa}_3 \leq 2) = 2n - 3$ for $n \geq 3$ and $n \neq 4$, and $f(n; \overline{\kappa}_3 \leq 2) = 2n - 2$ for $n = 4$. Furthermore, we characterized graphs attaining these values. For a general ℓ, we constructed graphs to show that $f(n; \overline{\kappa}_3 \leq \ell) \geq \frac{\ell+2}{2}(n-2) + \frac{1}{2}$ for both n and k odd, and $f(n; \overline{\kappa}_3 \leq \ell) \geq \frac{\ell+2}{2}(n-2) + 1$ otherwise.

We continue to study the above problems in this paper. The edge version of these problems are also introduced and investigated. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local edge-connectivity* $\lambda(S)$ is the maximum number of edge-disjoint trees connecting S in G. For an integer k with $2 \leq k \leq n$, the *generalized k-edge-connectivity* [27] is defined as $\lambda_k(G) = \min\{\lambda(S)|S \subseteq V(G), |S| = k\}$. The parameter $\overline{\lambda}_k(G) = \max\{\lambda(S)|S \subseteq V(G), |S| = k\}$ is called the *maximum generalized local edge-connectivity* of G. Similarly, $g(n; \overline{\lambda}_k \leq \ell) = \max\{e(G)||V(G)| = n \textit{ and } \overline{\lambda}_k(G) \leq \ell\}$, and $h_2(n; \overline{\lambda}_k \geq r)$ is the smallest number of edges, $h_2(n; \overline{\lambda}_k \geq r)$, which guarantees that any graph with n vertices and $h_2(n; \overline{\lambda}_k \geq r)$ edges will contain a set S of k vertices such that there are r edge-disjoint S-trees. Similarly, $h_2(n; \overline{\lambda}_k \geq \ell + 1) = g(n; \overline{\lambda}_k \leq \ell) + 1$ for $0 \leq \ell \leq n - \lceil k/2 \rceil - 1$.

The following result, due to Nash-Williams and Tutte, will be used later.

Theorem 1. *(Nash-Williams [32],Tutte [37]) A multigraph G contains a system of ℓ edge-disjoint spanning trees if and only if*

$$\|G/\mathscr{P}\| \geq \ell(|\mathscr{P}| - 1)$$

holds for every partition $\mathscr{P}$ of $V(G)$, where $\|G/\mathscr{P}\|$ denotes the number of edges in G between distinct blocks of $\mathscr{P}$.

The following corollary can be easily derived from Theorem 1.

Corollary 1. *Every 2ℓ-edge-connected graph contains a system of ℓ edge-disjoint spanning trees.*

A subset $S \subseteq V(G)$ is called *ℓ-edge-connected*, if $\lambda_G(x, y) \geq \ell$ for all $x \neq y$ in S. Kriesell [15] conjectured that this Corollary 1 can be generalized for Steiner trees.

Conjecture 1. *(Kriesell [15]) If a set S of vertices of G is 2ℓ-edge-connected, then there is a set of ℓ edge-disjoint Steiner trees connecting S in G.*

This conjecture has obtained wide attention and many results have been worked out; see [14, 15, 16, 17, 38].

With the help of Theorem 1, we determine the exact value of $f(n; \overline{\kappa}_k \leq \ell)$ and $g(n; \overline{\lambda}_k \leq \ell)$ for $k = n, n-1$. The graphs attaining these values are also characterized. It is not easy to solve these problems for a general k $(3 \leq k \leq n)$. So we construct a graph class to give them a sharp lower bound for a general k $(3 \leq k \leq n-2)$.

To start with, the following two observations are easily seen.

Observation 1. *Let G be a connected graph of order n. Then*

(1) $\kappa_k(G) \leq \lambda_k(G)$ *and* $\overline{\kappa}_k(G) \leq \overline{\lambda}_k(G)$;

(2) $\kappa_k(G) \leq \overline{\kappa}_k(G)$ *and* $\lambda_k(G) \leq \overline{\lambda}_k(G)$.

Observation 2. *If H is a spanning subgraph of G of order n, then $\kappa_k(H) \leq \kappa_k(G)$, $\lambda_k(H) \leq \lambda_k(G)$, $\overline{\kappa}_k(H) \leq \overline{\kappa}_k(G)$ and $\overline{\lambda}_k(H) \leq \overline{\lambda}_k(G)$.*

In [27], we obtained the exact value of $\lambda_k(K_n)$.

Lemma 1. *[27] Let k, n be two integers with $3 \leq k \leq n$. Then*

$$\lambda_k(K_n) = n - \lceil k/2 \rceil$$

From Lemma 1, we can derive sharp bounds of $\overline{\lambda}_k(G)$.

Observation 3. *Let k, n be two integers with $3 \leq k \leq n$, and let G be a connected graph G of order n. Then $1 \leq \overline{\lambda}_k(G) \leq n - \lceil k/2 \rceil$. Moreover, the upper and lower bounds are sharp.*

Proof. From the definitions of $\overline{\lambda}_k(G)$ and $\lambda_k(G)$ and the symmetry of a complete graph, $\overline{\lambda}_k(K_n) = \lambda_k(K_n) = n - \lceil \frac{k}{2} \rceil$. So for a connected graph G of order n it follows that $\overline{\lambda}_k(G) \leq \overline{\lambda}_k(K_n) = n - \lceil \frac{k}{2} \rceil$. Since G is connected, $\overline{\lambda}_k(G) \geq 1$. So $1 \leq \overline{\lambda}_k(G) \leq n - \lceil \frac{k}{2} \rceil$. $\square$

One can easily check that the complete K_n attains the upper bound and any tree T of order n attains the lower bound. Combining Observation 3 with (1) of Observation 1, the following observation is immediate.

Observation 4. *Let k, n be two integers with $3 \leq k \leq n$, and let G be a connected graph G of order n. Then $1 \leq \overline{\kappa}_k(G) \leq n - \lceil k/2 \rceil$. Moreover, the upper and lower bounds are sharp.*

2 The case $k = n$

In this section, we determine the exact value of $g(n; \overline{\lambda}_k \leq \ell)$ for the case $k = n$. This is also a preparation for the next section. From Observation 3, $1 \leq \overline{\lambda}_n(G) \leq \lfloor \frac{n}{2} \rfloor$. In order to make the parameter $g(n; \overline{\lambda}_n \leq \ell)$ to be meaningful, we assume that $1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor$. Let us focus on the case $1 \leq \ell \leq \lfloor \frac{n-4}{2} \rfloor$ and begin with a lemma derived from Theorem 1.

Lemma 2. *Let G be a connected graph of order n ($n \geq 5$). If $e(G) \geq \binom{n-1}{2} + \ell$ ($1 \leq \ell \leq \lfloor \frac{n-4}{2} \rfloor$) and $\delta(G) \geq \ell + 1$, then G contains $\ell + 1$ edge-disjoint spanning trees.*

Proof. Let $\mathscr{P} = \bigcup_{i=1}^{p} V_i$ be a partition of $V(G)$ with $|V_i| = n_i$ ($1 \leq i \leq p$), and $\mathcal{E}_p$ be the set of edges between distinct blocks of $\mathscr{P}$ in G. It suffices to show $|\mathcal{E}_p| \geq (\ell + 1)(p - 1)$ so that we can use Theorem 1.

The case $p = 1$ is trivial, thus we assume $p \geq 2$. For $p = 2$, we have $\mathscr{P} = V_1 \cup V_2$. Set $|V_1| = n_1$. Then $|V_2| = n - n_1$. If $n_1 = 1$ or $n_1 = n-1$, then $|\mathcal{E}_2| = |E_G[V_1, V_2]| \geq \ell+1$ since $\delta(G) \geq \ell+1$. Suppose $2 \leq n_1 \leq n-2$. Then $|\mathcal{E}_2| = |E_G[V_1, V_2]| \geq \binom{n-1}{2} + \ell - \binom{n_1}{2} - \binom{n-n_1}{2} = -n_1^2 + nn_1 + \ell - (n-1)$. Since $2 \leq n_1 \leq n - 2$, one can see that $|\mathcal{E}_2|$ attains its minimum value when $n_1 = 2$ or $n_1 = n - 2$. Thus $|\mathcal{E}_2| \geq n - 3 + \ell \geq \ell + 1$. So the conclusion is true for $p = 2$ by Theorem 1.

Consider the case $p = n$. To have $|\mathcal{E}_n| \geq (\ell + 1)(n - 1)$, we must have $\binom{n-1}{2} + \ell \geq (\ell + 1)(n - 1)$, that is, $(n - 2\ell - 3)(n - 2) \geq 2$. Since $\ell \leq \lfloor \frac{n-4}{2} \rfloor$, this inequality holds. The case $p = n - 1$ can be proved similarly. Since $|\mathcal{E}_{n-1}| \geq \binom{n-1}{2} + \ell - 1$, we need the inequality $\frac{(n-1)(n-2)}{2} + \ell - 1 \geq (\ell + 1)(n - 2)$, that is, $(n - 2\ell - 3)(n - 3) + (n - 5) \geq 0$. Since $\ell \leq \lfloor \frac{n-4}{2} \rfloor$ and $n \geq 5$, this inequality holds.

Let us consider the remaining case p for $3 \leq p \leq n - 2$. Clearly, $|\mathcal{E}_p| \geq e(G) - \sum_{i=1}^{p} \binom{n_i}{2} \geq \binom{n-1}{2} + \ell - \sum_{i=1}^{p} \binom{n_i}{2}$. We will show that $\binom{n-1}{2} + \ell - \sum_{i=1}^{p} \binom{n_i}{2} \geq (\ell+1)(p-1)$, that is, $\binom{n-1}{2} + \ell - (\ell + 1)(p - 1) \geq \sum_{i=1}^{p} \binom{n_i}{2}$. Actually, we only need to prove that $\frac{(n-1)(n-2)}{2} - (\ell + 1)(p - 2) - 1 \geq \max\{\sum_{i=1}^{p} \binom{n_i}{2}\}$. Since $f(n_1, n_2, \cdots, n_p) = \sum_{i=1}^{p} \binom{n_i}{2}$ achieves its maximum value when $n_1 = n_2 = \cdots = n_{p-1} = 1$ and $n_p = n - p + 1$, we need the inequality $\frac{(n-1)(n-2)}{2} - (\ell + 1)(p - 2) - 1 \geq \binom{1}{2}(p - 1) + \binom{n-p+1}{2}$, that is,

$(n-1)(n-2)-2(\ell+1)(p-2)-2 \geq (n-p+1)(n-p)$. Thus this inequality is equivalent to $(p-2)(2n-p-2\ell-3) \geq 2$. Since $1 \leq \ell \leq \lfloor \frac{n-4}{2} \rfloor$ and $3 \leq p \leq n-2$, one can see that the inequality holds. Thus, $|\mathcal{E}_p| \geq (\ell+1)(p-1)$. From Theorem 1, we know that there exist $\ell+1$ edge-disjoint spanning trees, as desired. $\square$

In [27], the graphs with $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil$ and $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$ were characterized, respectively.

Lemma 3. *[27] Let k, n be two integers with $3 \leq k \leq n$, and let G be a connected graph G of order n. Then $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil$ or $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$ if and only if $G = K_n$ for k even; $G = K_n \setminus M$ for k odd, where M is an edge set such that $0 \leq |M| \leq \frac{k-1}{2}$.*

Note that $\kappa_n(G) = \lambda_n(G) = \overline{\kappa}_n(G) = \overline{\lambda}_n(G)$. From the above lemma, we can derive the following corollary.

Corollary 2. *For a connected graph G of order n, $\kappa_n(G) = \overline{\kappa}_n(G) = \lambda_n(G) = \overline{\lambda}_n(G) = \lfloor \frac{n}{2} \rfloor$ if and only if $G = K_n$ for n even; $G = K_n \setminus M$ for n odd, where M is an edge set such that $0 \leq |M| \leq \frac{n-1}{2}$.*

Let $\mathcal{G}_n$ be a graph class obtained from a complete graph K_{n-1} by adding a vertex v and joining v to ℓ vertices of K_{n-1}.

Theorem 2. *Let G be a connected graph of order n ($n \geq 6$). If $\overline{\lambda}_n(G) \leq \ell$ ($1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor$), then*

$$e(G) \leq \begin{cases} \binom{n-1}{2} + \ell, & if\ 1 \leq \ell \leq \lfloor \frac{n-4}{2} \rfloor; \\ \binom{n-1}{2} + n - 2, & if\ \ell = \lfloor \frac{n-2}{2} \rfloor \text{ and } n \text{ is even}; \\ \binom{n-1}{2} + \frac{n-3}{2}, & if\ \ell = \lfloor \frac{n-2}{2} \rfloor \text{ and } n \text{ is odd}; \\ \binom{n}{2}, & if\ \ell = \lfloor \frac{n}{2} \rfloor. \end{cases}$$

with equality if and only if $G \in \mathcal{G}_n$ for $1 \leq \ell \leq \lfloor \frac{n-4}{2} \rfloor$; $G = K_n \setminus e$ where $e \in E(K_n)$ for $\ell = \lfloor \frac{n-2}{2} \rfloor$ and n even; $G = K_n \setminus M$ where $M \subseteq E(K_n)$ and $|M| = \frac{n+1}{2}$ for $\ell = \lfloor \frac{n-2}{2} \rfloor$ and n odd; $G = K_n$ for $\ell = \lfloor \frac{n}{2} \rfloor$.

Proof. For $1 \leq \ell \leq \lfloor \frac{n-4}{2} \rfloor$, if $e(G) \geq \binom{n-1}{2} + (\ell+1)$, then $\delta(G) \geq \ell+1$. From Lemma 2, $\overline{\lambda}_n(G) \geq \ell+1$, which contradicts to $\overline{\lambda}_n(G) \leq \ell$. So $e(G) \leq \binom{n-1}{2} + \ell$ for $1 \leq \ell \leq \lfloor \frac{n-4}{2} \rfloor$. For $\ell = \lfloor \frac{n-2}{2} \rfloor$ and n even, $e(G) \leq \binom{n-1}{2} + n - 2$ by Corollary 2. By the same reason, $e(G) \leq \binom{n-1}{2} + \frac{n-3}{2}$ for $\ell = \lfloor \frac{n-2}{2} \rfloor$ and n odd. If $\ell = \lfloor \frac{n}{2} \rfloor$, then for any connected graph G $\overline{\lambda}_n(G) \leq \ell$ by Observation 3. So $e(G) \leq \binom{n}{2}$.

