ON ADDITIVE BASES II

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Abstract. Let G be an additive finite abelian group, and let S be a sequence over G. We say that S is regular if for every proper subgroup $H \subseteq G$, S contains at most $|H|-1$ terms from H. Let $\mathsf{c}_0(G)$ be the smallest integer t such that every regular sequence S over G of length $|S| \geq t$ forms an additive basis of G, i.e., every element of G can be expressed as the sum over a nonempty subsequence of S. The constant $\mathsf{c}_0(G)$ has been determined previously only for the elementary abelian groups. In this paper, we determine $\mathsf{c}_0(G)$ for some groups including the cyclic groups, the groups of order even, the groups of rank at least five, and all the p-groups except $G = C_p \oplus C_{p^n}$ with $n \geq 2$.

1. Introduction

Let G be a finite abelian group, p be the smallest prime dividing $|G|$, and let $\mathsf{r}(G)$ denote the rank of G. Let S be a sequence over G. We say that S is an *additive basis* of G if every element of G can be expressed as the sum over a nonempty subsequence of S. Let $\mathsf{c}(G)$ denote the smallest integer t such that every subset of G of cardinality at least t is an additive basis of G. In 1964, Erdős and Heilbronn [1] proposed the problem to determine $\mathsf{c}(G)$, and it had been completely determined till 2009 through many authors' effort (see [5], [2] and their references). For every subgroup H of G, let S_H denote the subsequence of S consisting of all terms of S contained in H. We say that S is a regular sequence over G if $|S_H| \leq |H|-1$ holds for every subgroup $H \subsetneq G$. Let $\mathsf{c}_0(G)$ denote the smallest integer t such that every regular sequence over G of length at least t is an additive basis of G. The problem to determine $\mathsf{c}_0(G)$ was first proposed by Olson and then studied by Peng ([10], [11]) in 1987, he determined $\mathsf{c}_0(G)$ for all the elementary abelian groups. Let

$$m(G) = \begin{cases} |G|, & \text{if } G \text{ is cyclic,} \\ \frac{|G|}{p} + p - 1, & \text{if } G = C_p \oplus C_{\frac{|G|}{p}} \text{ and } p \mid \frac{|G|}{p}, \\ \frac{|G|}{p} + p - 2, & \text{otherwise.} \end{cases}$$

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In this paper we determine $\mathsf{c}_0(G)$ for more groups and our main result is the following.

Theorem 1.1. *Let G be a finite abelian group, and let p be the smallest prime dividing $|G|$. Then, $\mathsf{c}_0(G) = m(G)$ if one of the following conditions holds.*

(1) *G is cyclic;*
(2) *$|G|$ is even;*
(3) $\mathsf{r}(G) \geq 5$;
(4) $\mathsf{r}(G) \in \{3, 4\}$ *and $p \geq 17$;*
(5) $\mathsf{r}(G) \geq 2$ *and G is a p-group except $G = C_p \oplus C_{p^n}$ with $n \geq 2$.*

2. Preliminaries

Let G be an additive finite abelian group. A sequence S over G will be written in the form

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g \in G} g^{\mathsf{v}_g(S)}\,, \quad \text{with } \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ for all } g \in G\,,$$

and we call

$$|S| = \ell \in \mathbb{N}_0 \quad \text{the } \textit{length} \text{ and } \quad \sigma(S) = \sum_{i=1}^{\ell} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G \quad \text{the } \textit{sum} \text{ of } S\,.$$

Let $\mathrm{supp}(S) = \{g \in G : \mathsf{v}_g(S) > 0\}$. Define

$$\sum(S) = \{\sigma(T) : \ 1 \neq T \mid S\},$$

where $T \mid S$ means T is a subsequence of S, and 1 denotes the empty sequence.

We say that S is a zero-sum sequence if $\sigma(S) = 0$.

We say that a subset $A \subset G \setminus \{0\}$ is a 2-zero-sum free $|A|$-subset if A contains no two distinct elements with sum zero.

Let $A \subset \mathrm{supp}(S)$ be a subset of the maximal cardinality such that A is 2-zero-sum free. Define

$$|\mathrm{supp}^+(S)| = |A|.$$

Let $\mathsf{D}(G)$ denote the Davenport constant of G, which is defined as the smallest integer t such that, every sequence S over G of length $|S| \geq t$ contains a nonempty zero-sum subsequence.

For every subset A of G, denote by $\langle A \rangle$ the subgroup generated by A. Let $\mathrm{st}(A) = \{g \in G : g + A = A\}$. Then $\mathrm{st}(A)$ is the maximal subgroup H of G with $H + A = A$. We need the following well known Kneser's theorem. For the detailed proofs, the readers can refer to [6, 8, 9].

Lemma 2.1. *(Kneser) Let $A_1, \ldots, A_r$ be finite nonempty subsets of an abelian group, and let $H = \mathrm{st}(A_1 + \cdots + A_r)$. Then,*

$$|A_1 + \cdots + A_r| \geq |A_1 + H| + \cdots + |A_r + H| - (r-1)|H|.$$

Lemma 2.2. $\mathsf{c}_0(G) \geq m(G)$ *for every finite abelian group G.*

Proof. If G is cyclic then $m(G) = |G|$ by the definition. Let g be a generating element of G and $S = g^{|G|-1}$. Then S is regular and $0 \notin \sum(S)$. Therefore, $\mathsf{c}_0(G) \geq |S| + 1 = m(G)$.

If $G = C_p \oplus C_{\frac{|G|}{p}}$ with $p \mid \frac{|G|}{p}$, where p is the smallest prime dividing $|G|$, then $m(G) = \frac{|G|}{p} + p - 1$. Let $G = \langle e_1 \rangle \oplus \langle e_2 \rangle$ with $\mathrm{ord}(e_1) = p$ and $\mathrm{ord}(e_2) = \frac{|G|}{p}$. Let $S = e_1^{p-1} e_2^{\frac{|G|}{p}-1}$. Then, S is regular and $0 \notin \sum(S)$. Therefore, $\mathsf{c}_0(G) \geq |S| + 1 = m(G)$.

For all the other cases we have $m(G) = \frac{|G|}{p} + p - 2$. Let H be a subgroup of G with $|H| = \frac{|G|}{p}$, and let $g \in G \setminus H$. Take any $p-2$ distinct elements $h_1, \cdots, h_{p-2}$ from H. Let $S = (H \setminus \{0\}) \cup \{g + h_1, \cdots, g + h_{p-2}\}$. Then, S is a subset of G and therefore is a regular sequence over G. But $(-g + H) \cap \sum(S) = \emptyset$. Therefore, $\mathsf{c}_0(G) \geq |S| + 1 = m(G)$. □

The following result is crucial in the proof of Theorem 1.1.

Lemma 2.3. *Let G be a finite abelian group, and let p be the smallest prime dividing $|G|$. Let S be a regular sequence over G of length $|S| \geq \max\{\frac{|G|}{p} + p - 2, \mathsf{D}(G)\}$. If $\sum(S) \neq G$ then,*

(1) $\mathrm{st}(\sum(S)) = \{0\}$,
(2) $\mathrm{st}(\{0\} \cup \sum(T)) = \{0\}$ *and* $|\{0\} \cup \sum(T)| \geq |T| + 1$ *hold for every nonempty subsequence T of S.*

Proof. Write $S = g_1 \cdot \ldots \cdot g_\ell$. Since S is regular, $g_i \neq 0$ for all $1 \leq i \leq \ell$. Let $A_i = \{0, g_i\}$ for every $i \in [1, \ell]$. From $|S| \geq \max\{\frac{|G|}{p} + p - 2, \mathsf{D}(G)\} \geq \mathsf{D}(G)$, we know that $0 \in \sum(S)$. It follows that

$$\sum(S) = A_1 + \cdots + A_\ell.$$

Let $H = \mathrm{st}(\sum(S))$. From $\sum(S) \neq G$, we know that $H \neq G$. Suppose that $H \neq \{0\}$. Then by Lemma 2.1 and the fact that $|S_H| \leq |H| - 1$, we have

$$\begin{aligned}
|\sum(S)| &\geq |A_1 + H| + \cdots + |A_\ell + H| - (\ell - 1)|H| \\
&\geq (\ell + 2 - |H|)|H| \geq (|G|/p + p - |H|)|H| \\
&\geq \min\left((|G|/p + p - p)p, (|G|/p + p - |G|/p)\frac{|G|}{p}\right) = |G|,
\end{aligned}$$

a contradiction. This proves that $\mathrm{st}(\sum(S)) = \{0\}$.

By renumbering if necessary we assume that $T = g_1 \cdot \ldots \cdot g_t$, where $t = |T| \in [1, \ell]$. Let

$$B = A_1 + \cdots + A_t$$

and

$$C = (A_{t+1} + \cdots + A_\ell) \cup \{0\}.$$

Then, $B = \{0\} \cup \sum(T)$ and $\sum(S) = B + C$. It follows that $\mathrm{st}(B) \subset st(\sum(S))$. Therefore, $\mathrm{st}(B) = \{0\}$.

Again by Lemma 2.1, we have $|\{0\} \cup \sum(T)| = |A_1 + \cdots + A_t| \geq |A_1| + \cdots + |A_t| - (t-1) = |T| + 1$. $\square$

Lemma 2.4. $\mathsf{c}_0(G) \leq |G|$ *holds for every finite abelian group.*

Proof. Let S be an arbitrary regular sequence over G of length $|S| = |G|$. It follows from Lemma 2.3 that $\sum(S) = G$. Hence, $\mathsf{c}_0(G) \leq |G|$. $\square$

Lemma 2.5. ([13]) *Let H and K be two finite abelian groups with $1 < |H| \big| |K|$, and let $G = H \oplus K$. Then,* $\mathsf{D}(G) \leq |H| + |K| - 1$.

We need the following well known results on Davenport constant.

Lemma 2.6. ([13]) *Let p be a prime. Then,*

(1) $\mathsf{D}(C_p \oplus C_p \oplus C_p) = 3p - 2$*;*

(2) $\mathsf{D}(C_n) = n$*.*

(3) *If $G = C_{n_1} \oplus C_{n_2}$ with $1 < n_1 \mid n_2$ then $D(G) = n_1 + n_2 - 1$.*

Lemma 2.7. *If G is a finite abelian group then* $\mathsf{D}(G) \leq m(G)$*.*

Proof. Let $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$. Let p be the smallest prime dividing $|G|$.

If $r = 1$ then $\mathsf{D}(G) = |G| = m(G)$ by Lemma 2.6.

If $r = 2$ then $\mathsf{D}(G) = n_1 + n_2 - 1 = \frac{|G|}{n_1} + n_1 - 1$ by Lemma 2.6. Since p is the smallest prime dividing $|G|$, we have $m(G) \leq \frac{|G|}{p} + p - 1 \leq \frac{|G|}{n_1} + n_1 - 1 = \mathsf{D}(G)$.

If $r \geq 4$ then by Lemma 2.5 we derive that $\mathsf{D}(G) \leq \frac{|G|}{n_1 n_2} + n_1 n_2 - 1$ taking $H = C_{n_1} \oplus C_{n_2}$ and $K = C_{n_3} \oplus \cdots \oplus C_{n_r}$. Therefore, $m(G) = \frac{|G|}{p} + p - 2 < \frac{|G|}{n_1 n_2} + n_1 n_2 - 1 \leq \mathsf{D}(G)$. Now it remains to check the case that

$$r = 3.$$

If $p \neq n_2$ then $n_2 > p$. Taking $H = C_{n_2}$ and $K = C_{n_1} \oplus C_{n_3}$ in Lemma 2.5, we obtain that $D(G) \le \frac{|G|}{n_2} + n_2 - 1 \le \frac{|G|}{p} + p - 2 = m(G)$. So, we may assume that

$$n_1 = n_2 = p.$$

Write $n_3 = pu$. In this case we want to prove that

$$\mathsf{D}(G) \le (3p-2)u.$$

Let S be a sequence over G of length $|S| = (3p-2)u$. We need to show that S contains a nonempty zero-sum subsequence.

Let $\varphi : G = C_p \oplus C_p \oplus C_{pu} \to C_u$ be the natural homomorphism with $\ker(\varphi) = C_p \oplus C_p \oplus C_p$ (up to isomorphism). To apply $\mathsf{D}(\varphi(G)) = \mathsf{D}(C_u) = u$ on $\varphi(S)$ repeatedly, we can get a decomposition $S = S_1 \cdot \ldots \cdot S_{3p-2} \cdot S'$ with

$$|S_i| =\in [1, u],\ \sigma(S_i) \in \ker(\varphi) \text{ for every } i \in [1, 3p-2].$$

Applying $\mathsf{D}(\ker(\varphi)) = \mathsf{D}(C_p \oplus C_p \oplus C_p) = 3p - 2$ to the sequence $\sigma(S_1) \cdot \ldots \cdot \sigma(S_{3p-2})$ we obtain that, there is a nonempty subset $I \subset [1, 3p-2]$ such that $\sum_{i \in I} \sigma(S_i) = 0$. Now the sequence $\prod_{i \in I} S_i$ is a nonempty zero-sum subsequence of S. This proves that $\mathsf{D}(G) \le (3p-2)u$. Therefore, $\mathsf{D}(G) \le (3p-2)u \le p^2 u < p^2 u + p - 2 = m(G)$. □

3. Proof of Theorem 1.1 (1) and (2)

Proof of Theorem 1.1 (1). The result follows from Lemma 2.2 and Lemma 2.4. □

To prove Conclusion (2) of Theorem 1.1 we need the following technical result.

Lemma 3.1. *If A is a 2-zero-sum free subset of 3 elements from an abelian group, then either $|\sum(A) \setminus \{0\}| \ge 6$ or A contains some element with order two.*

Proof. Let $A = \{a, b, c\}$. If $a + b + c \neq 0$ then the result has been proved in [6, Proposition 5.3.2]. So we may assume that

$$a + b + c = 0.$$

Clearly, $a + b, a + c$ and $b + c$ are pairwise distinct nonzero elements. So, it suffices to prove that

$$\{a, b, c\} \cap \{a + b, a + c, b + c\} = \emptyset.$$

Assume to the contrary that, $\{a, b, c\} \cap \{a + b, a + c, b + c\} \neq \emptyset$. By renumbering we may assume that $a \in \{a + b, a + c, b + c\}$, which forces that $a = b + c$. This together with $a + b + c = 0$ gives that $2a = 0$. □

Proof of Theorem 1.1 (2). Let $n = |G|$. From Conclusion (1) of this theorem we may assume that

$$\mathsf{r}(G) \geq 2.$$

By Lemma 2.2, it suffices to prove $\mathsf{c}_0(G) \leq m(G)$. Let S be a regular sequence over G of length $|S| = m(G)$. We need to show that

$$\sum(S) = G.$$

Assume to the contrary that

$$\sum(S) \neq G.$$

By Lemma 2.3 we have

$$\mathrm{st}(\sum(S)) = \{0\}.$$

If there is some $g \in \mathrm{supp}(S)$ such that $2g = 0$, then $0 \neq g \in \mathrm{st}(\sum(S)) = \{0\}$ since $\sum(S) = \{0, g\} + \big(\sum(Sg^{-1}) \cup \{0\}\big)$ and $g + \{0, g\} = \{0, g\}$, a contradiction. So, every element $g \in \mathrm{supp}(S)$ satisfies that

$$2g \neq 0.$$

Now we distinguish several cases.

Case 1. $\max\{\mathsf{v}_g(S) + \mathsf{v}_{-g}(S) : g \in G\} \leq \frac{n}{6}$. Let $t \geq 0$ be the maximal integer such that S has a factorization

$$S = A_1 \cdots A_t T$$

with A_i is a 2-zero-sum free 3-subset of G for every $i \in [1, t]$.

We fix a factorization of S above so that $|\mathrm{supp}^+(T)|$ attains the maximal possible value. Clearly,

$$|\mathrm{supp}^+(T)| \leq 2.$$

We claim that

$$\mathsf{v}_g(T) + \mathsf{v}_{-g}(T) \leq 1$$

holds for every $g \in G$.