Now we characterize the graphs attaining the upper bounds. Consider the case $1 \leq \ell \leq \lfloor \frac{n-4}{2} \rfloor$. Suppose that G is a connected graph such that $e(G) = \binom{n-1}{2} + \ell$. Clearly,

$\delta(G) \geq \ell$. Assume $\delta(G) \geq \ell+1$. Since $e(G) = \binom{n-1}{2} + \ell$, G contains $\ell+1$ edge-disjoint spanning trees by Lemma 2, namely, $\overline{\lambda}_n(G) \geq \ell+1$, a contradiction. So $\delta(G) = \ell$, and hence there exists a vertex v such that $d_G(v) = \ell$. Clearly, $e(G-v) = \binom{n-1}{2}$. Thus $G-v$ is a clique of order $n-1$. Therefore, $G \in \mathcal{G}_n$. For n even and $\ell = \lfloor\frac{n-2}{2}\rfloor$, let $e(G) = \binom{n-1}{2} + n-2$. Obviously, $G = K_n \setminus e$, where $e \in E(K_n)$. For n odd and $\ell = \lfloor\frac{n-2}{2}\rfloor$, let $e(G) = \binom{n-1}{2} + \frac{n-3}{2}$. Clearly, $G = K_n \setminus M$, where $M \subseteq E(K_n)$ and $|M| = \frac{n+1}{2}$. For $\ell = \lfloor\frac{n}{2}\rfloor$, if $e(G) = \binom{n}{2}$, then $G = K_n$. □

Corollary 3. *For $1 \leq \ell \leq \lfloor\frac{n}{2}\rfloor$ and $n \geq 6$,*

$$f(n;\overline{\kappa}_n \leq \ell) = g(n;\overline{\lambda}_n \leq \ell) = \begin{cases} \binom{n-1}{2} + \ell, & if\ 1 \leq \ell \leq \lfloor\frac{n-4}{2}\rfloor \ or\ \ell = \lfloor\frac{n-2}{2}\rfloor \ and\ n\ is\ odd; \\ \binom{n-1}{2} + 2\ell, & if\ \ell = \lfloor\frac{n-2}{2}\rfloor \ and\ n\ is\ even; \\ \binom{n}{2}, & if\ \ell = \lfloor\frac{n}{2}\rfloor. \end{cases}$$

3 The case $k = n-1$

Before giving our main results, we need some preparations. From Observation 4, we know that $1 \leq \overline{\kappa}_{n-1}(G) \leq \lfloor\frac{n+1}{2}\rfloor$. So we only need to consider $1 \leq \ell \leq \lfloor\frac{n+1}{2}\rfloor$. In order to determine the exact value of $f(n;\overline{\kappa}_{n-1} \leq \ell)$ for a general ℓ ($1 \leq \ell \leq \lfloor\frac{n+1}{2}\rfloor$), we first focus on the cases $\ell = \lfloor\frac{n+1}{2}\rfloor$ and $\lfloor\frac{n-1}{2}\rfloor$. This is also because by characterizing the graphs with $\overline{\kappa}_{n-1}(G) = \lfloor\frac{n+1}{2}\rfloor$ and $\lfloor\frac{n-1}{2}\rfloor$, we can deal with the difficult case $\ell = \lfloor\frac{n-3}{2}\rfloor$.

3.1 The subcases $\ell = \lfloor\frac{n+1}{2}\rfloor$ and $\ell = \lfloor\frac{n-1}{2}\rfloor$

Let us begin this subsection with a useful lemma in [27].

Let $S \subseteq V(G)$ such that $|S| = k$, and $\mathscr{T}$ be a maximum set of edge-disjoint trees in G connecting S. Let $\mathscr{T}_1$ be the set of trees in $\mathscr{T}$ whose edges belong to $E(G[S])$, and $\mathscr{T}_2$ be the set of trees containing at least one edge of $E_G[S,\bar{S}]$, where $\bar{S} = V(G) \setminus S$. Thus, $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$ (Throughout this paper, $\mathscr{T}$, $\mathscr{T}_1$, $\mathscr{T}_2$ are defined in this way).

Lemma 4. *[27] Let $S \subseteq V(G)$, $|S| = k$ and T be a tree connecting S. If $T \in \mathscr{T}_1$, then T uses $k-1$ edges of $E(G[S]) \cup E_G[S,\bar{S}]$; If $T \in \mathscr{T}_2$, then T uses k edges of $E(G[S]) \cup E_G[S,\bar{S}]$.*

The following results can be derived from Lemma 4.

Lemma 5. *Let $G = K_n \setminus M$ be a connected graph of order n ($n \geq 4$), where $M \subseteq E(K_n)$.*

(1) *If n is odd and $|M| \geq 1$, then $\overline{\lambda}_{n-1}(G) < \frac{n+1}{2}$;*

(2) *If n is even and $|M| \geq \frac{n}{2}$, then $\overline{\lambda}_{n-1}(G) < \frac{n}{2}$.*

Proof. (1) For any $S \subseteq V(G)$ such that $|S| = n-1$, obviously, $|\bar{S}| = 1$ and $e \in E(G[S]) \cup E_G[S, \bar{S}]$ for all $e \in E(G)$. Let $|\mathscr{T}_1| = x$ and $|\mathscr{T}| = y$. Then $|\mathscr{T}_2| = y - x$. Clearly, $|\mathscr{T}_1| \leq \lfloor \frac{\binom{n-1}{2}}{n-2} \rfloor = \frac{n-1}{2}$. From Lemma 4, since $(n-2)|\mathscr{T}_1| + (n-1)|\mathscr{T}_2| \leq |E(G[S]) \cup E_G[S, \bar{S}]|$, it follows that $(n-2)x + (n-1)(y-x) \leq \binom{n}{2} - 1$. Then $\lambda(S) = |\mathscr{T}| = y \leq \frac{x}{n-1} + \frac{n}{2} - \frac{1}{n-1} \leq \frac{n+1}{2} - \frac{1}{n-1} < \frac{n+1}{2}$. So $\overline{\lambda}_{n-1}(G) < \frac{n+1}{2}$.

(2) In this case, for any $S \subseteq V(G)$ such that $|S| = n-1$, we have $|\bar{S}| = 1$ and $e \in E(G[S]) \cup E_G[S, \bar{S}]$ for all $e \in E(G)$. Let $|\mathscr{T}_1| = x$ and $|\mathscr{T}| = y$. Then $|\mathscr{T}_2| = y - x$. Clearly, $|\mathscr{T}_1| \leq \lfloor \frac{\binom{n-1}{2}}{n-2} \rfloor = \lfloor \frac{n-1}{2} \rfloor = \frac{n-2}{2}$. From Lemma 4, since $(n-2)|\mathscr{T}_1| + (n-1)|\mathscr{T}_2| \leq |E(G[S]) \cup E_G[S, \bar{S}]|$, it follows that $(n-2)x + (n-1)(y-x) \leq \binom{n}{2} - \frac{n}{2}$. Then $\lambda(S) = |\mathscr{T}| = y \leq \frac{x}{n-1} + \frac{n}{2} - \frac{n}{2(n-1)} \leq \frac{n}{2} - \frac{1}{n-1} < \frac{n}{2}$. So $\overline{\lambda}_{n-1}(G) < \frac{n}{2}$. □

With the help of Lemmas 3 and 5 and Observation 1, the graphs with $\overline{\kappa}_{n-1}(G) = \lfloor \frac{n+1}{2} \rfloor$ can be characterized now.

Proposition 1. *For a connected graph G of order n ($n \geq 4$), $\overline{\kappa}_{n-1}(G) = \lfloor \frac{n+1}{2} \rfloor$ if and only if $G = K_n$ for n odd; $G = K_n \setminus M$ for n even, where M is an edge set such that $0 \leq |M| \leq \frac{n-2}{2}$.*

Proof. Consider the case n odd. Suppose that G is a connected graph such that $\overline{\kappa}_{n-1}(G) = \frac{n+1}{2}$. In fact, the complete graph K_n is a unique graph attaining this value. Let $G = K_n \setminus e$ where $e \in E(K_n)$. From (1) of Lemma 5 and Observation 1, $\overline{\kappa}_{n-1}(G) \leq \overline{\lambda}_{n-1}(G) < \frac{n+1}{2}$, a contradiction. So $G = K_n$. Conversely, if $G = K_n$, then $\overline{\kappa}_{n-1}(G) \geq \kappa_{n-1}(G) = \frac{n+1}{2}$ by Lemma 3. Combining this with Observation 4, we have $\overline{\kappa}_{n-1}(G) = \frac{n+1}{2}$.

Now consider the case n even. Suppose that G is a connected graph such that $\overline{\kappa}_{n-1}(G) = \frac{n}{2}$. If $G = K_n \setminus M$ such that $|M| \geq \frac{n}{2}$, then $\overline{\kappa}_{n-1}(G) \leq \overline{\lambda}_{n-1}(G) < \frac{n}{2}$ by (2) of Lemma 5, a contradiction. So $G = K_n \setminus M$, where $0 \leq |M| \leq \frac{n-2}{2}$. Conversely, if $G = K_n \setminus M$ such that $0 \leq |M| \leq \frac{n-2}{2}$, then $\overline{\kappa}_{n-1}(G) \geq \kappa_{n-1}(G) = \frac{n}{2}$ by Lemma 3. From this together with Observation 4, we have $\overline{\kappa}_{n-1}(G) = \frac{n}{2}$. □

Furthermore, graphs with $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n+1}{2} \rfloor$ can also be characterized.

Proposition 2. *For a connected graph G of order n ($n \geq 4$), $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n+1}{2} \rfloor$ if and only if $G = K_n$ for n odd; $G = K_n \setminus M$ for n even, where M is an edge set such that $0 \leq |M| \leq \frac{n-2}{2}$.*

Proof. Assume that G is a connected graph satisfying the conditions of Proposition 2. From Observation 1 and Proposition 1, it follows that $\overline{\lambda}_{n-1}(G) \geq \overline{\kappa}_{n-1}(G) = \lfloor \frac{n+1}{2} \rfloor$. Combining this with Observation 3, $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n+1}{2} \rfloor$. Conversely, suppose $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n+1}{2} \rfloor$. For n odd, if $G = K_n \setminus e$ where $e \in E(K_n)$, then $\overline{\kappa}_{n-1}(G) \leq \overline{\lambda}_{n-1}(G) < \frac{n+1}{2}$ by

(1) of Lemma 5. So the complete graph K_n is a unique graph attaining this value. For n even, if $G = K_n \setminus M$ where $M \in E(K_n)$ such that $|M| \geq \frac{n}{2}$, then $\overline{\lambda}_{n-1}(G) < \lfloor \frac{n+1}{2} \rfloor$ by (2) of Lemma 5. So $G = K_n \setminus M$, where $0 \leq |M| \leq \frac{n-2}{2}$. $\square$

We now focus our attention on the case $\ell = \lfloor \frac{n-1}{2} \rfloor$. Before characterizing the graphs with $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n-1}{2} \rfloor$, we need the following four lemmas. The notion of a second minimal degree vertex in a graph G will be used in the sequel. If G has two or more minimum degree vertices, then, choosing one of them as the first minimum degree vertex, a *second minimal degree vertex* is defined as any one of the rest minimum degree vertices of G. If G has only one minimum degree vertex, then a *second minimal degree vertex* is as its name, defined as any one of vertices that have the second minimal degree. Note that a second minimal degree vertex is usually not unique.

Lemma 6. *Let $G = K_n \setminus M$ be a connected graph of order n, where $M \subseteq E(K_n)$.*

(1) *If n $(n \geq 10)$ is even and $|M| \geq \frac{3n-4}{2}$, then $\overline{\lambda}_{n-1}(G) < \frac{n-1}{2}$;*

(2) *If n $(n \geq 10)$ is even, $n+1 \leq |M| \leq \frac{3n-6}{2}$ and there is a second minimal degree vertex, say u_1, such that $d_G(u_1) \leq \frac{n-4}{2}$, then $\overline{\lambda}_{n-1}(G) < \frac{n-2}{2}$;*

(3) *If n $(n \geq 8)$ is odd and $|M| \geq n-1$, then $\overline{\lambda}_{n-1}(G) < \frac{n-1}{2}$.*

Proof. (1) For any $S \subseteq V(G)$ such that $|S| = n-1$, obviously, $|\bar{S}| = 1$ and $e \in E(G[S]) \cup E_G[S, \bar{S}]$ for all $e \in E(G)$. Set $S = V(G) \setminus v$ where $v \in V(G)$. Since G is connected graph, it follows that $d_G(v) \geq 1$ and hence $d_{K_n[M]}(v) \leq n-2$. So $|M \cap K_n[S]| \geq \frac{3n-4}{2} - (n-2) = \frac{n}{2}$ and $|E(G[S])| \leq \binom{n-1}{2} - \frac{n}{2}$. Therefore, $|\mathscr{T}_1| \leq \frac{\binom{n-1}{2} - \frac{n}{2}}{n-2} = \frac{n-2}{2} - \frac{1}{n-2} < \frac{n-2}{2}$, namely, $|\mathscr{T}_1| \leq \frac{n-4}{2}$. Let $|\mathscr{T}_1| = x$ and $|\mathscr{T}| = y$. Then $|\mathscr{T}_2| = y - x$. Since $(n-2)|\mathscr{T}_1| + (n-1)|\mathscr{T}_2| \leq |E(G[S]) \cup E_G[S, \bar{S}]|$, it follows that $(n-2)x + (n-1)(y-x) \leq \binom{n}{2} - \frac{3n-4}{2}$. Then $\lambda(S) = |\mathscr{T}| = y \leq \frac{x}{n-1} + \frac{n}{2} - \frac{3n-4}{2(n-1)} \leq \frac{n-2}{2} - \frac{1}{n-1} < \frac{n-2}{2}$. So $\overline{\lambda}_{n-1}(G) < \frac{n-2}{2}$.

(2) Let v be a vertex such that $d_G(v) = \delta(G)$. Then $d_G(v) \leq d_G(u_1) \leq \frac{n-4}{2}$. For any $S \subseteq V(G)$ with $|S| = n-1$, at least one of u_1, v belongs to S, say $u_1 \in S$. Hence $\lambda(S) \leq d_G(u_1) \leq \frac{n-4}{2} < \frac{n-2}{2}$. So $\overline{\lambda}_{n-1}(G) < \frac{n-2}{2}$.

(3) The proof of (3) is similar to that of (1), and thus omitted. $\square$

Lemma 7. *Let H be a connected graph of order $n-1$.*

(1) *If n $(n \geq 5)$ is odd, $e(H) \geq \binom{n-2}{2}$, $\delta(H) \geq \frac{n-3}{2}$ and any two vertices of degree $\frac{n-3}{2}$ are nonadjacent, then H contains $\frac{n-3}{2}$ edge-disjoint spanning trees.*

(2) *If n $(n \geq 7)$ is even, $e(H) \geq \binom{n-2}{2} - \frac{n-2}{2}$, $\delta(H) \geq \frac{n-4}{2}$ and any two vertices of degree $\frac{n-4}{2}$ are nonadjacent, then H contains $\frac{n-4}{2}$ edge-disjoint spanning trees.*

Proof. We only give the proof of (1), (2) can be proved similarly. Let $\mathscr{P} = \bigcup_{i=1}^{p} V_i$ be a partition of $V(H)$ with $|V_i| = n_i$ $(1 \leq i \leq p)$, and $\mathcal{E}_p$ be the set of edges between distinct blocks of $\mathscr{P}$ in H. It suffices to show $|\mathcal{E}_p| \geq \frac{n-3}{2}(|\mathscr{P}|-1)$ so that we can use Theorem 1.