Assume to the contrary that $\mathsf{v}_h(T) + \mathsf{v}_{-h}(T) \geq 2$ for some $h \in G$. We may assume that $\mathsf{v}_h(T) \geq 1$. Since A_1 is a 2-zero-sum free 3-set and $|\mathrm{supp}^+(T)| \leq 2$, we can choose some $x \in A_1$ such that neither x nor $-x$ occurs in T. We assert that

$$A_1 \cap \{h, -h\} \neq \emptyset.$$

Assume to the contrary that A_1 contains neither h nor $-h$. Let $A_1' = (A_1 \setminus \{x\}) \cup \{h\}$ and $T' = Txh^{-1}$. Then we obtains a factorization

$$S = A_1' A_2 \cdots A_t T'$$

with $A_1', A_2, \cdots, A_t$ are all 2-zero-sum free 3-subsets of G but $|\mathrm{supp}^+(T')| > |\mathrm{supp}^+(T)|$, a contradiction. Therefore, $A_1 \cap \{h, -h\} \neq \emptyset$. Similarly, $A_i \cap \{h, -h\} \neq \emptyset$ for every $i \in [2, t]$. It follows that

$$\max\{\mathsf{v}_g(S) + \mathsf{v}_{-g}(S) : g \in G\} \geq t + \frac{|T|}{|\mathrm{supp}^+(T)|} \geq t + \frac{|T|}{2}.$$

Note that $3t + |T| = |S| \geq \frac{n}{2}$. Therefore, $t + \frac{|T|}{3} \geq \frac{n}{6}$. Hence, $\max\{\mathsf{v}_g(S) + \mathsf{v}_{-g}(S) : g \in G\} \geq t + \frac{|T|}{2} > t + \frac{|T|}{3} \geq \frac{n}{6}$, a contradiction. This proves the claim. It follows that T is a subset of G and

$$|T| = |\mathrm{supp}(T)| = |\mathrm{supp}^+(T)| \leq 2.$$

Let $B_i = \{0\} \cup \sum(A_i)$ for every $i \in [1, t]$, and let $B = \{0\} \cup \sum(T)$. Then,

$$B_1 + \cdots + B_t + B = \sum(S).$$

From Lemma 3.1 we get that $|B_i| \geq 7$ for every $i \in [1, t]$. Since $\mathrm{st}(\sum(S)) = \{0\}$, by Lemma 2.1 we obtain that

$$|B_1 + \cdots + B_t + B| \geq |B_1| + \cdots + |B_t| + |B| - t \geq 6t + |B|.$$

Since $|T| = |\mathrm{supp}(T)| \leq 2$, T is a subset of G. It is easy to see that $|B| \geq 2|T|$. Note that $\sum(S) \neq G$. So we have

$$\begin{aligned} n - 1 &\geq |\sum(S)| = |B_1 + \cdots + B_t + B| \geq |B_1| + \cdots + |B_t| + |B| - t \\ &\geq 6t + |B| \geq 6t + 2|T| = 2|S| \geq n, \end{aligned}$$

a contradiction.

Case 2. $\max\{\mathsf{v}_g(S) + \mathsf{v}_{-g}(S) : g \in G\} > \frac{n}{6}$.

We first assume that

$$n \in [2, 11].$$

Since $\mathsf{r}(G) \geq 2$ we have that

$$n \in \{4, 8\}.$$

If $n = 8$ then $G \in \{C_2^3, C_2 \oplus C_4\}$. Since S contains no element of order two, it follows that $G = C_2 \oplus C_4$. Now $|S| = m(G) = 5$. Let $x_1, -x_1, x_2, -x_2$ be the only four elements of order four in G. Then, $\mathsf{v}_g(S) + \mathsf{v}_{-g}(S) \geq 3$ for some element $g \in \{x_1, x_2\}$. Let $K = \langle g \rangle$. By Lemma 2.3 $|\{0\} \cup \sum(S_K)| \geq |S_K| + 1 \geq 4 = |K|$. Therefore, $\{0\} \cup \sum(S_K) = K$ and hence $K = \mathrm{st}(\{0\} \cup \sum(S_K)) \subseteq \mathrm{st}(\sum(S)) = \{0\}$, a contradiction.

If $n = 4$ then $G = C_2 \oplus C_2$. Hence every term of S is of order two, a contradiction.

From now on we suppose that

$$|G| = n \geq 12. \tag{3.1}$$

Choose $h \in G$ such that $|S_{\langle h \rangle}|$ attains the maximal possible value. Then

$$|S_{\langle h \rangle}| \geq \max\{\mathsf{v}_g(S) + \mathsf{v}_{-g}(S) : g \in G\} \geq \frac{n+1}{6}.$$

Let

$$H = \langle h \rangle.$$

It follows that

$$|S_H| \geq 3.$$

Let

$$\overline{g} = g + H$$

for every $g \in G$. We distinguish two subcases:

Subcase 2.1. For any two terms g_1, g_2 of S with $g_1 g_2 | S$ we have $|\{\overline{0}\} \cup \sum(\overline{g_1}\ \overline{g_2})| \leq 2$. Then, for any two terms g_1, g_2 of SS_H^{-1} we have $\overline{g_1} = \overline{g_2}$ and $2\overline{g_1} = \overline{0}$. Therefore, for any term g_0 of SS_H^{-1} we have

$$\langle \mathrm{supp}(S) \rangle = \langle h, g_0 \rangle.$$

Since S is regular, $|\langle \mathrm{supp}(S) \rangle| \geq |S| + 1 > \frac{n}{2}$. Therefore,

$$G = \langle \mathrm{supp}(S) \rangle = \langle h, g_0 \rangle.$$

Since $2g_0 \in H = \langle h \rangle$, we infer that $|G| = 2|H|$ and $G = C_2 \oplus C_{n/2}$. Hence we have

$$|S| = m(G) = \frac{n}{2} + 1.$$

Let

$$T = g_0 S_H.$$

Let $t \geq 0$ be the maximal integer such that ST^{-1} has a factorization

$$ST^{-1} = A_1 \cdots A_t W$$

with A_i is a 2-zero-sum free 3-subset of G for every $i \in [1, t]$.

We fix a factorization of ST^{-1} above so that $|\mathrm{supp}^+(W)|$ attains the maximal possible value.

Clearly,

$$|\mathrm{supp}^+(W)| \leq 2.$$

Then S has a factorization

$$S = A_1 \cdots A_t W T$$

where $t \ge 0$, A_i is a 2-zero-sum free 3-subset of G, and W is a subsequence of S which contains no any 2-zero-sum free 3-subset of G. It follows that

$$3t + |W| + |T| = |S| \ge \frac{n}{2}$$

and

$$W | x_1^{\mathsf{v}_{x_1}(S)}(-x_1)^{\mathsf{v}_{-x_1}(S)} x_2^{\mathsf{v}_{x_2}(S)}(-x_2)^{\mathsf{v}_{-x_2}(S)}$$

for some distinct elements $x_1, x_2 \in G$.

Let $B_i = \{0\} \cup \sum(A_i)$ for every $i \in [1,t]$, $C = \{0\} \cup \sum(W)$, and let $D = \{0\} \cup \sum(T)$. From Lemma 3.1 we get that $|B_i| \ge 7$. It then follows from Lemmas 2.1 and 2.3 that

$$\begin{aligned} n-1 &\ge |\sum(S)| \ge |B_1| + \cdots + |B_t| + |C| + |D| - t - 1 \\ &\ge 7t + (|W|+1) + 2|T| - t - 1 = 6t + 2|W| + 2|T| - |W| \\ &= 2|S| - |W| = n + 2 - |W|. \end{aligned}$$

This gives that

$$|W| \ge 3.$$

Write $W = W_1 W_2$ with $W_1 | x_1^{\mathsf{v}_{x_1}(S)}(-x_1)^{\mathsf{v}_{-x_1}(S)}$ and $W_2 | x_2^{\mathsf{v}_{x_2}(S)}(-x_2)^{\mathsf{v}_{-x_2}(S)}$. Without loss of generality we may assume that

$$|W_1| \ge |W_2| \ge 0.$$

Since $|W_1| \ge \frac{|W|}{2} \ge \frac{3}{2}$, by the maximality of S_H, there is some element $y | S_H$ such that $y \notin \langle x_1 \rangle$. Let $U = W_1 y$ and let $T' = T y^{-1}$ and we obtain a factorization of S

$$S = A_1 \cdots A_t U W_2 T'.$$

Let $C_1 = \{0\} \cup \sum(U)$, $C_2 = \{0\} \cup \sum(W_2)$, and $D' = \{0\} \cup \sum(T')$. Similarly to above we obtain that

$$\begin{aligned} n-1 &\ge |\sum(S)| \ge |A_1| + \cdots + |A_t| + |C_1| + |C_2| + |D'| - t - 2 \\ &\ge 7t + 2|U| + |W_2| + 1 + 2|T'| - t - 2 = 2(3t + |U| + |W_2| + |T'|) - 1 - |W_2| \\ &= 2|S| - 1 - |W_2| = n + 1 - |W_2|. \end{aligned}$$

This gives that

$$|W_2| \ge 2.$$

By the maximality of S_H and $|S_H| \ge 3$, there is an element $z | S_H y^{-1}$ such that $z \notin \langle x_2 \rangle$. Let $V = z W_2$ and $T'' = T' z^{-1} = T(yz)^{-1}$. Then S has a factorization

$$S = A_1 \cdots A_t U V T''.$$

Let $C_2' = \{0\} \cup \sum(V)$ and $D'' = \{0\} \cup \sum(T'')$. Similarly to above we have

$$\begin{aligned} n-1 \geq |\sum(S)| &\geq |A_1| + \cdots + |A_t| + |C_1| + |C_2'| + |D''| - t - 2 \\ &\geq 7t + 2|U| + 2|V| + 2|T''| - t - 2 = 2|S| - 2 = n, \end{aligned}$$

a contradiction.

Subcase 2.2. There are two terms g_1, g_2 of S such that $g_1 g_2 | S$ and $|\{\overline{0}\} \cup \sum(\overline{g_1}\ \overline{g_2})| \geq 3$. Let $T = g_1 g_2 S_H$. Now S has a factorization

$$S = A_1 \cdots A_t W T$$

where $t \geq 0$, A_i is a 2-zero-sum free 3-subset of G, and W is a subsequence of S which contains no any 2-zero-sum free 3-subset of G. It follows that

$$3t + |W| + |T| = |S| \geq \frac{n}{2}$$

and

$$W | x_1^{\mathsf{v}_{x_1}(S)} (-x_1)^{\mathsf{v}_{-x_1}(S)} x_2^{\mathsf{v}_{x_2}(S)} (-x_2)^{\mathsf{v}_{-x_2}(S)}$$

for some distinct elements $x_1, x_2 \in G$. Let $B_i = \{0\} \cup \sum(A_i)$ for every $i \in [1,t]$, $C = \{0\} \cup \sum(W)$, and let $D = \{0\} \cup \sum(T)$. Then, $B_1 + \cdots + B_t + C + D = \sum(S)$. Since $\mathrm{st}(\sum(S)) = \{0\}$, by Kneser's theorem we obtain that

$$\begin{aligned} &n - 1 \geq |\sum(S)| \geq |B_1| + \cdots + |B_t| + |C| + |D| - t - 1 \\ &\geq 7t + (|W| + 1) + (3|T| - 3) - t - 1 = 6t + 2|W| + 2|T| + (|T| - 3 - |W|) \\ &= 2|S| + (|T| - 3 - |W|) \geq n + (|T| - 3 - |W|). \end{aligned}$$

This gives that

$$|W| \geq |T| - 2 \geq 3.$$

Write $W = W_1 W_2$ with $W_1 | x_1^{\mathsf{v}_{x_1}(S)} (-x_1)^{\mathsf{v}_{-x_1}(S)}$ and $W_2 | x_2^{\mathsf{v}_{x_2}(S)} (-x_2)^{\mathsf{v}_{-x_2}(S)}$. Without loss of generality we may assume that

$$|W_1| \geq |W_2| \geq 0.$$

Since $|W_1| \geq \frac{|W|}{2} \geq \frac{3}{2}$, by the maximality of S_H, there is some element $y | S_H$ such that $y \notin \langle x_1 \rangle$. Let $U = W_1 y$ and let $T' = T y^{-1}$ and we obtain a factorization of S

$$S = A_1 \cdots A_t U W_2 T'.$$

Let $C_1 = \{0\} \cup \sum(U)$, $C_2 = \{0\} \cup \sum(W_2)$, and $D' = \{0\} \cup \sum(T')$. Similarly to above we obtain that

$$\begin{aligned} n - 1 \geq |\sum(S)| &\geq |B_1| + \cdots + |B_t| + |C_1| + |C_2| + |D'| - t - 2 \\ &\geq 7t + 2|U| + |W_2| + 1 + 3|T'| - 3 - t - 2 = 6t + 2|W_1| + |W_2| + 3|T| - 5 \\ &= 6t + 2|W| + 2|T| + (|T| - 5 - |W_2|) \geq n + (|T| - 5 - |W_2|). \end{aligned}$$

This gives that

$$|W_2| \geq |T| - 4 \geq 1.$$

Therefore

$$|W_1| \geq 2, |W_2| \geq 1.$$

By the maximality of S_H, there is some element $y|S_H$ such that $y \notin \langle x_2 \rangle$. Let $U = W_2 y$ and let $T' = Ty^{-1}$. Again by the maximality of S_H and $|S_H| \geq 3$, there is an element $z|S_H y^{-1}$ such that $z \notin \langle x_1 \rangle$. Let $V = zW_1$ and $T'' = T'z^{-1} = T(yz)^{-1}$. Then S has a factorization

$$S = A_1 \cdots A_t UVT''.$$

Let $C_1' = \{0\} \cup \sum(U), C_2' = \{0\} \cup \sum(V)$ and $D'' = \{0\} \cup \sum(T'')$. Similarly to above we have

$$\begin{aligned} n - 1 \geq |\sum(S)| &\geq |B_1| + \cdots + |B_t| + |C_1| + |C_2'| + |D''| - t - 2 \\ &\geq 7t + 2|U| + 2|V| + 3|T''| - 3 - t - 2 \\ &= 6t + 2|W| + 2|T| + (|T| - 7) = 2|S| + (|T| - 7) \geq 2m(G) + (|T| - 7). \end{aligned}$$

This gives that $|T| \leq n + 6 - 2m(G)$. Therefore,

$$\frac{n+1}{6} \leq |S_H| \leq n + 4 - 2m(G). \tag{3.2}$$

If $m(G) \geq \frac{n}{2} + 1$ then $n \leq 11$ follows from (3.2), a contradiction on (3.1). Therefore,

$$m(G) = \frac{n}{2}. \tag{3.3}$$

It follows from (3.2) that $n \leq 23$. Since n is even, we have

$$n \leq 22. \tag{3.4}$$

By (3.1), (3.3) and (3.4), to complete the proof of this subcase it remains to consider the cases

$$n \in [12, 22] \text{ and } m(G) = \frac{n}{2}. \tag{3.5}$$

Since $\mathsf{r}(G) \geq 2$ we have that $n \notin \{14, 22\}$. So, it remains to check that

$$n \in \{12, 16, 18, 20\}.$$

If $n \in \{12, 20\}$ then $G = C_2 \oplus C_t$ with $t = 6$ or 10. Hence we get $m(G) = \frac{n}{2} + 1$. This is not any case listed in (3.5).

If $n = 18$ then $G = C_3 \oplus C_6$. Now we have $|S| \geq m(G) = 9$, $|S_H| \geq 4$, and there are two terms g_1, g_2 of S such that $g_1 g_2 | S S_H^{-1}$ and $|\{\overline{0}\} \cup \sum(\overline{g_1}\ \overline{g_2})| \geq 3$.

Let $T = g_1 g_2 S_H$. Then $|T| \geq 6$ and $|ST^{-1}| \leq 3$. Let $A = ST^{-1}$. Then S has a factorization

$$S = AT.$$

Let $B = \{0\} \cup \sum(A)$, and $D = \{0\} \cup \sum(T)$. Then, $B + D = \sum(S)$. So by Lemmas 2.1 and 2.3, we have that

$$|\sum(S)| \geq |B| + |D| - 1 \geq |A| + 1 + (3|T| - 3) - 1 = |S| + 2|T| - 3 \geq 18.$$

Therefore $\sum(S) = G$, a contradiction.