The case $p = 1$ is trivial, thus we assume $p \geq 2$. For $p = 2$, we have $\mathscr{P} = V_1 \cup V_2$. Set $|V_1| = n_1$. Then $|V_2| = n - 1 - n_1$. If $n_1 = 1$ or $n_1 = n-2$, then $|\mathcal{E}_2| = |E_G[V_1, V_2]| \geq \frac{n-3}{2}$ since $\delta(H) \geq \frac{n-3}{2}$. Suppose $2 \leq n_1 \leq n-3$. Clearly, $|\mathcal{E}_2| = |E_G[V_1, V_2]| \geq \binom{n-2}{2} - \binom{n_1}{2} - \binom{n-1-n_1}{2} = -n_1^2 + (n-1)n_1 - (n-2)$. Since $2 \leq n_1 \leq n-3$, one can see that $|\mathcal{E}_2|$ attains its minimum value when $n_1 = 2$ or $n_1 = n-3$. Thus $|\mathcal{E}_2| \geq n - 4 \geq \frac{n-3}{2}$ since $n \geq 5$. So the conclusion holds for $p = 2$ by Theorem 1.

Now consider the remaining case p with $3 \leq p \leq n-1$. Since $|\mathcal{E}_p| \geq e(H) - \sum_{i=1}^{p}\binom{n_i}{2} \geq \binom{n-2}{2} - \sum_{i=1}^{p}\binom{n_i}{2}$, we need to show that $\binom{n-2}{2} - \sum_{i=1}^{p}\binom{n_i}{2} \geq \frac{n-3}{2}(p-1)$, that is, $\binom{n-2}{2} - \frac{n-3}{2}(p-1) \geq \sum_{i=1}^{p}\binom{n_i}{2}$. Furthermore, we only need to prove that $\binom{n-2}{2} - \frac{n-3}{2}(p-1) \geq max\{\sum_{i=1}^{p}\binom{n_i}{2}\}$. Since $f(n_1, n_2, \cdots, n_p) = \sum_{i=1}^{p}\binom{n_i}{2}$ attains its maximum value when $n_1 = n_2 = \cdots = n_{p-1} = 1$ and $n_p = n - p$, we need the inequality $\binom{n-2}{2} - \frac{n-3}{2}(p-1) \geq \binom{1}{2}(p-1) + \binom{n-p}{2}$, that is, $(p-3)(n-p-1) \geq 0$. Since $3 \leq p \leq n-1$, one can see that the inequality holds. Thus, $|\mathcal{E}_p| \geq \frac{n-3}{2}(p-1)$. From Theorem 1, there exist $\frac{n-3}{2}$ edge-disjoint spanning trees, as desired. □

The following theorem, due to Dirac, is well-known.

Theorem 3. *[7](p-485) Let G be a simple graph of order n $(n \geq 3)$ and minimum degree δ. If $\delta \geq \frac{n}{2}$, then G is Hamiltonian.*

Lemma 8. *If n $(n \geq 8)$ is odd and $G = K_n \setminus M$ such that $|M| = n-2$, then $\overline{\kappa}_{n-1}(G) \geq \frac{n-1}{2}$.*

Proof. Clearly, $e(G) = \binom{n-1}{2} + 1$. Let $\delta(G) = r$ and v be a vertex such that $d_G(v) = \delta(G) = r$. Choose $S = V(G) \setminus v$. Then $|S| = n-1$. We distinguish the following two cases to show this lemma.

Case 1. $1 \leq \delta(G) \leq \frac{n-1}{2}$.

If $\delta(G) = r = 1$, then $e(G-v) = \binom{n-1}{2}$, which implies that $G - v$ is a clique of order $n-1$. Obviously, $G - v$ contains $\frac{n-1}{2}$ edge-disjoint spanning trees connecting S, namely, $\overline{\kappa}_{n-1}(G-v) \geq \frac{n-1}{2}$. Therefore, $\overline{\kappa}_{n-1}(G) \geq \frac{n-1}{2}$, as desired.

Suppose $\delta(G) = r \geq 2$. Since $d_G(v) \leq \frac{n-1}{2}$, it follows that $d_{K_n[M]}(v) \geq n - 1 - \frac{n-1}{2} = \frac{n-1}{2}$. Combining this with $|M| = n-2$, we have $|M \cap E(K_n[S])| \leq n - 2 - \frac{n-1}{2} \leq \frac{n-3}{2}$, namely, $G[S]$ is a graph obtained from a clique of order $n-1$ by deleting at most $\frac{n-3}{2}$ edges. So $\delta(G[S]) \geq n - 2 - \frac{n-3}{2} = \frac{n-1}{2}$. We claim that there exists at most one vertex in $G[S]$ such that its degree is $\frac{n-1}{2}$ or $\frac{n+1}{2}$. Assume, to the contrary, that there exist two vertices in S, say u_1, u_2, such that $d_{G[S]}(u_j) \leq \frac{n+1}{2}$ $(j = 1, 2)$. Then $d_G(u_j) \leq \frac{n+3}{2}$, and hence $d_{K_n[M]}(u_j) \geq n-1-\frac{n+3}{2} = \frac{n-5}{2}$. Therefore, $|M| \geq d_{K_n[M]}(v) + d_{K_n[M]}(u_1) + d_{K_n[M]}(u_2) \geq \frac{n-1}{2} + 2 \cdot \frac{n-5}{2} = \frac{3n-11}{2} > n-2$, a contradiction. So we conclude that there exists at most one vertex in $G[S]$ such that its degree is $\frac{n-1}{2}$ or $\frac{n+1}{2}$. Since $\delta(G[S]) \geq \frac{n-1}{2}$, from Theorem 3 $G[S]$ is Hamiltonian and hence $G[S]$ contains a Hamilton cycle, say C. Let $S = \{u_1, u_2, \cdots, u_{n-1}\}$ such that $vu_j \in E(G)$ $(1 \leq j \leq r)$. Clearly, $vu_j \in M$ $(r+1 \leq j \leq n-1)$. Then the vertices $u_1, u_2, \cdots, u_r$ divide the cycle C into r paths, say $P_1, P_2, \cdots, P_r$; see Figure 1 (a).

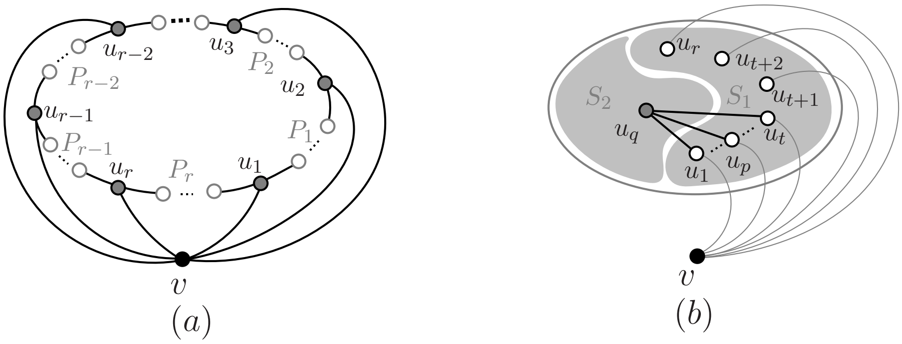

Figure 1. Graphs for Lemmas 8 and 9.

Now we find a Steiner tree connecting S with its root v in G, say T, such that $G_1[S]$ satisfies the conditions of (1) in Lemma 7, where $G_1 = G \setminus E(T)$. If there exists a vertex $u_s \in S$ of degree $\frac{n-1}{2}$ in $G[S]$ such that $u_s \in \{u_1, u_2, \cdots, u_r\}$ and $e(P_{s-1}) = e(P_s) = 1$, then there exists a vertex $u_t \in \{u_{r+1}, u_{r+2}, \cdots, u_{n-1}\}$ such that $u_s u_t \in E(G[S])$ since $d_{G[S]}(u_s) = \frac{n-1}{2}$ and $r \leq \frac{n-1}{2}$. Then u_t is an internal vertex of some path, without loss of generality, let $u_t \in V(P_q)$ $(1 \leq q \leq r, q \neq s-1, s)$. For each path P_i $(1 \leq i \leq r)$, we choose one edge $e_i \in E(P_i)$ $(1 \leq i \leq r)$ to delete. Since u_t is an internal vertex of P_q, it follows that, after deleting the edge e_q from P_q, there exists an edge e'_q in P_q that is incident with u_t such that e_q and e'_q lie in different sides of u_t in P_q. Then the tree $T = (vu_1 \cup vu_2 \cup \cdots vu_r \cup (P_1 \setminus e_1) \cup (P_2 \setminus e_2) \cdots (P_r \setminus e_r) \cup u_s u_t) \setminus e'_q$ is our desired tree. Set $G_1 = G \setminus E(T)$. Observe that $\delta(G_1[S]) \geq \frac{n-3}{2}$ and there is at most one vertex of degree $\frac{n-3}{2}$ in $G_1[S]$. Combining this with $e(G_1[S]) = e(G) - (n-1) = \binom{n-1}{2} - (n-2) = \binom{n-2}{2}$, $G_1[S]$ contains $\frac{n-3}{2}$ spanning trees by (1) of Lemma 7. These trees together with the tree T are $\frac{n-1}{2}$ internally disjoint trees connecting S, namely, $\overline{\kappa}_{n-1}(G) \geq \frac{n-1}{2}$, as desired.

Except the above case, we also have the following five cases to consider. For each case, we choose one edge $e_i \in E(P_i)$ $(1 \leq i \leq r)$ to delete that satisfies the following conditions:

❶ if there is no vertex of degree $\frac{n-1}{2}$ in $G[S]$, then e_i $(1 \leq i \leq r)$ is chosen as any edge in P_i;

❷ if there exists a vertex u_s of degree $\frac{n-1}{2}$ in $G[S]$ such that $u_s \in \{u_{r+1}, u_{r+2}, \cdots, u_{n-1}\}$, then e_i $(1 \leq i \leq r)$ is chosen as any edge in P_i;

❸ if there exists a vertex u_s of degree $\frac{n-1}{2}$ in $G[S]$ such that $u_s \in \{u_1, u_2, \cdots, u_r\}$, $e(P_{s-1}) \geq 2$ and $e(P_s) \geq 2$, then e_{s-1} is the edge that is incident with u_{s-1}, e_s is the edge that is incident with u_{s+1}, and e_i $(1 \leq i \leq r, i \neq s-1, s)$ is chosen as any edge in P_i;

❹ if there exists a vertex u_s of degree $\frac{n-1}{2}$ in $G[S]$ such that $u_s \in \{u_1, u_2, \cdots, u_r\}$, $e(P_{s-1}) \geq 2$ and $e(P_s) = 1$, then e_{s-1} is the edge that is incident with u_{s-1}, and e_i $(1 \leq i \leq r, i \neq s-1)$ is chosen as any edge in P_i;

❺ if there exists a vertex u_s of degree $\frac{n-1}{2}$ in $G[S]$ such that $u_s \in \{u_1, u_2, \cdots, u_r\}$, $e(P_{s-1}) = 1$ and $e(P_s) \geq 2$, then e_s is the edge that is incident with u_{s+1}, and e_i $(1 \leq i \leq r, i \neq s)$ is chosen as any edge in P_i;

Then $T = vu_1 \cup vu_2 \cup \cdots vu_r \cup (P_1 \backslash e_1) \cup (P_2 \backslash e_2) \cdots (P_r \backslash e_r)$ is a Steiner tree connecting S. Set $G_1 = G \backslash E(T)$. One can also check that $\delta(G_1[S]) \geq \frac{n-3}{2}$ and there is at most one vertex of degree $\frac{n-3}{2}$. Combining this with $e(G_1[S]) = e(G) - (n-1) = \binom{n-1}{2} - (n-2) = \binom{n-2}{2}$, $G_1[S]$ contains $\frac{n-3}{2}$ spanning trees by (1) of Lemma 7. These trees together with the tree T are $\frac{n-1}{2}$ internally disjoint trees connecting S, namely, $\overline{\kappa}_{n-1}(G) \geq \frac{n-1}{2}$, as desired.

Case 2. $\frac{n+1}{2} \leq \delta(G) \leq n-1$.

Let $S = V(G) \backslash v = \{u_1, \cdots, u_{n-1}\}$. Without loss of generality, let $S_1 = \{u_1, u_2, \cdots, u_r\}$ such that $vu_j \in E(G)$ $(1 \leq j \leq r)$. Then $\frac{n+1}{2} \leq r \leq n-1$, and $S_2 = S \setminus S_1 = \{u_{r+1}, u_{r+2}, \cdots, u_{n-1}\}$. Since $d_G(v) = \delta(G) \geq \frac{n+1}{2}$, it follows that $|S_1| = r = \delta(G) \geq \frac{n+1}{2}$ and $|S_2| = n-1-r \leq n-1-\frac{n+1}{2} = \frac{n-3}{2}$. For each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$, u_j has at most $\frac{n-5}{2}$ neighbors in S_2 and hence $|E_G[u_j, S_1]| \geq \frac{n+1}{2} - \frac{n-5}{2} = 3$ since $d_G(u_j) \geq \delta(G) \geq \frac{n+1}{2}$. Clearly, the tree $T' = vu_1 \cup vu_2 \cup \cdots \cup vu_r$ is a Steiner tree connecting S_1. Our idea is to seek for $n-1-r$ edges in $E_G[S_1, S_2]$ and combine them with T' to form a Steiner tree connecting S. Choose the one with the smallest subscript among all the vertices of S_2 with the maximum degree in $G[S]$, say u'_1. Then we search for the vertex adjacent to u'_1 with the smallest subscript among all the vertices of S_1 with the maximum degree in $G[S]$, say u''_1. Let $e_1 = u'_1u''_1$. Consider the graph $G_1 = G \setminus e_1$, and pick up the one with the smallest subscript among all the vertices of $S_2 \setminus u'_1$ with the maximum degree in $G_1[S]$, say u'_2. Then we search for the vertex adjacent to u'_2 with the smallest subscript among all the vertices of S_1 with the maximum degree in $G_1[S]$, say u''_2. Set $e_2 = u'_2u''_2$. We consider the graph $G_2 = G_1 \setminus e_1 = G \setminus \{e_1, e_2\}$. Choose the one with the smallest subscript among all the vertices of $S_2 \setminus \{u'_1, u'_2\}$ with the maximum degree in $G_2[S]$, say u'_3, and search for the vertex adjacent to u'_3 with the smallest subscript among all the vertices of S_1 with the maximum degree in $G_2[S]$, say u''_3. Set $e_3 = u'_3u''_3$. We now consider the graph $G_3 = G_2 \setminus e_3 = G \setminus \{e_1, e_2, e_3\}$. For each $u_i \in S_2$ $(r+1 \leq i \leq n-1)$, we proceed to find $e_4, e_5, \cdots, e_{n-1-r}$ in the same way. Let $M' = \{e_1, e_2, \cdots, e_{n-1-r}\}$ and $G_{n-1-r} = G \setminus M'$.