If $n = 16$ then $G \in \{C_2^4, C_2^2 \oplus C_4, C_4^2, C_2 \oplus C_8\}$. Since $m(G) = \frac{n}{2}$, we may assume that $G \neq C_2 \oplus C_8$. Therefore, $G \in \{C_2^4, C_2^2 \oplus C_4, C_4^2\}$. If $G = C_2^4$ then every term of S is of order two, a contradiction. So, $G = C_2^2 \oplus C_4$ or $G = C_4^2$. Since $\max\{\mathsf{v}_g(S) + \mathsf{v}_{-g}(S) : g \in G\} \geq \frac{n+1}{6} = \frac{16+1}{6}$, we have that $\mathsf{v}_g(S) + \mathsf{v}_{-g}(S) \geq 3$ for some element g of order 4. Let $K = \langle g \rangle$. By Lemma 2.3 $|\{0\} \cup \sum(S_K)| \geq |S_K| + 1 \geq 4 = |K|$. Therefore, $\{0\} \cup \sum(S_K) = K$ and hence $K = \mathrm{st}(\{0\} \cup \sum(S_K)) \subseteq \mathrm{st}(\sum(S)) = \{0\}$, a contradiction. This completes the proof. $\square$

4. Proof of Theorem 1.1 (3) and (4)

In this section we shall prove Conclusions (3) and (4) of Theorem 1.1 by employing group algebras as a tool.

Let $G = \bigoplus_{i=1}^{r} C_{n_i}$ with $1 < n_1 | n_2 | \ldots | n_r$, and let K be a field. The *group algebra* $K[G]$ is a vector space over K with K-basis $\{X^g \mid g \in G\}$ (built with a symbol X), where multiplication is defined by

$$\Big(\sum_{g \in G} a_g X^g\Big)\Big(\sum_{g \in G} b_g X^g\Big) = \sum_{g \in G} \Big(\sum_{h \in G} a_h b_{g-h}\Big) X^g.$$

More precisely, $K[G]$ consists of all formal expression of the form $f = \sum_{g \in G} c_g X^g$ with $c_g \in K$. For more detailed background information, we refer the readers to [6, 7, 8].

Choose a prime q so that $q \equiv 1 \pmod{n_r}$. Consider the group algebra $\mathbb{F}_q[G]$. For any $\alpha \in \mathbb{F}_q[G]$, denote by L_α the set of elements $g \in G$ such that $\alpha(a - X^g) = 0$ holds for some $a \in \mathbb{F}_q$.

Lemma 4.1. 1. *For any $\alpha \in \mathbb{F}_q[G]$, L_α is a subgroup of G.*
2. *If $\alpha \neq 0$ and $L_\alpha = G$, then $\alpha = \sum\limits_{g \in G} a_g X^g$ with $0 \neq a_g \in \mathbb{F}_q$ holds for all $g \in G$.*
3. *Let $S = g_1 \cdot \ldots \cdot g_l$ be a sequence over G. If there exist $a_1, \ldots, a_t \in \mathbb{F}_q^*$ such that $\alpha = \prod\limits_{i=1}^{l} (a_i - X^{g_i}) \neq 0$ and $L_\alpha = G$, then $G \setminus \{0\} \subset \sum(S)$.*

Proof. Conclusion 1 and 2 has been proved in [4, Lemma 3.1]. Here we only give a proof of Conclusion 3. Let $0 \neq \alpha = \prod_{i=1}^{l}(a_i - X^{g_i}) = \sum_{g \in G} a_g X^g$. By Conclusion 2, $a_g \neq 0$ for all $g \in G$. This implies that $g \in \sum(S)$ for all $g \in G \setminus \{0\}$. Therefore, $G \setminus \{0\} \subset \sum(S)$. $\square$

Lemma 4.2. ([4]) *Let S be a sequence of elements in G of length $l \geq n_r(1 + \log n_1 \cdots n_{r-1})$. Suppose that S contains at least one non-zero term. Then, one can find a subsequence $T = g_1 \cdot \ldots \cdot g_t$ of S of length $t \leq n_r(1 + \log n_1 \cdots n_{r-1}) - 1$ and $a_1, \ldots, a_t \in \mathbb{F}_q^*$ such that*

$$\alpha = (a_1 - X^{g_1}) \cdots (a_t - X^{g_t}) \neq 0$$

and all terms of ST^{-1} are in L_α.

Proof. The lemma has been proved in [4, Lemma 3.2]. But there is some typo in [4], in which $\log n/\log m$ has to be replaced by $\log(n/m)$. $\square$

Let $a \neq 0$ be a real number, and let $r \geq 3$ be an integer. Define the following function on r variables $y_1, \ldots, y_r$ by

$$f_a(y_1, \ldots, y_r) := \frac{y_1 \cdots y_r}{a} + a - 2 - 2y_r(1 + \log y_1 \cdots y_{r-1}) - \frac{y_1 \cdots y_r}{a^2}.$$

Lemma 4.3. *If $y_i \geq a \geq 3$ for all $i \in [1, r]$ then, $f_a(y_1, \ldots, y_r) \geq 0$ provided that one of the following conditions holds.*

(1) $r \geq 5$;
(2) $r \in \{3, 4\}$ *and* $a \geq 17$.

Proof. First, we compute the partial derivatives of $f_a(y_1, ..., y_r)$. By straightforward calculations, we have

$$\frac{\partial f_a}{\partial y_i} = \frac{y_1 \cdots y_r}{a^2 y_i}(a-1) - 2\frac{y_r}{y_i} \geq \frac{y_r}{y_i}(\frac{y_1 \cdots y_{r-1}}{a^2}(a-1) - 2) \geq \frac{y_r}{y_i}(a-3) \geq 0$$

for $1 \leq i \leq r-1$, and

$$\frac{\partial f_a}{\partial y_r} = \frac{y_1 \cdots y_{r-1}}{a^2}(a-1) - 2 - 2\log(y_1 \cdots y_{r-1}).$$

It is easy to see that $g(x) = \frac{x}{a^2}(a-1) - 2 - 2\log x$ is increasing when $x \geq a^2$.

(1). If $r \geq 5$, then

$$\begin{aligned}\frac{\partial f_a}{\partial y_r} &= \frac{y_1 \cdots y_{r-1}}{a^2}(a-1) - 2 - 2\log(y_1 \cdots y_{r-1}) \\ &\geq a^{r-3}(a-1) - 2 - 2(r-1)\log a \\ &\geq a^2(a-1) - 2 - 8\log a > 0.\end{aligned}$$

So we have

$$\begin{aligned} f_a(y_1, ..., y_r) &\geq f_a(a, ..., a) = a^{r-2}(a-1) + a - 2 - 2a(1 + \log a^{r-1}) \\ &\geq a^3(a-1) + a - 2 - 2a - 8a \log a \\ &= a(a^2(a-1) - 2 - 8 \log a) + a - 2 \geq a - 2 \geq 1. \end{aligned}$$

(2). If $a \geq 17$ and $r \in \{3, 4\}$, then

$$\begin{aligned} \frac{\partial f_a}{\partial y_r} &= \frac{y_1 \cdots y_{r-1}}{a^2}(a-1) - 2 - 2\log(y_1 \cdots y_{r-1}) \\ &\geq a - 3 - 4 \log a > 0 \end{aligned} \tag{4.1}$$

since $f(x) = x - 3 - 4 \log x$ is an increasing function of $x \geq 17$. We get $f_a(y_1, ..., y_r) \geq f_a(a, ..., a) = a^{r-2}(a-1) + a - 2 - 2a(1 + \log a^{r-1}) \geq a(a-1)+a-2-2a-4a \log a$, since $f_a(a, ..., a) = a^{r-2}(a-1)+a-2-2a(1+\log a^{r-1})$ is an increasing function of $r \geq 3$. By (4.1), we obtain that

$$f_a(y_1, ..., y_r) \geq f_a(a, a, a) = a(a - 3 - 4 \log a) + a - 2 \geq a - 2 \geq 15$$

as desired. This completes the proof. □

Proof of Theorem 1.1 (3) and (4). Suppose that $G = C_{n_i} \oplus \cdots \oplus C_{n_r}$ where $1 < n_1 | n_2 | \ldots | n_r$. By Lemma 2.2 and Conclusion (2) of Theorem 1.1, it suffices to prove that $\mathsf{c}_0(G) \leq m(G) = \frac{|G|}{p} + p - 2$ for $p \geq 3$. To do this, let S be a regular sequence over G of length $|S| = \frac{|G|}{p} + p - 2$. We need to prove that $\sum(S) = G$.

Assume to the contrary that

$$\sum(S) \neq G.$$

By Lemma 4.3, we can deduce that $|S| \geq n_r(1 + \log n_1 \cdots n_{r-1})$. Then by Lemma 4.2, one can find a subsequence $T = g_1 \cdot \ldots \cdot g_t$ of S with $t \leq n_r(1 + \log n_1 \ldots n_{r-1}) - 1$ and $a_1, \ldots, a_t \in \mathbb{F}_q^*$ such that

$$\alpha = (a_1 - X^{g_1}) \cdots (a_t - X^{g_t}) \neq 0$$

and all terms of ST^{-1} are in L_α.

Since S is regular, again by Lemma 4.3 we have

$$|L_\alpha| - 1 \geq |S_{L_\alpha}| \geq |ST^{-1}| \geq \frac{n_1 \cdots n_r}{p} + p - 2 - 2n_r(1 + \log n_1 \cdots n_{r-1}) \geq \frac{n_1 \cdots n_r}{p^2}.$$

Together with Lemma 4.1, we get that $|L_\alpha| = \frac{|G|}{p_1}$ for some prime divisor p_1 of $|G|$ with $p \leq p_1 < p^2$. It follows that L_α as a subgroup of G must be isomorphic to the group of the following form

$$\bigoplus_{i=1, i \neq i_0}^{r} C_{n_i} \bigoplus C_{n_{i_0}/p_1},$$

where $1 \leq i_0 \leq r$.

Let $L_\alpha = \bigoplus_{j=1}^{s} C_{m_j}$ with $1 < m_1 | \cdots | m_s$.

We claim that

$$m_s(1 + \log m_1 \cdots m_{s-1}) \le n_r(1 + \log n_1 \cdots n_{r-1}).$$

If $1 \le i_0 \le r-1$, then

$$m_s(1 + \log m_1 \cdots m_{s-1}) = n_r(1 + \log \frac{n_1 \cdots n_{r-1}}{p_1}) \le n_r(1 + \log n_1 \cdots n_{r-1}).$$

If $i_0 = r$, then

$$m_s(1 + \log m_1 \cdots m_{s-1}) \le m_s(1 + \log n_1 \cdots n_{r-1}) \le n_r(1 + \log n_1 \cdots n_{r-1}).$$

This proves the claim.

By lemma 4.3, we get $|ST^{-1}| \ge n_r(1+\log n_1 \cdots n_{r-1}) \ge m_s(1+\log m_1 \cdots m_{s-1})$. Since ST^{-1} is a sequence over L_α, by lemma 4.2, we can find a subsequence $S_1 = h_1 \cdot \ldots \cdot h_u$ of ST^{-1} with $u \le m_s(1 + \log m_1 \cdots m_{s-1}) - 1$ and $b_1, \ldots, b_u \in \mathbb{F}_q^*$ such that

$$\beta = (b_1 - X^{h_1}) \cdots (b_u - X^{h_u}) \ne 0$$

and all terms of $ST^{-1}S_1^{-1}$ are in L_β, where L_β denotes the set of elements $g \in L_\alpha$ such that $\beta(a - X^g) = 0$ holds for some $a \in \mathbb{F}_q^*$.

Since S is regular, by Lemma 4.3 we have

$$\begin{aligned} |L_\beta| - 1 &\ge |(ST^{-1})_{L_\beta}| \ge |ST^{-1}S_1^{-1}| \\ &\ge \frac{n_1 \cdots n_r}{p} + p - 2 - n_r(1 + \log n_1 \cdots n_{r-1}) - m_s(1 + \log m_1 \cdots m_{s-1}) \\ &\ge \frac{n_1 \cdots n_r}{p^2}. \end{aligned}$$

This implies $|L_\beta| = \frac{|G|}{p_1} = |L_\alpha|$. Hence $L_\beta = L_\alpha$. Since $\beta = \prod_{i=1}^{u}(b_i - X^{h_i})$, we deduce from Lemma 4.1 that $\{0\} \cup \sum(S_1) = L_\beta = L_\alpha$. Therefore, $L_\alpha = L_\beta = \mathrm{st}(\{0\} \cup \sum(S_1))$, a contradiction to Lemma 2.3. This completes the proof. $\square$

5. Proof of Theorem 1.1 (5)

Let p be a prime. In this section we shall prove Conclusion (5) of Theorem 1.1 by using group algebras as in Section 4.

Let $G = \bigoplus_{i=1}^{r} C_{p^{n_i}} = \bigoplus_{i=1}^{r} \langle e_i \rangle$, where $C_{p^{n_i}} = \langle e_i \rangle$ for $1 \le i \le r$ and $1 \le n_1 \le \cdots \le n_r$.

Consider the group algebra $\mathbb{F}_p[G]$ over $\mathbb{F}_p$. As a vector space over $\mathbb{F}_p$, $\mathbb{F}_p[G]$ has a basis

$$\Big\{ \prod_{i=1}^{r} (1 - X^{e_i})^{k_i} \;\Big|\; k_i \in [0, p^{n_i} - 1] \text{ for all } i \in [1, r] \Big\},$$

see for example [6]. So any $\alpha \in \mathbb{F}_p[G]$ can be written uniquely in the form $\alpha = \sum \sigma_{k_1,\ldots,k_r}(1-X^{e_1})^{k_1}\cdots(1-X^{e_r})^{k_r}$, $\sigma_{k_1,\ldots,k_r} \in \mathbb{F}_p$.

For any sequence $S = g_1 \cdot \ldots \cdot g_l$ over G, let

$$\prod(S) = \prod_{i=1}^{l}(1-X^{g_i}).$$

Let $g \in G$ and $a \in \mathbb{F}_p$. Since 1 is the only $\exp(G)$-th root in $\mathbb{F}_p$, the element $a - X^g$ is invertible in $\mathbb{F}_p[G]$ if and only if $a \neq 1$. Thus it follows that

$$\begin{aligned} L_\alpha &= \{g \in G : \text{there is an } a \in \mathbb{F}_p \text{ such that } \alpha(a - X^g) = 0\} \\ &= \{g \in G : \alpha(1-X^g) = 0\}. \end{aligned}$$

Lemma 5.1. ([11]) *Let S be a sequence over G. Then $L_{\prod(S)} = G$ if and only if $\prod(S) = \sigma\prod_{i=1}^r(1-X^{e_i})^{p^{n_i}-1}$ for some $\sigma \in \mathbb{F}_p$. In particular, if $|S| = \sum_{i=1}^r(p^{n_i}-1)$, then $\prod(S) = \sigma\prod_{i=1}^r(1-X^{e_i})^{p^{n_i}-1}$. Furthermore, if $\sigma \neq 0$ then $G \setminus \{0\} \subseteq \sum(S)$.*

Lemma 5.2. ([6, Proposition 5.5.8], [12]) *Let S be a sequence over G of length $|S| \geq \sum\limits_{i=1}^{r}(p^{n_i}-1)+1$. Then*

$$\prod(S) = 0.$$

Let a be a real number and let $r \geq 2$ be an integer. Define

$$f_a(y_1,...,y_r) := a^{\sum_{i=1}^r y_i - 1} + a - 2 - \sum_{i=1}^{r}(a^{y_i}-1) - \sum_{i=2}^{r}(a^{y_i}-1) - (a^{y_1-1}-1) - a^{\sum_{i=1}^r y_i - 2} + 3,$$

where $y_1,...,y_r$ are real variables.

Lemma 5.3. *Let $p \geq 3$ be a prime, and let $r \geq 2$ be an integer. Let $n_1,\ldots,n_r$ be positive integers.*

(1) *If $r \geq 3$, then $f_p(n_1,\ldots,n_r) \geq 0$.*
(2) *If $r = 2$ and $n_2 \geq n_1 \geq 2$, then $f_p(n_1,n_2) > 0$ except for the case $p = 3$ and $n_1 = 2$, in which case $f_p(n_1,n_2) = -4 < 0$.*

Proof. First, we compute the partial derivatives of $f_p(y_1,\ldots,y_r)$ at the point $(n_1,\ldots,n_r)$. By calculations, we have

$$\frac{\partial f_p}{\partial n_1} = p^{n_1-1}\log p\big(p^{\sum_{i=2}^r n_i - 1}(p-1) - p - 1\big) \geq p(p-2) - 1 > 0,$$

and for $2 \leq i \leq r$ we have that

$$\frac{\partial f_p}{\partial n_i} = p^{n_i-1}\log p\big(p^{\sum_{j=1,j\neq i}^r n_j - 1}(p-1) - 2p\big) \geq p(p-3) \geq 0$$

if either $r \geq 3$, or $r = 2$ and $n_2 \geq n_1 \geq 2$.