Then $G_{n-1-r}[S] = G[S] \setminus M'$ and the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_r \cup e_1 \cup e_2 \cup \cdots \cup e_{n-1-r}$ is our desired tree. Set $G' = G \setminus E(T)$ (note that $G'[S] = G_{n-1-r}[S]$).

Claim 1. For each $u_j \in S_1$ $(1 \leq j \leq r)$, $d_{G'[S]}(u_j) \geq \frac{n-1}{2}$.

Proof of Claim 1. Assume, to the contrary, that there exists one vertex $u_p \in S_1$ such that $d_{G'[S]}(u_p) \leq \frac{n-3}{2}$. By the above procedure, there exists a vertex $u_q \in S_2$ such that when we pick up the edge $e_i = u_p u_q$ from $G_{i-1}[S]$ the degree of u_p in $G_i[S]$ is equal to $\frac{n-3}{2}$. That is $d_{G_i[S]}(u_p) = \frac{n-3}{2}$ and $d_{G_{i-1}[S]}(u_p) = \frac{n-1}{2}$. From our procedure, $|E_G[u_q, S_1]| = |E_{G_{i-1}}[u_q, S_1]|$. Without loss of generality, let $|E_G[u_q, S_1]| = t$ and $u_q u_j \in E(G)$ for $1 \leq j \leq t$; see Figure 1 (b). Thus $u_p \in \{u_1, u_2, \cdots, u_t\}$. Recall that $|E_G[u_j, S_1]| \geq 3$ for each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$. Since $u_q \in S_2$, we have $t \geq 3$. Clearly, $u_q u_j \notin E(G)$ and hence $u_q u_j \in M$ for $t+1 \leq j \leq r$ by our procedure, namely, $|E_{K_n[M]}[u_q, S_1]| = r - t$. Since $d_{G_{i-1}[S]}(u_p) = \frac{n-1}{2}$, by our procedure $d_{G_{i-1}[S]}(u_j) \leq \frac{n-1}{2}$ for each $u_j \in S_1$ $(1 \leq j \leq t)$. Assume, to the contrary, that there is a vertex u_s $(1 \leq s \leq t)$ such that $d_{G_{i-1}[S]}(u_s) \geq \frac{n+1}{2}$. Then we should choose the edge $u_q u_s$ instead of $e_i = u_q u_p$ by our procedure, a contradiction. We conclude that $d_{G_{i-1}[S]}(u_j) \leq \frac{n-1}{2}$ for each $u_j \in S_1$ $(1 \leq j \leq t)$. Clearly, there are at least $n - 2 - \frac{n-1}{2}$ edges incident to each u_j $(1 \leq j \leq t)$ that belong to $M \cup \{e_1, e_2, \cdots, e_{i-1}\}$. Since $i \leq n - 1 - r$, we have $\sum_{j=1}^{t} d_{K_n[M]}(u_j) \geq (n - 2 - \frac{n-1}{2})t - (i-1) > \frac{n-3}{2}t - (n-1-r)$ and hence $|M| \geq d_{K_n[M]}(v) + \sum_{j=1}^{t} d_{K_n[M]}(u_j) + |E_{K_n[M]}[u_q, S_1]| > (n-1-r) + \frac{n-3}{2}t - (n-1-r) + (r-t) = r + \frac{n-5}{2}t \geq \frac{n+1}{2} + \frac{3(n-5)}{2} = 2n - 7$, which contradicts to $|M| = n - 2$.

From Claim 1, $d_{G'[S]}(u_j) \geq \frac{n-1}{2}$ for each $u_j \in S_1$ $(1 \leq j \leq r)$. For each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$, $d_{G'[S]}(u_j) = d_{G[S]}(u_j) - 1 = d_G(u_j) - 1 \geq \delta(G) - 1 \geq \frac{n-1}{2}$. So $\delta(G'[S]) \geq \frac{n-1}{2}$. Combining this with $e(G'[S]) = e(G) - (n-1) = \binom{n-2}{2}$, $G'[S]$ contains $\frac{n-3}{2}$ spanning trees by (1) of Lemma 7. These trees together with the tree T are $\frac{n-1}{2}$ trees connecting S, namely, $\overline{\kappa}_{n-1}(G) \geq \frac{n-1}{2}$, as desired. □

Lemma 9. *If n $(n \geq 10)$ is even and $G = K_n \setminus M$ such that $|M| = \frac{3n-6}{2}$ and $d_G(u_1) \geq \frac{n-2}{2}$, then $\overline{\kappa}_{n-2}(G) \geq \frac{n-2}{2}$, where u_1 is a second minimal degree vertex in G.*

Proof. It is clear that $e(G) = \binom{n-2}{2} + \frac{n}{2} = \binom{n-1}{2} - \frac{n-4}{2}$. Let $\delta(G) = r$ and v be a vertex such that $d_G(v) = \delta(G) = r$. Let $S = V(G) \setminus v = \{u_1, u_2, \cdots, u_{n-1}\}$. Without loss of generality, let $S_1 = \{u_1, u_2, \cdots, u_r\}$ such that $vu_j \in E(G)$ $(1 \leq j \leq r)$. Then $S_2 = S \setminus S_1 = \{u_{r+1}, u_{r+2}, \cdots, u_{n-1}\}$ such that $vu_j \in M$ $(r+1 \leq j \leq n-1)$. We have the following two cases to consider.

Case 1. $1 \leq \delta(G) \leq \frac{n-2}{2}$.

If $d_G(v) = \delta(G) = 1$, then $e(G - v) = \binom{n-1}{2} - \frac{n-2}{2}$, which implies that $G - v$ is a graph obtained from a clique of order $n - 1$ by deleting $\frac{n-2}{2}$ edges. From Corollary 2,

$\overline{\kappa}_{n-1}(G-v)=\frac{n-2}{2}$. Therefore, $\overline{\kappa}_{n-1}(G)\geq\frac{n-2}{2}$. Suppose $\delta(G)\geq 2$. Since $\delta(G)\leq\frac{n-2}{2}$, it follows that $d_{K_n[M]}(v)\geq n-1-\frac{n-2}{2}=\frac{n}{2}$ and hence $|M\cap K_n[S]|\leq n-3$. Since $d_G(u_1)\geq\frac{n-2}{2}$ where u_1 is a second minimal degree vertex, we have $\delta(G[S])\geq\frac{n-4}{2}$.

First, we consider the case $\delta(G[S])\geq\frac{n}{2}$. We claim that there are at most two vertices of degree $\frac{n}{2}$ in $G[S]$. Assume, to the contrary, that there are three vertices of degree $\frac{n}{2}$ in $G[S]$, say u_1,u_2,u_3. Then $d_G(u_j)\leq\frac{n+2}{2}$ for $j=1,2,3$ and hence $d_{K_n[M]}(u_j)\geq\frac{n-4}{2}$. Therefore, $|M|\geq d_{K_n[M]}(v)+\sum_{j=1}^{3}d_{K_n[M]}(u_j)\geq\frac{n}{2}+3\cdot\frac{n-4}{2}=\frac{4n-12}{2}=2n-6>\frac{3n-6}{2}$, a contradiction. From the above, we conclude that there exist at most two vertices of degree $\frac{n}{2}$ in $G[S]$. Since $\delta(G[S])\geq\frac{n}{2}>\frac{n-1}{2}$, from Theorem 3 $G[S]$ is Hamiltonian and hence $G[S]$ contains a Hamilton cycle, say C. Then the vertices $u_1,u_2,\cdots,u_r$ divide the cycle C into r paths, say $P_1,P_2,\cdots,P_r$; see Figure 1 (a).

Now we find a Steiner tree connecting S with its root v in G, say T, such that $G_1[S]$ satisfies the conditions of (2) in Lemma 7, where $G_1=G\backslash E(T)$. If there exist two adjacent vertices $u_s,u_p\in S$ of degree $\frac{n}{2}$ in $G[S]$ such that $u_s,u_p\in\{u_1,u_2,\cdots,u_r\}$, then $p=s+1$ and $P_s=u_su_{s+1}$ $(1\leq s\leq r-1)$. Since $d_{G[S]}(u_s)=\frac{n}{2}$ and $r\leq\frac{n-2}{2}$, it follows that there exists a vertex $u_t\in\{u_{r+1},u_{r+2},\cdots,u_{n-1}\}$ such that $u_su_t\in E(G)$. It is clear that u_t is an internal vertex of some path, without loss of generality, let $u_t\in V(P_q)$ $(1\leq q\leq r,q\neq s)$. For each path P_i $(1\leq i\leq r)$, we choose one edge $e_i\in E(P_i)$ $(1\leq i\leq r)$ to delete. Since u_t is an internal vertex of P_q, it follows that, after deleting the edge e_q in P_q, there exists an edge e_q' in P_q that is incident with u_t such that e_q and e_q' lie in different sides of u_t in P_q. Then the tree $T=(vu_1\cup vu_2\cup\cdots vu_r\cup(P_1\backslash e_1)\cup(P_2\backslash e_2)\cdots(P_r\backslash e_r)\cup u_su_t)\backslash e_q'$ is our desired tree. Set $G_1=G\setminus E(T)$. Observe that $\delta(G_1[S])\geq\frac{n-4}{2}$ and there is at most one vertex of degree $\frac{n-4}{2}$ in $G_1[S]$. Combining this with $e(G_1[S])=e(G)-(n-1)=\binom{n-2}{2}-\frac{n-2}{2}$, $G_1[S]$ contains $\frac{n-4}{2}$ spanning trees by (2) of Lemma 7. These trees together with the tree T are $\frac{n-2}{2}$ trees connecting S, namely, $\overline{\kappa}_{n-1}(G)\geq\frac{n-2}{2}$, as desired.

Except the above case, we also have the following five cases to consider. For each case, we choose one edge $e_i\in E(P_i)$ $(1\leq i\leq r)$ to delete that satisfies the following conditions:

❶ if there is at most one vertex of degree $\frac{n}{2}$, then e_i $(1\leq i\leq r)$ is chosen as any edge in P_i.

❷ if there exist two adjacent vertices $u_s,u_t\in S$ of degree $\frac{n}{2}$ in $G[S]$ such that $u_s\in\{u_1,u_2,\cdots,u_r\}$ and $u_t\in\{u_{r+1},u_{r+2},\cdots,u_{n-1}\}$, then e_s is the edge that is incident with u_{s+1}, and e_i $(1\leq i\leq r,i\neq s)$ is chosen as any edge in P_i;

❸ if there exist two adjacent vertices $u_s,u_t\in S$ of degree $\frac{n}{2}$ in $G[S]$ such that $u_s\in\{u_{r+1},u_{r+2},\cdots,u_{n-1}\}$ and $u_t\in\{u_1,u_2,\cdots,u_r\}$, then e_{t-1} is the edge that is incident with u_{t-1}, and e_i $(1\leq i\leq r,i\neq t-1)$ is chosen as any edge in P_i;

❹ if there exist two adjacent vertices $u_s,u_t\in S$ of degree $\frac{n}{2}$ in $G[S]$ such that $u_s,u_t\in$

$\{u_{r+1}, u_{r+2}, \cdots, u_{n-1}\}$, then e_i $(1 \leq i \leq r)$ is chosen as any edge in P_i;

❺ if there exist two nonadjacent vertices $u_s, u_t \in S$ of degree $\frac{n-1}{2}$ in $G[S]$, then e_i $(1 \leq i \leq r)$ is chosen as any edge in P_i;

Then $T = vu_1 \cup vu_2 \cup \cdots vu_r \cup (P_1 \setminus e_1) \cup (P_2 \setminus e_2) \cdots (P_r \setminus e_r)$ is a Steiner tree connecting S. Set $G_1 = G \setminus E(T)$. Obviously, $\delta(G_1[S]) \geq \frac{n-4}{2}$ and there is at most one vertex of degree $\frac{n-4}{2}$. Combining this with $e(G_1[S]) = e(G) - (n-1) = \binom{n-2}{2} - \frac{n-2}{2}$, $G_1[S]$ contains $\frac{n-4}{2}$ spanning trees by (2) of Lemma 7. These trees together with the tree T are $\frac{n-2}{2}$ trees connecting S, namely, $\overline{\kappa}_{n-1}(G) \geq \frac{n-2}{2}$, as desired.

Next, we focus on the case that $\delta(G[S]) = \frac{n-2}{2}$ and $\delta(G[S]) = \frac{n-4}{2}$. If $\delta(G[S]) = \frac{n-4}{2}$, then there exists a vertex, say u_1, such that $d_{G[S]}(u_1) = \frac{n-4}{2}$. Since the degree of a second minimal degree vertex is not less than $\frac{n-2}{2}$, we have $u_1 \in S_1$. Thus $d_G(u_1) = \frac{n-2}{2}$ and $u_1 \in S_1$. If $\delta(G[S]) = \frac{n-2}{2}$, then there exists a vertex, say u_1, such that $d_{G[S]}(u_1) = \frac{n-2}{2}$ and $u_1 \in S_1$, or $d_{G[S]}(u_1) = \frac{n-2}{2}$ and $u_1 \in S_2$. Thus $d_G(u_1) = \frac{n}{2}$ and $u_1 \in S_1$, or $d_G(u_1) = \frac{n-2}{2}$ and $u_1 \in S_2$. We only give the proof of the case that $d_G(u_1) = \frac{n}{2}$ and $u_1 \in S_1$. The other two cases can be proved similarly.

Suppose $d_G(u_1) = \frac{n}{2}$ and $u_1 \in S_1$. Similar to the proof of Lemma 8, we want to find out a tree connecting S with root v, say T. Let $G_1 = G \setminus E(T)$. We hope that the graph $G_1[S]$ satisfies the conditions of (2) in Lemma 7. Then there are $\frac{n-4}{2}$ spanning trees connecting S in $G_1[S]$, and these trees together with the tree T are $\frac{n-2}{2}$ internally disjoint trees connecting S, namely, $\overline{\kappa}_{n-1}(G) \geq \frac{n-2}{2}$. Let $S_1' = S_1 \setminus u_1$ and $S' = S_1' \cup S_2$. Let us focus on the graph $G[S_1']$. If $r = 2$, then $G[S']$ is a graph obtained from a clique of order $n-2$ by deleting one edge since $d_{K_n[M]}(u_1) = \frac{n-2}{2}$ and $d_{K_n[M]}(v) = n-3$ and $|M| = \frac{3n-6}{2}$. Without loss of generality, let $N_G(v) = \{u_1, u_2\}$. Clearly, $G[S']$ contains a Hamilton path P with u_2 as one of its endpoints. Then $T = vu_1 \cup vu_2 \cup P$. Set $G_1 = G \setminus E(T)$. Thus $\delta(G_1[S']) = \delta(G[S']) - 2 \geq n-4-2 = n-6 \geq \frac{n-2}{2}$. Combining this with $d_{G_1[S]}(u_1) = \frac{n-2}{2}$, the result follows by (2) of Lemma 7. We now assume $r \geq 3$. Since $d_{K_n[M]}(u_1) = \frac{n-2}{2}$, $d_{K_n[M]}(v) \geq \frac{n}{2}$ and $|M| = \frac{3n-6}{2}$, $G[S']$ is a graph obtained from the complete graph K_{n-2} by deleting at most $\frac{n-4}{2}$ edges and hence $\delta(G[S']) \geq n-3-\frac{n-4}{2} = \frac{n-2}{2}$. It is clear that there exist at least two vertices of degree $n-3$ in $G[S']$, and there is also at most one vertex of degree $\frac{n-2}{2}$ in $G[S']$. Without loss of generality, let u_{i_1}, u_{i_2} be two vertices of degree $n-3$.