(1). If $r \geq 3$, then $f_p(n_1, ..., n_{r-1}, n_r) \geq f_p(1, ..., 1)$. Thus it remains to prove that $f_p(1, ..., 1) \geq 0$. It is easy to see that $g(r) := f_p(1, ..., 1) = p^{r-2}(p-1) - (2r-2)p + 2r$ is an increasing function of r, since $g'(r) = (p-1)(p^{r-2}\log p - 2) > 0$ when $p \geq 3$ and $r \geq 3$. So we get $f_p(1, ..., 1) \geq g(3) = (p-2)(p-3) \geq 0$ as desired.

(2). If $p \geq 5$, then we have

$$f_p(n_1, n_2) = p^{n_1+n_2-2}(p-1) - 2p^{n_2} - p^{n_1} - p^{n_1-1} + p + 5 \geq p + 5 > 0.$$

By calculations, we obtain that

(i) $f_3(2, n_2) = -4$ for all $n_2 \geq 2$, and

(ii) $f_3(n_1, n_2) \geq f_3(3, 3) = 80 > 0$ for any two integers n_1, n_2 with $n_2 \geq n_1 \geq 3$. This completes the proof. $\square$

Lemma 5.4. *Let p be a prime, and $n_1, \ldots, n_r$ be positive integers. Let $G = \bigoplus_{i=1}^{r} C_{p^{n_i}}$. If either $r \geq 3$, or $r = 2, n_2 \geq n_1 \geq 2$ and $(p, n_1) \neq (3, 2)$, then we have*

$$\mathsf{c}_0(G) = \frac{|G|}{p} + p - 2.$$

Proof. By Lemma 2.2, it suffices to prove that $\mathsf{c}_0(G) \leq m(G) = \frac{|G|}{p} + p - 2$. To do this, let S be a regular sequence over G of length $|S| = \frac{|G|}{p} + p - 2$. We need to prove that $\sum(S) = G$. Since $|S| \geq \mathsf{D}(G)$, by Lemma 2.7 we have

$$0 \in \sum(S).$$

Assume to the contrary that

$$\sum(S) \neq G.$$

Then by Lemma 2.3, we have $\mathrm{st}(\sum(S)) = \{0\}$. Let S_0 be the maximal subsequence of S such that $\prod(S_0) \neq 0$. By Lemma 5.2, we have that $|S_0| \leq \sum_{i=1}^{r}(p^{n_i} - 1)$. If $|S_0| = \sum_{i=1}^{r}(p^{n_i} - 1)$, then by Lemma 5.1 we have $G \setminus \{0\} \subset \sum(S_0)$. It follows from $0 \in \sum(S)$ that $\sum(S) = G$, a contradiction. Therefore,

$$|S_0| \leq \sum_{i=1}^{r}(p^{n_i} - 1) - 1.$$

Let $H = L_{\prod(S_0)}$ and $T = S{S_0}^{-1}$. By the maximality of S_0, we know that every term of T belongs to H and T is a regular sequence over the subgroup H of G. By Lemma 5.3 we obtain that

$$|H| - 1 \geq |S_H| \geq |S - S_0| \geq \frac{|G|}{p} + p - 2 - \sum_{i=1}^{r}(p^{n_i} - 1) \geq \frac{|G|}{p^2}.$$

Together with Lemma 5.1, we deduce $|H| = \frac{|G|}{p}$. Since H is a subgroup of G with $|H| = \frac{|G|}{p}$, H must be isomorphic to the group of the following form

$$\bigoplus_{i=1, i\neq i_0}^{r} C_{p^{n_i}} \bigoplus C_{p^{n_{i_0}-1}},$$

where $1 \leq i_0 \leq r$.

Since $n_1 \leq n_2 \leq \cdots \leq n_r$, we can easily deduce that

$$\begin{aligned} \mathsf{D}(H) - 1 &= \sum_{i=1, i\neq i_0}^{r} (p^{n_i} - 1) + (p^{n_{i_0}-1} - 1) \\ &\leq \sum_{i=2}^{r} (p^{n_i} - 1) + p^{n_1 - 1} - 1. \end{aligned} \tag{5.1}$$

Let S_1 be the maximal subsequence of T such that $\prod(S_1) \neq 0$. By Lemma 5.2, we have $|S_1| \leq \mathsf{D}(H) - 1$. If $|S_1| = \mathsf{D}(H) - 1$ then by Lemma 5.1 we get $\{0\} \cup \sum(S_1) = H$. Therefore, $H = \mathrm{st}\big(\{0\} \cup \sum(S_1)\big)$. But $|H| = |G|/p \geq p^2$, a contradiction to Lemma 2.3. Therefore,

$$|S_1| \leq \mathsf{D}(H) - 2.$$

Let $T_1 = TS_1^{-1} = S(S_0 S_1)^{-1}$ and let $N = L_{\prod(S_1)}$. By the maximality of S_1 we have that T_1 is a sequence over N. By (5.1) and Lemma 5.3 we obtain that $|T_1| \geq \frac{|G|}{p^2} - 1$. If $N = H$ then by Lemma 5.1 we have, $\{0\} \cup \sum(S_1) = H = \mathrm{st}(\{0\} \cup \sum(S_1))$, again a contradiction to Lemma 2.3. Therefore,

$$N \neq H.$$

But $|N| - 1 \geq |T| - |S_1| = |T_1| \geq |G|/p^2 - 1$. This forces that $|N| = |G|/p^2$. On the other hand, using Lemma 2.3, we have $|\{0\} \cup \sum(T_1)| \geq |T_1| + 1 \geq |G|/p^2 = |N|$. Hence $\{0\} \cup \sum(T_1) = N$, which implies that $N = \mathrm{st}(\{0\} \cup \sum(T_1))$. But $|N| = |G|/p^2 > 1$, a contradiction to Lemma 2.3. $\square$

In what follows, by using group algebras and the method used in Section 3 we determine $\mathsf{c}_0(G)$ for $G = C_{3^2} \bigoplus C_{3^n}$ with $n \geq 2$.

Lemma 5.5. *Let $G = C_{3^2} \bigoplus C_{3^n}$ with $n \geq 2$. Then*

$$\mathsf{c}_0(G) = 3^{n+1} + 1.$$

Proof. Let S be a regular sequence over G of length $|S| = m(G) = 3^{n+1} + 1$. We need to show $\sum(S) = G$. Assume to the contrary that,

$$\sum(S) \neq G.$$

Note that $|S| \geq \mathsf{D}(G)$. So we have

$$0 \in \sum(S). \tag{5.2}$$

Let S_1 be the maximal subsequence of S such that $\prod(S_1) \neq 0$. Clearly, $|S_1| \leq \mathsf{D}(G)-1 = 9-1+3^n-1 = 3^n+7$. If $|S_1| = 3^n+7$ then $G\setminus\{0\} \subset \sum(S_1)$ by Lemma 5.1. It follows from (5.2) that $\sum(S) = G$, a contradiction. So we have

$$|S_1| \leq 3^n + 6.$$

Let $H = L_{\prod(S_1)}$. Since S_1 is maximal, every term of SS_1^{-1} is in H. Note that S is regular. We have

$$|H| - 1 \geq |S_H| \geq |SS_1^{-1}| \geq 3^{n+1} + 1 - (3^n + 6) = 2 \times 3^n - 5.$$

Hence

$$3^{n+1} \geq |H| > 2 \times 3^n - 5.$$

It follows from $n \geq 2$ that

$$|H| = 3^{n+1}.$$

This implies

$$H = C_3 \bigoplus C_{3^n} \text{ or } C_{3^2} \bigoplus C_{3^{n-1}}.$$

Therefore,

$$\mathsf{D}(H) \leq 3^n + 2.$$

We show next that

$$c_0(H) \leq 2 \times 3^n - 5, \tag{5.3}$$

which implies that $\sum(S_H) = H$, a contradiction to lemma 2.3. Thus it follows from Lemma 2.2 that $c_0(G) = 3^{n+1} + 1$ completing the proof.