If $u_{i_1}, u_{i_2} \in S_1'$, without loss of generality, let $u_{i_1} = u_2$ and $u_{i_2} = u_r$, then the tree $T = vu_1 \cup \cdots \cup vu_r \cup u_2u_{r+1} \cup \cdots \cup u_2u_{r+\frac{n-4}{2}} \cup u_ru_{r+\frac{n-4}{2}+1} \cup \cdots \cup u_ru_{n-1}$ is a Steiner tree connecting S; see Figure 2 (a). Set $G_1 = G \setminus E(T)$. Observe that $d_{G_1[S]}(u_1) = \frac{n-2}{2}$, $d_{G_1[S]}(u_2) \geq n-3-\frac{n-4}{2} = \frac{n-2}{2}$ and $d_{G_1[S]}(u_r) = (n-3)-(n-1-r-\frac{n-4}{2}) = r-2+\frac{n-4}{2} \geq \frac{n-2}{2}$. For $u_j \in S_2$ $(r+1 \leq j \leq n-1)$, $d_{G_1[S]}(u_j) \geq \frac{n-4}{2}$ and there is at most one vertex of degree $\frac{n-4}{2}$ in $G_1[S]$. So $\delta(G_1[S]) \geq \frac{n-4}{2}$ and there is at most one vertex of degree $\frac{n-4}{2}$ in $G_1[S]$, as desired. If

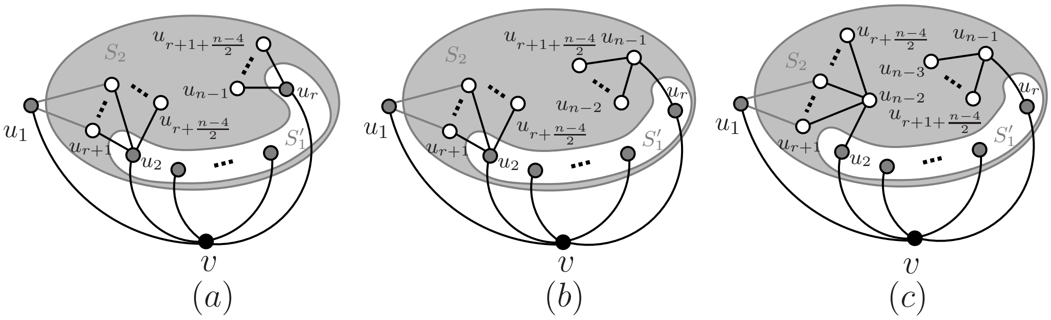

Figure 2. Graphs for Case 1 of Lemma 9.

$u_{i_1} \in S_1'$ and $u_{i_2} \in S_2$, without loss of generality, let $u_{i_1} = u_2$ and $u_{i_2} = u_{n-1}$, then the tree $T = vu_1 \cup \cdots \cup vu_r \cup u_2u_{r+1} \cup \cdots \cup u_2u_{r+\frac{n-4}{2}} \cup u_{n-1}u_{r+\frac{n-4}{2}+1} \cup \cdots \cup u_{n-1}u_{n-2} \cup u_{n-1}u_r$ is our desired tree; see Figure 2 (b). Set $G_1 = G \setminus E(T)$. One can see that $\delta(G_1[S]) \geq \frac{n-4}{2}$ and there is at most one vertex of degree $\frac{n-4}{2}$ in $G_1[S]$, as desired. Let us consider the remaining case $u_{i_1}, u_{i_2} \in S_2$. Without loss of generality, let $u_{i_1} = u_{n-1}$ and $u_{i_2} = u_{n-2}$. The tree $T = vu_1 \cup \cdots \cup vu_r \cup u_{n-2}u_{r+1} \cup \cdots \cup u_{n-2}u_{r+\frac{n-4}{2}} \cup u_{n-1}u_{r+\frac{n-4}{2}+1} \cup \cdots \cup u_{n-1}u_{n-3} \cup u_2u_{n-2} \cup u_{n-1}u_r$ is our desired tree; see Figure 2 (c). Set $G_1 = G \setminus E(T)$. One can see that $\delta(G_1[S]) \geq \frac{n-4}{2}$ and there is at most one vertex of degree $\frac{n-4}{2}$ in $G_1[S]$. Using (2) of Lemma 7, we can get $\overline{\kappa}_{n-1}(G) \geq \frac{n-2}{2}$, as desired.

Case 2. $\frac{n}{2} \leq \delta(G) \leq n-1$.

Recall that $S_1 = \{u_1, u_2, \cdots, u_r\}$ with $vu_j \in E(G)$ $(1 \leq j \leq r)$ and $S_2 = S \setminus S_1 = \{u_{r+1}, u_{r+2}, \cdots, u_{n-1}\}$. Obviously, $|S_1| = r = \delta(G) \geq \frac{n}{2}$ and $|S_2| = n-1-r \leq n-1-\frac{n}{2} = \frac{n-2}{2}$. For each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$, u_j has at most $\frac{n-4}{2}$ neighbors in S_2 and hence $|E_G[u_j, S_1]| \geq \frac{n}{2} - \frac{n-4}{2} = 2$ since $d_G(u_j) \geq \delta(G) \geq \frac{n}{2}$. Clearly, the tree $T' = vu_1 \cup vu_2 \cup \cdots \cup vu_r$ is a Steiner tree connecting S_1. Our idea is to seek for $n-1-r$ edges in $E_G[S_1, S_2]$ and combine them with T' to form a Steiner tree connecting S. We employ the method used in Case 2 of Lemma 8. Choose the one with the smallest subscript among all the vertices of S_2 with the maximum degree in $G[S]$, say u_1'. Then we search for the vertex adjacent to u_1' with the smallest subscript among all the vertices of S_1 with the maximum degree in $G[S]$, say u_1''. Let $e_1 = u_1'u_1''$. Consider the graph $G_1 = G \setminus e_1$, and pick up the one with the smallest subscript among all the vertices of $S_2 \setminus u_1'$ with the maximum degree in $G_1[S]$, say u_2'. Then we search for the vertex adjacent to u_2' with the smallest subscript among all the vertices of S_1 with the maximum degree in $G_1[S]$, say u_2''. Set $e_2 = u_2'u_2''$. We consider the graph $G_2 = G_1 \setminus e_1 = G \setminus \{e_1, e_2\}$. For each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$, we proceed to find $e_3, e_4, \cdots, e_{n-1-r}$ in the same way. Let $M' = \{e_1, e_2, \cdots, e_{n-1-r}\}$ and $G_{n-1-r} = G \setminus M'$. Then $G_{n-1-r}[S] = G[S] \setminus M'$ and the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_r \cup e_1 \cup e_2 \cup \cdots \cup e_{n-1-r}$ is our desired tree. Set $G' = G \setminus E(T)$ (note that $G'[S] = G_{n-1-r}[S]$).

Claim 2. For each $u_j \in S_1$ $(1 \leq j \leq r)$, $d_{G'[S]}(u_j) \geq \frac{n-4}{2}$ and there exists at most one vertex of degree $\frac{n-4}{2}$ in $G'[S]$.

Proof of Claim 2. First, we prove that for each $u_j \in S_1$ $(1 \leq j \leq r)$, $d_{G'[S]}(u_j) \geq \frac{n-4}{2}$. Assume, to the contrary, that there exists one vertex $u_p \in S_1$ such that $d_{G'[S]}(u_p) \leq \frac{n-6}{2}$. By the above procedure, there exists a vertex $u_q \in S_2$ such that when we pick up the edge $e_i = u_p u_q$ from $G_{i-1}[S]$ the degree of u_p in $G_i[S]$ is equal to $\frac{n-6}{2}$. That is $d_{G_i[S]}(u_p) = \frac{n-6}{2}$ and $d_{G_{i-1}[S]}(u_p) = \frac{n-4}{2}$. From our procedure, $|E_G[u_q, S_1]| = |E_{G_{i-1}}[u_q, S_1]|$. Without loss of generality, let $|E_G[u_q, S_1]| = t$ and $u_q u_j \in E(G)$ for $1 \leq j \leq t$; see Figure 1 (b). Thus $u_p \in \{u_1, u_2, \cdots, u_t\}$. Recall that $|E_G[u_j, S_1]| \geq 2$ for each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$. Since $u_q \in S_2$, we have $t \geq 2$. Observe that $u_q u_j \notin E(G)$ and hence $u_q u_j \in M$ for $t+1 \leq j \leq r$ by our procedure, namely, $|E_{K_n[M]}[u_q, S_1]| = r - t$. Since $d_{G_{i-1}[S]}(u_p) = \frac{n-4}{2}$, by our procedure $d_{G_{i-1}[S]}(u_j) \leq \frac{n-4}{2}$ for each $u_j \in S_1$ $(1 \leq j \leq t)$. Assume, to the contrary, that there is a vertex u_s $(1 \leq s \leq t)$ such that $d_{G_{i-1}[S]}(u_s) \geq \frac{n-2}{2}$. Then we should choose the edge $u_q u_s$ instead of $e_i = u_q u_p$ by our procedure, a contradiction. We conclude that $d_{G_{i-1}[S]}(u_j) \leq \frac{n-4}{2}$ for each $u_j \in S_1$ $(1 \leq j \leq t)$. Clearly, there are at least $n-2-\frac{n-4}{2}$ edges incident to each u_j $(1 \leq j \leq t)$ that belong to $M \cup \{e_1, e_2, \cdots, e_{i-1}\}$. Since $i \leq n-1-r$, we have $\sum_{j=1}^{t} d_{K_n[M]}(u_j) \geq (n-2-\frac{n-4}{2})t-(i-1) \geq \frac{n}{2}t-(n-2-r)$ and hence $|M| \geq d_{K_n[M]}(v)+\sum_{j=1}^{t} d_{K_n[M]}(u_j)+|E_{K_n[M]}[u_q, S_1]| \geq (n-1-r)+\frac{n}{2}t-(n-2-r)+(r-t) = r+1+\frac{n-2}{2}t \geq \frac{n}{2}+1+\frac{2(n-2)}{2} = \frac{3n-2}{2}$, which contradicts to $|M| = \frac{3n-6}{2}$.

Next, we consider to prove that there exists at most one vertex of degree $\frac{n-4}{2}$ in $G'[S]$. Assume, to the contrary, that there exist two vertices of degree $\frac{n-4}{2}$ in $G'[S]$, say $u_{p'}, u_p$. By the above procedure, there exists a vertex $u_{q'} \in S_2$ such that when we pick up the edge $e_{i'} = u_{p'} u_{q'}$ from $G_{i'-1}[S]$ the degree of u_p in $G_{i'}[S]$ is equal to $\frac{n-4}{2}$, that is $d_{G_{i'}[S]}(u_{p'}) = \frac{n-4}{2}$. By the same reason, there exists a vertex $u_q \in S_2$ such that when we pick up the edge $e_i = u_p u_q$ from $G_{i-1}[S]$ the degree of u_p in $G_i[S]$ is equal to $\frac{n-4}{2}$, that is, $d_{G_i[S]}(u_p) = \frac{n-4}{2}$ and $d_{G_{i-1}[S]}(u_p) = \frac{n-2}{2}$. Without loss of generality, let $i' < i$. From our procedure, $|E_G[u_q, S_1]| = |E_{G_{i-1}}[u_q, S_1]|$. Without loss of generality, let $|E_G[u_q, S_1]| = t$ and $u_q u_j \in E(G)$ for $1 \leq j \leq t$; see Figure 1 (b). Thus $u_p \in \{u_1, u_2, \cdots, u_t\}$. Recall that $|E_G[u_j, S_1]| \geq 2$ for each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$. Since $u_q \in S_2$, we have $t \geq 2$. Then $u_q u_j \notin E(G)$ and hence $u_q u_j \in M$ for $t+1 \leq j \leq r$ by our procedure, namely, $|E_{K_n[M]}[u_q, S_1]| = r - t$. Since $d_{G_{i-1}[S]}(u_p) = \frac{n-2}{2}$, by our procedure $d_{G_{i-1}[S]}(u_j) \leq \frac{n-2}{2}$ for each $u_j \in S_1$ $(1 \leq j \leq t)$. Assume, to the contrary, that there is a vertex u_s $(1 \leq s \leq t)$ such that $d_{G_{i-1}[S]}(u_s) \geq \frac{n}{2}$. Then we should choose the edge $u_q u_s$ instead of $e_i = u_q u_p$ by our procedure, a contradiction. We conclude that $d_{G_{i-1}[S]}(u_j) \leq \frac{n-2}{2}$ for each $u_j \in S_1$ $(1 \leq j \leq t)$. If $u_{p'} \in \{u_1, \cdots, u_t\}$, without loss of generality, let $u_{p'} = u_1$, then $d_{K_n[M]}(u_1) + \sum_{j=2}^{t} d_{K_n[M]}(u_j) \geq (n-2-d_{G_{i-1}[S]}(u_1)) + (n-2-\frac{n-2}{2})(t-1) - (i-1) \geq (n-2-d_{G_{i'}[S]}(u_1)) + \frac{n-2}{2}(t-1) - (i-1) \geq (n-2-\frac{n-4}{2}) + \frac{n-2}{2}(t-1) - (n-2-r) = \frac{n-2}{2}t - n + 3 + r$ since $i \leq n-1-r$. Since $t \geq 2$ and $r \geq \frac{n}{2}$, we have

$|M| \geq d_{K_n[M]}(v)+d_{K_n[M]}(u_1)+\sum_{j=2}^{t} d_{K_n[M]}(u_j)+|E_{K_n[M]}[u_q,S_1]| \geq (n-1-r)+(\frac{n-2}{2}t-n+3+r)+(r-t) = \frac{n-4}{2}t+r+2 \geq \frac{2(n-4)}{2}+\frac{n}{2}+2 \geq \frac{3n-4}{2}$, which contradicts to $|M| = \frac{3n-6}{2}$. If $u_{p'} \notin \{u_1, \cdots, u_t\}$, then $u_{p'} \in \{u_{t+1}, \cdots, u_r\}$ and $d_{K_n[M]}(u_{p'}) + \sum_{j=1}^{t} d_{K_n[M]}(u_j) \geq (n-2-d_{G_{i-1}[S]}(u_{p'})) + (n-2-\frac{n-2}{2})t-(i-1) \geq (n-2-d_{G_{i'}[S]}(u_{p'})) + \frac{n-2}{2}t-(i-1) \geq (n-2-\frac{n-4}{2})+\frac{n-2}{2}t-(n-2-r) = \frac{n-2}{2}(t+1)-n+3+r$ since $i \leq n-1-r$. Since $t \geq 2$ and $r \geq \frac{n}{2}$, we have $|M| \geq d_{K_n[M]}(v)+d_{K_n[M]}(u_{p'})+\sum_{j=1}^{p} d_{K_n[M]}(u_j)+(|E_{K_n[M]}[u_q,S_1]|-1) \geq (n-1-r)+\frac{n-2}{2}(t+1)-n+3+r+(r-1-t) = r+1+\frac{n-4}{2}t+\frac{n-2}{2} \geq \frac{n}{2}+1+\frac{2(n-4)}{2}+\frac{n-2}{2} = 2n-4$, which contradicts to $|M| = \frac{3n-6}{2}$. The proof of this claim is complete.

From Claim 2, $d_{G'[S]}(u_j) \geq \frac{n-4}{2}$ for each $u_j \in S_1$ $(1 \leq j \leq r)$ and and there exists at most one vertex of degree $\frac{n-4}{2}$ in $G'[S]$. For each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$, $d_{G'[S]}(u_j) = d_{G[S]}(u_j)-1 = d_G(u_j)-1 \geq \delta(G)-1 \geq \frac{n-2}{2}$. So $\delta(G'[S]) \geq \frac{n-4}{2}$ and there exists at most one vertex of degree $\frac{n-4}{2}$ in $G'[S]$. Combining this with $e(G'[S]) = e(G)-(n-1) = \binom{n-2}{2}-\frac{n-2}{2}$, $G'[S]$ contains $\frac{n-4}{2}$ spanning trees by (2) of Lemma 7. These trees together with the tree T are $\frac{n-2}{2}$ trees connecting S, namely, $\overline{\kappa}_{n-1}(G) \geq \frac{n-2}{2}$, as desired. □

After above preparations, we now characterize the graphs with $\overline{\kappa}_{n-1}(G) = \lfloor\frac{n-1}{2}\rfloor$.

Proposition 3. *For a connected graph G of order n $(n \geq 11)$, $\overline{\kappa}_{n-1}(G) = \lfloor\frac{n-1}{2}\rfloor$ if and only if $G = K_n \setminus M$ and $M \subseteq E(K_n)$ satisfies one of the following conditions:*

- $1 \leq |M| \leq n-2$ *for n odd;*
- $\frac{n}{2} \leq |M| \leq n$ *for n even;*
- $n+1 \leq |M| \leq \frac{3n-6}{2}$ *and $d_G(u_1) \geq \frac{n-2}{2}$ where u_1 is a second minimal degree vertex in G for n even.*

Proof. For n odd, if G is a connected graph of order n such that $\overline{\kappa}_{n-1}(G) = \frac{n-1}{2}$, then we can consider G as the graph obtained from a complete graph K_n by deleting some edges. Set $G = K_n \setminus M$ where $M \subseteq E(K_n)$. From Proposition 1, $|M| \geq 1$. Combining this with (3) of Lemma 6, $1 \leq |M| \leq n-2$. For n even, if G is a connected graph of order n such that $\overline{\kappa}_{n-1}(G) = \frac{n-2}{2}$, then we let $G = K_n \setminus M$, where $M \subseteq E(K_n)$. From Proposition 1, $|M| \geq \frac{n}{2}$. Combining this with (1) of Lemma 6, $\frac{n}{2} \leq |M| \leq \frac{3n-6}{2}$. Furthermore, for $n+1 \leq |M| \leq \frac{3n-6}{2}$ we have $d_G(u_1) \geq \frac{n-2}{2}$ by (2) of Lemma 6, where u_1 is a second minimal degree vertex. So $\frac{n}{2} \leq |M| \leq n$, or $n+1 \leq |M| \leq \frac{3n-6}{2}$ and $d_G(u_1) \geq \frac{n-2}{2}$.

Conversely, assume that G is a graph satisfying one of the conditions of this proposition. Then we will show $\overline{\kappa}_{n-1}(G) = \lfloor\frac{n-1}{2}\rfloor$. For n odd, $G = K_n \setminus M$ and $M \subseteq E(K_n)$ such that $1 \leq |M| \leq n-2$. In fact, we only need to show that $\overline{\kappa}_{n-1}(G) \geq \lfloor\frac{n-1}{2}\rfloor$ for $|M| = n-2$. It follows by Lemma 8. Combining with Proposition 1, $\overline{\kappa}_{n-1}(G) = \lfloor\frac{n-1}{2}\rfloor$. For n even, $G = K_n \setminus M$ and $M \subseteq E(K_n)$ such that $\frac{n}{2} \leq |M| \leq n$, or $n+1 \leq |M| \leq \frac{3n-6}{2}$ and

$d_G(u_1) \geq \frac{n-2}{2}$ where u_1 is a second minimal degree vertex. Actually, for $\frac{n}{2} \leq |M| \leq n$, we claim that $d_G(u_1) \geq \frac{n-2}{2}$, where u_1 is a second minimal degree vertex. Otherwise, let $d_G(u_1) \leq \frac{n-4}{2}$. Let v be a vertex in G such that $d_G(v) = \delta(G)$. From the definition of the second minimal degree vertex, $d_G(v) \leq d_G(u_1) \leq \frac{n-4}{2}$ and hence $d_{K_n[M]}(v) \geq d_{K_n[M]}(u_1) \geq n-1-\frac{n-4}{2} = \frac{n+2}{2}$. Therefore, $|M| \geq d_{K_n[M]}(v) + d_{K_n[M]}(u_1) \geq n+2$, a contradiction. So we only need to show that $\overline{\kappa}_{n-1}(G) \geq \lfloor \frac{n-1}{2} \rfloor$ for $|M| = \frac{3n-6}{2}$ and $d_G(u_1) \geq \frac{n-2}{2}$ where u_1 is a second minimal degree vertex. It follows by Lemma 9. From this together with Proposition 1, $\overline{\kappa}_{n-1}(G) = \lfloor \frac{n-1}{2} \rfloor$. □

Furthermore, graphs with $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n-1}{2} \rfloor$ can also be characterized.

Proposition 4. *For a connected graph G of order n ($n \geq 11$), $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n-1}{2} \rfloor$ if and only if $G = K_n \setminus M$ and $M \subseteq E(K_n)$ satisfies one of the following conditions.*

- *$1 \leq |M| \leq n-2$ for n odd;*
- *$\frac{n}{2} \leq |M| \leq n$ for n even;*
- *$n+1 \leq |M| \leq \frac{3n-6}{2}$ and $d_G(u_1) \geq \frac{n-2}{2}$ where u_1 is a second minimal degree vertex in G for n even.*

Proof. Assume that G is a connected graph satisfying the conditions of Proposition 4. From Observation 1 and Proposition 3, it follows that $\overline{\lambda}_{n-1}(G) \geq \overline{\kappa}_{n-1}(G) = \lfloor \frac{n-1}{2} \rfloor$. Combining this with Proposition 2, $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n-1}{2} \rfloor$. Conversely, if $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n-1}{2} \rfloor$, then from Lemma 6 we have $G = K_n \setminus M$ for n odd, where M is an edge set such that $1 \leq |M| \leq n-2$; $G = K_n \setminus M$ for n even, where M is an edge set such that $\frac{n}{2} \leq |M| \leq n$, or $n+1 \leq |M| \leq \frac{3n-6}{2}$ and $d_G(u_1) \geq \frac{n-2}{2}$. □

3.2 The subcase $1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$

Now we consider the case $1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$.

Lemma 10. *Let H is a connected graph of order $n-1$ ($n \geq 12$). If $e(H) = \binom{n-2}{2} + 2\ell - (n-1)$ ($1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$) and $\delta(H) \geq \ell$ and any two vertices of degree ℓ are nonadjacent, then H contains ℓ edge-disjoint spanning trees.*

Proof. Let $\mathscr{P} = \bigcup_{i=1}^{p} V_i$ be a partition of $V(G)$ with $|V_i| = n_i$ ($1 \leq i \leq p$), and $\mathcal{E}_p$ be the set of edges between distinct blocks of $\mathscr{P}$ in G. It suffices to show $|\mathcal{E}_p| \geq \ell(|\mathscr{P}|-1)$ so that we can use Theorem 1.

The case $p = 1$ is trivial, thus we assume $p \geq 2$. For $p = 2$, we have $\mathscr{P} = V_1 \cup V_2$. Set $|V_1| = n_1$. Then $|V_2| = n-1-n_1$. If $n_1 = 1, 2, n-2, n-1$, then $|\mathcal{E}_2| = |E_G[V_1, V_2]| \geq \ell$ since

$\delta(H) \geq \ell$ and any two vertices of degree ℓ are nonadjacent. Suppose $3 \leq n_1 \leq n-4$. Then $|\mathcal{E}_2| = |E_G[V_1, V_2]| \geq \binom{n-2}{2} + 2\ell - (n-1) - \binom{n_1}{2} - \binom{n-1-n_1}{2} = -n_1^2 + (n-1)n_1 + 2\ell - (2n-3)$. Since $3 \leq n_1 \leq n-4$, one can see that $|\mathcal{E}_2|$ attains its minimum value when $n_1 = 3$ or $n_1 = n-4$. Thus $|\mathcal{E}_2| \geq n - 9 + 2\ell \geq \ell$. So the conclusion holds for $p = 2$ by Theorem 1.

Consider the case $p = 3$. We will show $|\mathcal{E}_3| \geq 2\ell$. Let $\mathscr{P} = V_1 \cup V_2 \cup V_3$ and $|V_i| = n_i$ $(i = 1, 2, 3)$ where $n_1 + n_2 + n_3 = n - 1$. If there are two of n_1, n_2, n_3 that equals 1, say $n_1 = n_2 = 1$, then $|\mathcal{E}_3| \geq 2\ell$ since $\delta(H) \geq \ell$ and any two vertices of degree ℓ are nonadjacent. If there is at most one of n_1, n_2, n_3 that equals 1, then we need to prove that $|\mathcal{E}_3| \geq \binom{n-2}{2} + 2\ell - (n-1) - \sum_{i=1}^{3} \binom{n_i}{2} \geq 2\ell$. Since $f(n_1, n_2, n_3) = \sum_{i=1}^{3} \binom{n_i}{2}$ attains its maximum value when $n_1 = 1$, $n_2 = 2$ and $n_3 = n-4$, we need the inequality $\binom{n-2}{2} + 2\ell - (n-1) - \binom{n-4}{2} - 1 \geq 2\ell$. Since $n \geq 12$, the inequality holds. So the conclusion holds for $p = 3$ by Theorem 1. For $p = n-1$, we will show $|\mathcal{E}_{n-1}| \geq \ell(n-2)$ so that we can use Theorem 1. That is $\binom{n-2}{2} + 2\ell - (n-1) \geq \ell(n-2)$. Thus we need the inequality $(n-2-2\ell)(n-4) - n \geq 0$. Since $\ell \leq \lfloor \frac{n-5}{2} \rfloor$, the inequality holds. For $p = n-2$, it suffices to prove $|\mathcal{E}_{n-2}| \geq \ell(n-3)$. Clearly, $|\mathcal{E}_{n-2}| \geq \binom{n-2}{2} + 2\ell - (n-1) - 1 \geq \ell(n-3)$. Thus we need the inequality $(n-2-2\ell)(n-5) - 4 \geq 0$. Since $\ell \leq \lfloor \frac{n-5}{2} \rfloor$, this inequality holds.

Let us consider the remaining case p with $4 \leq p \leq n-4$. Clearly, we need to prove that $|\mathcal{E}_p| \geq \binom{n-2}{2} + 2\ell - (n-1) - \sum_{i=1}^{p} \binom{n_i}{2} \geq \ell(p-1)$, that is, $\frac{(n-2)(n-3)}{2} + 2\ell - (n-1) - \ell p + \ell \geq \sum_{i=1}^{p} \binom{n_i}{2}$. Since $f(n_1, n_2, \cdots, n_p) = \sum_{i=1}^{p} \binom{n_i}{2}$ achieves its maximum value when $n_1 = n_2 = \cdots = n_{p-1} = 1$ and $n_p = n - p$, we need the inequality $\frac{(n-2)(n-3)}{2} + 3\ell - (n-1) - \ell p \geq \frac{(n-p)(n-p-1)}{2}$. It is equivalent to $(2n - 2\ell - p - 4)(p-3) \geq 4$. One can see that the inequality holds since $\ell \leq \frac{n-5}{2}$ and $4 \leq p \leq n-4$. From Theorem 1, we know that there exist ℓ edge-disjoint spanning trees. $\square$

Lemma 11. *Let G be a connected graph of order n $(n \geq 12)$. If $e(G) \geq \binom{n-2}{2} + 2\ell$ $(1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor)$, $\delta(G) \geq \ell + 1$ and any two vertices of degree $\ell + 1$ are nonadjacent, then $\overline{\kappa}_{n-1}(G) \geq \ell + 1$.*

Proof. The following claim can be easily proved.

Claim 3. $\Delta(G) \geq n - 4$.

Proof of Claim 3. Assume, to the contrary, that $\Delta(G) \leq n-5$. Then $(n-2)(n-3) + 4\ell = 2e(G) \leq n\Delta(G) \leq n(n-5)$, which implies that $4\ell + 6 \leq 0$, a contradiction.

From Claim 3, $n - 4 \leq \Delta(G) \leq n - 1$. Our basic idea is to find out a Steiner tree T connecting $S = V(G) \setminus v$, where $v \in V(G)$ such that $d_G(v) = \Delta(G)$. Let $G_1 = G \setminus E(T)$. Then we prove that $G_1[S]$ satisfies the conditions of Lemma 10 so that $G_1[S]$ contains ℓ edge-disjoint spanning trees. These trees together with the tree T are $\ell + 1$ internally disjoint trees connecting S, which implies that $\overline{\kappa}_{n-1}(G) \geq \ell+1$, as desired. We distinguish the following four cases to show this lemma.

If $\Delta(G) = n-1$, then there exists a vertex $v \in V(G)$ such that $d_G(v) = n-1$. Let $S = V(G) \setminus v = \{u_1, u_2, \cdots, u_{n-1}\}$. Then the tree $T = u_1v \cup u_2v \cup \cdots \cup u_{n-1}v$ is a Steiner tree connecting S. Set $G_1 = G \setminus E(T)$. Since $\delta(G) \geq \ell+1$ and any two vertices of degree $\ell+1$ are nonadjacent, it follows that $\delta(G_1[S]) \geq \ell$ and any two vertices of degree ℓ are nonadjacent. From Lemma 10, $G_1[S]$ contains ℓ edge-disjoint spanning trees, as desired.