To prove (5.3), let S' be a regular sequence over H of length $|S'| = 2 \times 3^n - 5$. We need to show that $\sum(S') = H$. Assume to the contrary that,

$$\sum(S') \neq H.$$

Since $|S'| = 2 \times 3^n - 5 \geq m(H)$, by Lemmas 2.3 and 2.7 we obtain that

$$\mathrm{st}(\sum(S')) = \{0\} \text{ and } 0 \in \sum(S').$$

Let S_2 be the maximal subsequence of S' such that $\prod(S_2) \neq 0$. Similarly to above we derive that $|S_2| \leq \mathsf{D}(H) - 2 \leq 3^n$.

Let $H_1 = L_{\prod(S_2)}$. Similarly to above, we have

$$|H_1| - 1 \geq |S'_{H_1}| \geq |S'S_2^{-1}| \geq 2 \times 3^n - 5 - 3^n = 3^n - 5.$$

This implies that

$$|H_1| = 3^n.$$

Choose a subgroup K of H with $|K| = 3^n$ such that $|S'_K|$ is maximal. Since S' is regular, we have that $|S'_K| \le |K| - 1 \le 3^n - 1$. By the maximality of S'_K, $3^n - 5 \le |S'_{H_1}| \le |S'_K|$. Therefore,

$$3^n - 5 \le |S'_K| \le 3^n - 1.$$

Let $\overline{g} = g + K$ for every $g \in H$.

Since $|H| = 3^{n+1}$, we can always choose two terms g_1, g_2 of S' not in K such that $g_1 g_2 | S'$ and $|\{\overline{0}\} \cup \sum(\overline{g_1}\ \overline{g_2})| \ge 3$. We distinguish two cases.

Case 1. $3^n - 1 \ge |S'_K| \ge 3^n - 3$.

Take a subsequence $W_1 | S'_K$ with $|W_1| = 3^n - 3$. Let $T = g_1 g_2 W_1$ and $T_1 = S' T^{-1}$. Then we have

$$|T| = 3^n - 1$$

and

$$|T_1| = |S' T^{-1}| = 2 \times 3^n - 5 - 3^n + 1 = 3^n - 4 \ge 5.$$

Subcase 1.1. $\mathsf{v}_g(T_1) + \mathsf{v}_{-g}(T_1) \le 2$ for all $g \in H$.

Since $|T_1| \ge 5$, T_1 contains a 2-zero-sum free 3-subset A of H. Let

$$W = S' T^{-1} A^{-1}.$$

Then

$$|W| \ge 2.$$

Now S' has a factorization

$$S' = AWT.$$

Let $B = \{0\} \cup \sum(A)$, $C = \{0\} \cup \sum(W)$, and let $D = \{0\} \cup \sum(T)$. Then, $B + C + D = \sum(S')$. Since $\mathrm{st}(\sum(S')) = \{0\}$ and S' is regular, by Kneser's theorem we obtain that

$$\begin{aligned} |H| - 1 \ge |\sum(S')| &\ge |B| + |C| + |D| - 2 \\ &\ge 7 + (|W| + 1) + (3|T| - 3) - 2 \\ &\ge 7 + 3 + 3^{n+1} - 6 - 2 \ge 3^{n+1} = |H|, \end{aligned}$$

a contradiction.

Subcase 1.2. $\mathsf{v}_g(T_1) + \mathsf{v}_{-g}(T_1) \ge 3$ for some $g \in H$.

Since S' is regular over H, there is some term y of W_1 such that $y \notin \langle g \rangle$. Otherwise $|S'_{\langle g \rangle}| \ge 3^n \ge |\langle g \rangle|$, which is a contradiction. Let $T_2 = T y^{-1}$. Then

$$|T_2| = 3^n - 2$$

and

$$|S' T_2^{-1}| = 2 \times 3^n - 5 - 3^n + 2 = 3^n - 3.$$

Since $\mathsf{v}_g(T_1)+\mathsf{v}_{-g}(T_1)\geq 3$, there is a subsequence $A_1=g^a(-g)^b$ of T_1 with $a+b=3$. Let

$$A'=A_1y$$

and

$$W'=S'T_2^{-1}A'^{-1}.$$

Then

$$|W'|\geq 2.$$

Now S' has a factorization

$$S'=A'W'T_2.$$

Let $B=\{0\}\cup\sum(A')$, $C=\{0\}\cup\sum(W')$, and let $D=\{0\}\cup\sum(T_2)$. Then, $B+C+D=\sum(S')$. Since $\mathrm{st}(\sum(S'))=\{0\}$ and S' is regular, by Kneser's theorem we obtain that

$$\begin{aligned}|H|-1\geq |\sum(S')| &\geq |B|+|C|+|D|-2\\ &\geq 2(|A_1|+1)+(|W'|+1)+(3|T_2|-3)-2\\ &\geq 8+3+3^{n+1}-9-2=3^{n+1}=|H|,\end{aligned}$$

a contradiction.

Case 2. $3^n-5\leq |S'_K|\leq 3^n-4$.

Take a subsequence $W_1|S'_K$ with $|W_1|=3^n-5$. Let $T=g_1g_2W_1$ and $T_1=S'T^{-1}$. Then we have

$$|T|=3^n-3$$

and

$$|T_1|=|S'T^{-1}|=2\times 3^n-5-3^n+3=3^n-2\geq 7.$$

Subcase 2.1. $\mathsf{v}_g(T_1)+\mathsf{v}_{-g}(T_1)\leq 2$ for all $g\in H$.

Since $|T_1|\geq 7$, there are two 2-zero-sum free 3-sets A_1 and A_2 of H such that $A_1A_2|T_1$. Let $W=S'T^{-1}A_1^{-1}A_2^{-1}$. Then $|W|\geq 1$.

Now S' has a factorization

$$S'=A_1A_2WT.$$

Let $B_i=\{0\}\cup\sum(A_i)$ for $i\in\{1,2\}$, $C=\{0\}\cup\sum(W)$, and let $D=\{0\}\cup\sum(T)$. Then, $B_1+B_2+C+D=\sum(S')$. Since $\mathrm{st}(\sum(S'))=\{0\}$ and S' is regular, by Kneser's theorem we obtain that

$$\begin{aligned}|H|-1\geq |\sum(S')| &\geq |B_1|+|B_2|+|C|+|D|-3\\ &\geq 7+7+(|W|+1)+(3|T|-3)-3\\ &\geq 7+7+2+3^{n+1}-12-3\geq 3^{n+1}=|H|,\end{aligned}$$

a contradiction.

Subcase 2.2. $\mathsf{v}_g(T_1)+\mathsf{v}_{-g}(T_1)\geq 3$ for some $g\in H$.

Since $|T_1| = 3^n - 2$, there are two elements $y_1, y_2 \notin \langle g \rangle$ such that $y_1 y_2 | T_1$. Otherwise, $|S'_{\langle g \rangle}| \geq |T_1| - 1 = 3^n - 3 > |S'_K|$. This contradicts to the maximality of S'_K.

Since $\mathsf{v}_g(T_1) + \mathsf{v}_{-g}(T_1) \geq 3$, there is a subsequence $A_1 = g^a(-g)^b$ of T_1 with $a + b = 3$ and $a, b \geq 0$. Let

$$A' = A_1 y_1 y_2$$

and

$$W' = S' T^{-1} A'^{-1}.$$

Then

$$|W'| \geq 2.$$

Now S' has a factorization

$$S' = A' W' T.$$

Let $B = \{0\} \cup \sum(A')$, $C = \{0\} \cup \sum(W')$, and let $D = \{0\} \cup \sum(T)$. Then, $B + C + D = \sum(S')$. Since $\mathrm{st}(\sum(S')) = \{0\}$ and S' is regular, by Kneser's theorem we obtain that

$$\begin{aligned} |H| - 1 \geq |\sum(S')| &\geq |B| + |C| + |D| - 2 \\ &\geq 3(|A_1| + 1) + (|W'| + 1) + (3|T| - 3) - 2 \\ &\geq 12 + 3 + 3^{n+1} - 12 - 2 > 3^{n+1} = |H|, \end{aligned}$$

a contradiction. □

Proof of Theorem 1.1 (5). If $G = C_p \oplus C_p$, then $\mathsf{c}_0(G) = m(G) = 2p - 1$ by a result of Peng [10]. For the other cases, the result follows from Lemma 5.4 and Lemma 5.5. □

We end this section with the following

Conjecture 5.6. $\mathsf{c}_0(G) = m(G)$ for all finite abelian groups.

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