Consider the case $\Delta(G) = n-4$. We claim that $\delta(G) \geq \ell+4$. Otherwise, let $\delta(G) \leq \ell+3$. Then there exists a vertex u such that $d_G(u) \leq \ell+3$. Then $2[\binom{n-2}{2}+2\ell] = 2e(G) = \sum_{u\in V(G)} d(u) \leq d_G(u) + (n-1)\Delta(G) \leq (\ell+3)+(n-1)(n-4)$, which results in $\ell \leq \frac{1}{3}$, a contradiction. So $\delta(G) \geq \ell+4$. Since $\Delta(G) = n-4$, there exists a vertex $v \in V(G)$ such that $d_G(v) = n-4$. Let $S = V(G) \setminus v = \{u_1, \cdots, u_{n-1}\}$ such that $vu_{n-1}, vu_{n-2}, vu_{n-3} \notin E(G)$. Pick up $u_i \in N_G(u_{n-1}), u_j \in N_G(u_{n-2}), u_k \in N_G(u_{n-3})$ (note that u_i, u_j, u_k are not necessarily different). Then the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-4} \cup u_iu_{n-1} \cup u_ju_{n-2} \cup u_ku_{n-1}$ is our desired tree. Set $G_1 = G \setminus E(T)$. Since $\delta(G) \geq \ell+4$, one can check that $\delta(G_1) \geq \ell+4$ and there is at most one vertex of degree ℓ in $G_1[S]$, as desired.

If $\Delta(G) = n-2$, then there exists a vertex of degree $n-2$ in G, say v. Let $S = G \setminus v = \{u_1, u_2, \cdots, u_{n-1}\}$ such that u_{n-1} is the unique vertex with $u_{n-1}v \notin E(G)$. Let $d_G(u_{n-1}) = x$. Without loss of generality, let $N_G(u_{n-1}) = \{u_1, \cdots, u_x\}$. Since $\delta(G) \geq \ell+1$, it follows that $x \geq \ell+1 \geq 2$. First, we consider the case $x \geq 3$. We claim that there exists a vertex, say u_i $(1 \leq i \leq x)$, such that $d_G(u_i) \geq \ell+3$. Assume, to the contrary, that $d_G(u_j) \leq \ell+2$ for each u_j $(1 \leq j \leq x)$. Then $(n-2)(n-3)+4\ell = 2e(G) \leq d_G(u_{n-1}) + d_G(v) + \sum_{j=1}^{x} d_G(u_j) + \sum_{j=x+1}^{n-2} d_G(u_j) \leq x+(n-2)+(\ell+2)x+(n-2-x)(n-2)$ and hence $x \leq \frac{2n-4\ell-4}{n-\ell-5}$. Since $x \geq 3$, it follows that $n+\ell-11 \leq 0$, which contradicts to $n \geq 12$. So there exists a vertex, say u_i $(1 \leq i \leq x)$, such that $d_G(u_i) \geq \ell+3$. Then the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-2} \cup u_{n-1}u_i$ is our desired tree. Set $G_1 = G \setminus E(T)$. Since $\delta(G) \geq \ell+1$ and any two vertices of degree $\ell+1$ are nonadjacent, one can check that $\delta(G_1[S]) \geq \ell$ and any two vertices of degree ℓ are nonadjacent, as desired. Next, we consider the case $x = 2$. Since $\ell+1 \leq x$, it follows that $\ell = 1$ and hence $d_G(u_{n-1}) = 2$ and $N_G(u_{n-1}) = \{u_1, u_2\}$. Let p be the number of vertices of degree 2 in G. We claim that $0 \leq p \leq 3$. Otherwise, let $p \geq 4$. Then $2\binom{n-2}{2} + 4 = 2e(G) = \sum_{v\in V(G)} d(v) \leq 2p+(n-p)(n-2)$ and hence $p \leq \frac{3n-10}{n-4}$. Since $p \geq 4$, it follows that $n \leq 6$, a contradiction. So $0 \leq p \leq 3$. If $p = 3$, then there are three vertices of degree 2, say v_1, v_2, v_3. Let $G_1 = G \setminus \{v_1, v_2, v_3\}$. Since the three vertices are pairwise nonadjacent, $|V(G_1)| = n-3$ and $e(G_1) = \binom{n-2}{2} + 2 - 6 = \binom{n-2}{2} - 4 > \binom{n-3}{2}$, a contradiction. So we can assume $0 \leq p \leq 2$. If $p = 2$, then there are two vertices of degree 2, say v_1, v_2. Let $G_1 = G \setminus \{v_1, v_2\}$. Then G_1 is a graph obtained from a clique of order $n-2$ by deleting 2 edges and hence

$\overline{\kappa}_{n-2}(G_1) \geq \lfloor \frac{n-2}{2} \rfloor - 2 \geq 2$, that is, G_1 contains two edge-disjoint spanning trees, say T_1', T_2'. Let $N_G(v_1) = \{u_1, u_2\}$, the trees $T_i = T_i' \cup v_1u_i$ $(i = 1, 2)$ are two internally disjoint Steiner trees connecting $S = V(G) \setminus v_2$, which implies that $\overline{\kappa}_{n-1}(G) \geq 2$, as desired. So we now assume $0 \leq p \leq 1$. Consider the case $p = 1$. If $d_G(u_{n-1}) = 2$, then $d_G(u_j) \geq 3$ for each u_j $(1 \leq j \leq n-2)$. Recall that $N_G(u_{n-1}) = \{u_1, u_2\}$, certainly we have $d_G(u_j) \geq 3$ $(j = 1, 2)$. Then the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-2} \cup u_1u_{n-1}$ is a Steiner tree connecting $S = V(G) \setminus v$. Set $G_1 = G \setminus E(T)$. Clearly, $d_{G_1[S]}(u_1) \geq 1$, $d_{G_1[S]}(u_{n-1}) = 1$ and $u_1u_{n-1} \notin E(G_1[S])$. In addition, the degree of the other vertices in $G_1[S]$ is at least 2, as desired. Suppose $d_G(u_{n-1}) \geq 3$. Let u_i be the vertex of degree 2 in $V(G) \setminus \{v, u_{n-1}\}$. If $u_i \in N_G(u_{n-1})$, then there is another vertex $u_j \in N_G(u_{n-1})$ such that $d_G(u_j) \geq 3$ since $p = 1$. Then the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-2} \cup u_ju_{n-1}$ is our desired tree. Set $G_1 = G \setminus E(T)$. Obviously, $d_{G_1[S]}(u_i) = 1$, $d_{G_1[S]}(u_j) \geq 1$, $d_{G_1[S]}(u_{n-1}) \geq 2$, $u_iu_j \notin E(G_1[S])$ and the degree of the other vertices in $G_1[S]$ is at least 2, as desired. If $u_i \notin N_G(u_{n-1})$, then there exists a vertex $u_j \in N_G(u_{n-1})$ such that $d_G(u_j) \geq 3$ and $u_iu_j \notin E(G)$. Thus the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-2} \cup u_ju_{n-1}$ is our desired tree. Set $G_1 = G \setminus E(T)$. Clearly, $d_{G_1[S]}(u_i) = 1$, $d_{G_1[S]}(u_t) \geq 1$, $d_{G_1[S]}(u_{n-1}) \geq 2$, $u_iu_j \notin E(G_1[S])$ and the degree of the other vertices in $G_1[S]$ is at least 2, as desired. For the remaining case $p = 0$, we choose a vertex $u_j \in N_G(u_{n-1})$ and the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-2} \cup u_ju_{n-1}$ is our desired tree. Set $G_1 = G \setminus E(T)$. Clearly, $\delta(G_1[S]) \geq 1$ and there is at most one vertex of degree 1, as desired.

Let us consider the remaining case $\Delta(G) = n-3$. Then there exists a vertex of degree $n-3$, say v. Let p be the number of vertices of degree $\ell+1$. Since $(n-2)(n-3)+4\ell = 2e(G) \leq p(\ell+1)+(n-p)(n-3)$, it follows that $p \leq \frac{2n-4\ell-6}{n-\ell-4}$. Consider the case $\ell \geq 2$. Since $p \leq \frac{2n-4\ell-6}{n-\ell-4}$, if $p \geq 2$ then $\ell \leq 1$, a contradiction. So $0 \leq p \leq 1$ for $2 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$. Let $V(G) \setminus v = \{u_1, u_2, \cdots, u_{n-1}\}$ such that $vu_{n-1}, vu_{n-2} \notin E(G)$. Without loss of generality, let $d_G(u_{n-1}) \geq d_G(u_{n-2})$. For the vertex $v \in V(G)$, we choose $\ell+1$ vertices in $N_G(v)$ and the following claim can be easily proved.

Claim 4. For $\ell \geq 2$ and any $\ell+1$ vertices in $N_G(v)$, there exists one of them, say u_i, such that $d_G(u_i) \geq \ell+4$.

Proof of Claim 4. Assume, to the contrary, that for any $\ell+1$ vertices in $N_G(v)$, say $u_1, u_2, \cdots, u_{\ell+1}$, $d_G(u_j) \leq \ell+3$ $(1 \leq j \leq \ell+1)$. Then $(n-2)(n-3)+4\ell = 2e(G) \leq (\ell+1)(\ell+3)+(n-\ell-1)(n-3)$ and hence $(\ell-1)(n-3) \leq \ell^2+3$. So $n-3 \leq \frac{\ell^2+3}{\ell-1} = \ell+1+\frac{4}{\ell-1} \leq \ell+5 \leq \frac{n+5}{2}$, which contradicts to $n \geq 12$.

First, we consider the case $u_{n-1}u_{n-2} \in E(G)$. Recall that $0 \leq p \leq 1$ for $2 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$, that is, there is at most one vertex of degree $\ell+1$ in G. If $d_G(u_{n-2}) = \ell+1$, then $d_G(u_{n-1}) \geq d_G(u_{n-2}) = \ell+2$ and hence there exists a vertex $u_i \in N_G(u_{n-1}) \setminus u_{n-2}$ such that $d_G(u_i) \geq \ell+4$ by Claim 4, where $1 \leq i \leq \ell+1$. Then the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-3} \cup$

$u_iu_{n-1} \cup u_{n-1}u_{n-2}$ is a Steiner tree connecting $S = V(G) \setminus v$. Let $G_1 = G \setminus E(T)$. Observe that $d_{G_1[S]}(u_{n-1}) \geq d_G(u_{n-1}) - 2 \geq \ell$, $d_{G_1[S]}(u_{n-2}) = d_G(u_{n-2}) - 1 = \ell$ and $u_{n-2}u_{n-1} \notin E(G_1)$. In addition, $d_{G_1[S]}(u_i) \geq d_G(u_i) - 2 \geq \ell + 2$ and $d_{G_1[S]}(u_j) \geq d_G(u_j) - 1 \geq \ell + 1$ for each $u_j \in V(G) \setminus \{u_{n-1}, u_{n-2}, u_i, v\}$. Thus $\delta(G_1[S]) \geq \ell$ and any two vertices of degree ℓ are nonadjacent, as desired. If $d_G(u_{n-2}) \geq \ell + 2$, then $d_G(u_{n-1}) \geq d_G(u_{n-2}) \geq \ell + 2$. From Claim 4, there exist two vertices, say u_i, u_j, such that $u_i \in N_G(u_{n-1}) \setminus u_{n-2}$, $u_j \in N_G(u_{n-2}) \setminus u_{n-1}$, $d_G(u_i) \geq \ell + 4$ and $d_G(u_j) \geq \ell + 4$ (note that u_i, u_j are not necessarily different). Then the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-3} \cup u_iu_{n-1} \cup u_ju_{n-2}$ is our desired. Set $G_1 = G \setminus E(T)$. One can see that $G_1[S]$ satisfies the conditions of Lemma 10. Next, we consider the case $u_{n-1}u_{n-2} \notin E(G)$. Then $d_G(u_{n-1}) \geq d_G(u_{n-2}) \geq \ell + 1$. From Claim 4, there exist two vertices, say u_i, u_j, such that $u_i \in N_G(u_{n-1}) \setminus u_{n-2}$, $u_j \in N_G(u_{n-2}) \setminus u_{n-1}$, $d_G(u_i) \geq \ell+4$ and $d_G(u_j) \geq \ell+4$ (note that u_i, u_j are not necessarily different). Thus the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-3} \cup u_iu_{n-1} \cup u_ju_{n-2}$ is our desired tree. Set $G_1 = G \setminus E(T)$ and $S = V(G) \setminus v$. One can check that $\delta(G_1[S]) \geq \ell$ and there is at most one vertex of degree ℓ, as desired. Similar to the proof of the case $\Delta(G) = n - 2$, we can prove that the conclusion holds for $\ell = 1$. The proof is now complete. $\square$

3.3 Results for the maximum generalized local (edge-)connectivity

Let $\mathcal{H}_n$ be a graph class obtained from the complete graph of order $n-2$ by adding two nonadjacent vertices and joining each of them to any ℓ vertices of K_{n-2}. The following theorem summarizes the results for a general ℓ.

Theorem 4. *Let G be a connected graph of order n ($n \geq 12$). If $\overline{\kappa}_{n-1}(G) \leq \ell$ ($1 \leq \ell \leq \lfloor \frac{n+1}{2} \rfloor$), then*

$$e(G) \leq \begin{cases} \binom{n-2}{2} + 2\ell, & if\ 1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor; \\ \binom{n-2}{2} + n - 2, & if\ \ell = \lfloor \frac{n-3}{2} \rfloor \text{ and } n \text{ is odd}; \\ \binom{n-2}{2} + n - 4, & if\ \ell = \lfloor \frac{n-3}{2} \rfloor \text{ and } n \text{ is even}; \\ \binom{n-1}{2} + n - 2, & if\ \ell = \lfloor \frac{n-1}{2} \rfloor \text{ and } n \text{ is odd}; \\ \binom{n-1}{2} + \frac{n-2}{2}, & if\ \ell = \lfloor \frac{n-1}{2} \rfloor \text{ and } n \text{ is even}; \\ \binom{n}{2}, & if\ \ell = \lfloor \frac{n+1}{2} \rfloor. \end{cases}$$

with equality if and only if $G \in \mathcal{H}_n$ for $1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$; $G = K_n \setminus M$ where $|M| = n - 1$ for $\ell = \lfloor \frac{n-3}{2} \rfloor$ and n odd; $G \in \mathcal{H}_n$ for $\ell = \lfloor \frac{n-3}{2} \rfloor$ and n even; $G = K_n \setminus e$ where $e \in E(K_n)$ for $\ell = \lfloor \frac{n-1}{2} \rfloor$ and n odd; $G = K_n \setminus M$ where $|M| = \frac{n}{2}$ for $\ell = \lfloor \frac{n-1}{2} \rfloor$ and n even; $G = K_n$ for $\ell = \lfloor \frac{n+1}{2} \rfloor$.

Proof. For $1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$, we assume that $e(G) \geq \binom{n-2}{2} + 2\ell + 1$. Then the following claim is immediate.

Claim 5. $\delta(G) \geq \ell + 1$.

Proof of Claim 5. Assume, to the contrary, that $\delta(G) \leq \ell$. Then there exists a vertex $v \in V(G)$ such that $d_G(v) = \delta(G) \leq \ell$, which results in $e(G-v) \geq e(G)-\ell \geq \binom{n-2}{2}+\ell+1$. Since $1 \leq \ell \leq \lfloor\frac{n-5}{2}\rfloor$, it follows that $\overline{\kappa}_{n-1}(G-v) \geq \ell+1$ by Theorem 2, which results in $\overline{\kappa}_{n-1}(G) \geq \ell+1$, a contradiction.

From Claim 5, $\delta(G) \geq \ell+1$. If any two vertices of degree $\ell+1$ are nonadjacent, then $\overline{\kappa}_{n-1}(G) \geq \ell+1$ by Lemma 11, a contradiction. So we assume that v_1 and v_2 are two vertices of degree $\ell+1$ such that $v_1v_2 \in E(G)$. Let $G_1 = G \setminus \{v_1, v_2\}$ and $V(G_1) = \{u_1, u_2, \cdots, u_{n-2}\}$. Then $e(G_1) \geq e(G) - (2\ell+1) = \binom{n-2}{2}$ and hence G_1 is a clique of order $n-2$. Furthermore, G_1 contains $\lfloor\frac{n-2}{2}\rfloor \geq \ell+1$ edge-disjoint spanning trees, say $T_1', T_2', \cdots, T_{\ell+1}'$. Without loss of generality, let $N_G(v_1) = \{u_1, u_2, \cdots, u_\ell, v_2\}$. Choose $S = \{u_1, u_2, \cdots, u_{n-2}, v_1\}$. Then the trees $T_i = T_i' \cup v_1u_i$ $(1 \leq i \leq \ell)$ together with $T_{\ell+1} = T_{\ell+1}' \cup v_1v_2 \cup v_2u_t$ are $\ell+1$ internally disjoint trees connecting S where $u_t \in N_G(v_2)\backslash v_1$, which implies that $\overline{\kappa}_{n-1}(G) \geq \ell+1$, a contradiction. So $e(G) \leq \binom{n-2}{2}+2\ell$ for $1 \leq \ell \leq \lfloor\frac{n-5}{2}\rfloor$. From Proposition 3, $e(G) \leq \binom{n-2}{2}+n-2$ for $\ell = \lfloor\frac{n-3}{2}\rfloor$ and n odd, and $e(G) \leq \binom{n-2}{2}+n-4$ for $\ell = \lfloor\frac{n-3}{2}\rfloor$ and n even. From Proposition 1, $e(G) \leq \binom{n-1}{2}+n-2$ for $\ell = \lfloor\frac{n-1}{2}\rfloor$ and n odd, and $e(G) \leq \binom{n-1}{2}+\frac{n-2}{2}$ for $\ell = \lfloor\frac{n-1}{2}\rfloor$ and n even. If $\ell = \lfloor\frac{n+1}{2}\rfloor$, then for any connected graph G it follows that $\overline{\kappa}_{n-1}(G) \leq \ell$ by Observation 4 and hence $e(G) \leq \binom{n}{2}$.

Now we characterize the graphs attaining these upper bounds. For $\ell = \lfloor\frac{n+1}{2}\rfloor$, if $e(G) = \binom{n}{2}$, then $G = K_n$. For $\ell = \lfloor\frac{n-1}{2}\rfloor$ and n odd, if $e(G) = \binom{n-1}{2}+n-2$, then $G = K_n \setminus e$ where $e \in E(K_n)$. For $\ell = \lfloor\frac{n-1}{2}\rfloor$ and n even, if $e(G) = \binom{n-1}{2}+\frac{n-2}{2}$, then $G = K_n \setminus M$ where $|M| = \frac{n}{2}$. For $\ell = \lfloor\frac{n-3}{2}\rfloor$ and n odd, if $e(G) = \binom{n-2}{2}+n-2$, then $G = K_n \setminus M$ where $|M| = n-1$. Suppose that $e(G) = \binom{n-2}{2}+n-4$ for $\ell = \lfloor\frac{n-3}{2}\rfloor$ and n even. From Proposition 3, G is a graph obtained from a complete graph K_{n-2} by adding two nonadjacent vertices and adding $\frac{n-4}{2}$ edges between each of them and the complete graph K_{n-2}, that is, $G \in \mathcal{H}_n$.

Let us now focus on the case $1 \leq \ell \leq \lfloor\frac{n-5}{2}\rfloor$. Suppose $e(G) = \binom{n-2}{2}+2\ell$. Similar to the proof of Claim 5, we can get $\delta(G) \geq \ell$. Furthermore, we prove that $\delta(G) = \ell$. If $\delta(G) \geq \ell+2$, or $\delta(G) = \ell+1$ and any two vertices of degree $\ell+1$ are nonadjacent, then $\overline{\kappa}_{n-1}(G) \geq \ell+1$ by Lemma 11, a contradiction. If there exist two vertices of degree $\ell+1$, say v_1 and v_2, such that $v_1v_2 \in E(G)$, then $G_1 = G \setminus \{v_1, v_2\}$ is a graph obtained from a complete graph of order $n-2$ by deleting an edge. For n odd, from Corollary 2 we have $\overline{\kappa}_{n-2}(G_1) = \lfloor\frac{n-2}{2}\rfloor = \frac{n-3}{2} \geq \ell+1$ since $\ell \leq \lfloor\frac{n-5}{2}\rfloor = \frac{n-5}{2}$. For n even, from Corollary 2, it follows that $\overline{\kappa}_{n-2}(G_1) \geq \lfloor\frac{n-2}{2}\rfloor - 1 = \frac{n-4}{2} \geq \ell+1$ since $\ell \leq \lfloor\frac{n-5}{2}\rfloor = \frac{n-6}{2}$. We conclude that G_1 contains $\ell+1$ edge-disjoint spanning trees, say $T_1', T_2', \cdots, T_{\ell+1}'$. Set $N_G(v_1) = \{u_1, u_2, \cdots, u_\ell, v_2\}$. Then the trees $T_i = T_i' \cup v_1u_i$ $(1 \leq i \leq \ell)$ together with

$T_{\ell+1} = T'_{\ell+1} \cup v_1v_2 \cup v_1u_t$ are $\ell+1$ internally disjoint trees connecting $S = V(G) \setminus v_2$ where $u_t \in N_G(v_2) \setminus v_1$, which implies that $\overline{\kappa}_{n-1}(G) \geq \ell+1$, a contradiction. So $\delta(G) = \ell$. If there exist two vertices of degree ℓ, say v_1, v_2, such that $v_1v_2 \in E(G)$, then we set $G_1 = G \setminus \{v_1, v_2\}$. Thus $|V(G_1)| = n-2$ and $e(G_1) = \binom{n-2}{2} + 1$, a contradiction.

So we assume that any two vertices of degree ℓ are nonadjacent in G. Let p be the number of vertices of degree ℓ. The following claim can be easily proved.

Claim 6. $2 \leq p \leq 3$.

Proof of Claim 6. Assume $p \geq 4$. Then $2\binom{n-2}{2} + 4\ell = 2e(G) = \sum_{v \in V(G)} d(v) \leq p\ell + (n-p)(n-1)$ and hence $p \leq \frac{4n-4\ell-6}{n-\ell-1}$. Since $p \geq 4$, it follows that $4n-4\ell-4 \leq 4n-4\ell-6$, a contradiction. Assume $p = 1$, that is, G contains exactly one vertex of degree ℓ, say v_1. Set $G_1 = G \setminus v_1$. Clearly, $e(G_1) = e(G) - \ell = \binom{n-2}{2} + \ell$. Since $\overline{\kappa}_{n-1}(G) \leq \ell$, it follows that $\overline{\kappa}_{n-1}(G_1) \leq \overline{\kappa}_{n-1}(G) \leq \ell$. From Theorem 2, G_1 is a graph obtained from a clique of order $n-2$ by adding a vertex of degree ℓ, say v_2. Since $p = 1$, we have $d_G(v_1) = \ell$, $d_G(v_2) = \ell+1$ and $v_1v_2 \in E(G)$. Observe that $G_2 = G \setminus \{v_1, v_2\}$ is a clique of order $n-2$. Thus G_2 contains $\lfloor \frac{n-2}{2} \rfloor \geq \ell+1$ edge-disjoint spanning trees, say $T'_1, T'_2, \cdots, T'_{\ell+1}$. Without loss of generality, let $N_G(v_1) = \{v_2, u_1, u_2, \cdots, u_\ell\}$. Then the trees $T_i = v_1u_i \cup T'_i$ $(1 \leq i \leq \ell)$ together with $T_{\ell+1} = T'_{\ell+1} \cup v_1v_2 \cup v_2u_t$ form $\ell+1$ edge-disjoint trees connecting $S = V(G) \setminus v_2$, where $u_t \in N_G(v_2) \setminus v_1$. This implies that $\overline{\kappa}_{n-1}(G) \geq \ell+1$, a contradiction.

From Claim 6, we know that $p = 2, 3$. If $p = 3$, then G contains three vertices of degree ℓ, say v_1, v_2, v_3. Set $G_1 = G \setminus \{v_1, v_2, v_3\}$. Then $|V(G_1)| = n-3$ and $e(G_1) = \binom{n-2}{2} + 2\ell - 3\ell = \binom{n-2}{2} - \ell > \binom{n-3}{2}$ since $1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$, a contradiction. If $p = 2$, then G contains two vertices of degree ℓ, say v_1, v_2. Set $G_1 = G \setminus \{v_1, v_2\}$. Since v_1 and v_2 are nonadjacent, it follows that $e(G_1) = e(G) - 2\ell = \binom{n-2}{2}$ and hence G_1 is a complete graph of order $n-2$, which implies that $G \in \mathcal{H}_n$. □

The following corollary is immediate from Theorem 4.

Corollary 4. *For $1 \leq \ell \leq \lfloor \frac{n+1}{2} \rfloor$ and $n \geq 12$,*

$$f(n; \overline{\kappa}_{n-1} \leq \ell) = \begin{cases} \binom{n-2}{2} + 2\ell, & if\ 1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor,\ or\ \ell = \lfloor \frac{n-3}{2} \rfloor \text{ and } n \text{ is even;} \\ \binom{n-2}{2} + 2\ell + 1, & if\ \ell = \lfloor \frac{n-3}{2} \rfloor \text{ and } n \text{ is odd;} \\ \binom{n-1}{2} + \ell, & if\ \ell = \lfloor \frac{n-1}{2} \rfloor \text{ and } n \text{ is even;} \\ \binom{n-1}{2} + 2\ell - 1, & if\ \ell = \lfloor \frac{n-1}{2} \rfloor \text{ and } n \text{ is odd;} \\ \binom{n}{2}, & if\ \ell = \lfloor \frac{n+1}{2} \rfloor. \end{cases}$$

Now we focus on the edge case.

Theorem 5. *Let G be a connected graph of order n $(n \geq 12)$. If $\overline{\lambda}_{n-1}(G) \leq \ell$ $(1 \leq \ell \leq \lfloor \frac{n+1}{2} \rfloor)$, then*

$$e(G) \leq \begin{cases} \binom{n-2}{2} + 2\ell, & if\ 1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor; \\ \binom{n-2}{2} + n - 2, & if\ \ell = \lfloor \frac{n-3}{2} \rfloor \text{ and } n \text{ is odd}; \\ \binom{n-2}{2} + n - 4, & if\ \ell = \lfloor \frac{n-3}{2} \rfloor \text{ and } n \text{ is even}; \\ \binom{n-1}{2} + n - 2, & if\ \ell = \lfloor \frac{n-1}{2} \rfloor \text{ and } n \text{ is odd}; \\ \binom{n-1}{2} + \frac{n-2}{2}, & if\ \ell = \lfloor \frac{n-1}{2} \rfloor \text{ and } n \text{ is even}; \\ \binom{n}{2}, & if\ \ell = \lfloor \frac{n+1}{2} \rfloor. \end{cases}$$

with equality if and only if $G \in \mathcal{H}_n$ for $1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$; $G = K_n \setminus M$ where $|M| = n - 1$ for $\ell = \lfloor \frac{n-3}{2} \rfloor$ and n odd; $G \in \mathcal{H}_n$ for $\ell = \lfloor \frac{n-3}{2} \rfloor$ and n even; $G = K_n \setminus e$ where $e \in E(K_n)$ for $\ell = \lfloor \frac{n-1}{2} \rfloor$ and n odd; $G = K_n \setminus M$ where $|M| = \frac{n}{2}$ for $\ell = \lfloor \frac{n-1}{2} \rfloor$ and n even; $G = K_n$ for $\ell = \lfloor \frac{n+1}{2} \rfloor$.

Proof. Since $\overline{\lambda}_{n-1}(G) \leq \ell$ $(1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor)$, it follows that $\overline{\kappa}_{n-1}(G) \leq \overline{\lambda}_{n-1}(G) \leq \ell$ and hence $e(G) \leq \binom{n-2}{2} + 2\ell$ by Theorem 4. Suppose $e(G) = \binom{n-2}{2} + 2\ell$. Since $\overline{\kappa}_{n-1}(G) \leq \overline{\lambda}_{n-1}(G) \leq \ell$, we have $G \in \mathcal{H}_n$ by Theorem 4. For $\ell = \lfloor \frac{n+1}{2} \rfloor$, $\lfloor \frac{n-1}{2} \rfloor$ and $\lfloor \frac{n-3}{2} \rfloor$, respectively, the conclusion holds by Propositions 2 and 4.

□

Corollary 5. *For $1 \leq \ell \leq \lfloor \frac{n+1}{2} \rfloor$ and $n \geq 12$,*

$$g(n; \overline{\lambda}_{n-1} \leq \ell) = \begin{cases} \binom{n-2}{2} + 2\ell, & if\ 1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor, \text{ or } \ell = \lfloor \frac{n-3}{2} \rfloor \text{ and } n \text{ is even}; \\ \binom{n-2}{2} + 2\ell + 1, & if\ \ell = \lfloor \frac{n-3}{2} \rfloor \text{ and } n \text{ is odd}; \\ \binom{n-1}{2} + \ell, & if\ \ell = \lfloor \frac{n-1}{2} \rfloor \text{ and } n \text{ is even}; \\ \binom{n-1}{2} + 2\ell - 1, & if\ \ell = \lfloor \frac{n-1}{2} \rfloor \text{ and } n \text{ is odd}; \\ \binom{n}{2}, & if\ \ell = \lfloor \frac{n+1}{2} \rfloor. \end{cases}$$

Remark. It is not easy to determine the exact value of $f(n; \overline{\kappa}_k \leq \ell)$ and $g(n; \overline{\lambda}_k \leq \ell)$ for a general k. So we hope to give a sharp lower bound of them. We construct a graph G of order n as follows: Choose a complete graph K_{k-1} $(1 \leq \ell \leq \lfloor \frac{k-1}{2} \rfloor)$. For the remaining $n - k + 1$ vertices, we join each of them to any ℓ vertices of K_{k-1}. Clearly, $\overline{\kappa}_{n-1}(G) \leq \overline{\lambda}_{n-1}(G) \leq \ell$ and $e(G) = \binom{k-1}{2} + (n - k + 1)\ell$. So $f(n; \overline{\kappa}_k \leq \ell) \geq \binom{k-1}{2} + (n - k + 1)\ell$ and $g(n; \overline{\lambda}_k \leq \ell) \geq \binom{k-1}{2} + (n - k + 1)\ell$. From Theorems 4 and 5, we know that these two bounds are sharp for $k = n, n - 1$.

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