Some Motzkin-Straus type results for non-uniform hypergraphs*

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Abstract

A remarkable connection between the order of a maximum clique and the Lagrangian of a graph was established by Motzkin and Straus in 1965. This connection and its extensions were applied in Turán problems of graphs and uniform hypergraphs. Very recently, the study of Turán densities of non-uniform hypergraphs has been motivated by extremal poset problems. Peng et al. showed a generalization of Motzkin-Straus result for $\{1, 2\}$-graphs. In this paper, we attempt to explore the relationship between the Lagrangian of a non-uniform hypergraph and the order of its maximum cliques. We give a Motzkin-Straus type result for $\{1, r\}$-graphs. Moreover, we also give an extension of Motzkin-Straus theorem for $\{1, r_2, \cdots, r_l\}$-graphs.

Keywords: Lagrangians of hypergraphs; Turán problems; extremal problems

AMS Subject Classification (2010): 05C65, 05D05

1 Introduction

In 1965, Motzkin and Straus [20] established a connection between the order of a maximum clique and the Lagrangian of a graph, which was used to give another

*Supported by NSFC and the 973 program.

proof of Turán's theorem. This type of connection aroused interests in the study of Lagrangians of uniform hypergraphs. Actually, the Lagrangian of a hypergraph has been a useful tool in hypergraph extremal problems. Very recently, the study of Turán densities of non-uniform hypergraphs has been motivated by extremal poset problems; see [13, 14]. In this paper, we intend to study the connection between the order of a maximum clique and the Lagrangian of a non-uniform hypergraph.

A *hypergraph* is a pair $H = (V(H), E(H))$ consisting of a vertex set $V(H)$ and an edge set $E(H)$, where each edge is a subset of $V(H)$. The set $R(H) = \{|f| : f \in E(H)\}$ is called the set of *edge types* of H. We also say that H is an $R(H)$-graph. For example, if $R(H) = \{1, 3\}$, then we say that H is a $\{1, 3\}$-graph. If all edges have the same cardinality r, then H is an r-uniform hypergraph, which is simply written as r-graph. A 2-uniform hypergraph is exactly a simple graph. A hypergraph is non-uniform if it has at least two edge types. For any $r \in R(H)$, the *r-level hypergraph* H^r is the hypergraph consisting of all edges with r vertices of H. We also use notation E^r to denote the set of all edges with r vertices of H. We write H^R_n for a hypergraph H on n vertices with $R(H) = R$. For convenience, an edge $\{i_1, i_2, \ldots, i_r\}$ in a hypergraph is simply written as $i_1 i_2 \ldots i_r$ throughout the paper.

For an integer n, let $[n]$ denote the set $\{1, 2, \cdots, n\}$. The complete hypergraph K^R_n is a hypergraph on vertex set $[n]$ with edge set $\bigcup_{i\in R} \binom{[n]}{i}$. For example, $K^{\{r\}}_n$ is the complete r-uniform hypergraph on n vertices. $K^{[r]}_n$ is the non-uniform hypergraph with all possible edges of cardinality at most r. Let $[n]^R$ represent the complete R-type hypergraph on vertex set $[n]$. For example, $[n]^{\{1,3\}}$ represents the complete $\{1, 3\}$-hypergraph on vertex set $[n]$. We also let $[n]^{(r)}$ represent the complete r-uniform hypergraph on vertex set $[n]$.

Definition 1 *For an r-uniform hypergraph G with vertex set $\{1, 2, \cdots, n\}$, edge set $E(G)$ and a vector $\vec{x} = (x_1, \ldots, x_n) \in R^n$, define*

$$\lambda(G, \vec{x}) = \sum_{i_1 i_2 \ldots i_r \in E(G)} x_{i_1} x_{i_2} \ldots x_{i_r}.$$

Definition 2 *Let $S = \{\vec{x} = (x_1, \cdots, x_n) : \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0$ for $i = 1, 2, \cdots, n\}$. The Lagrangian of G, denoted by $\lambda(G)$, is defined as*

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.$$

The value x_i is called the weight of the vertex i and any vector $\vec{x} \in S$ is called a *legal weighting*. A weighting $\vec{y} \in S$ is called an optimal weighting for G if $\lambda(G, \vec{y}) = \lambda(G)$.

Motzkin and Straus in [20] proved the following result for the Lagrangian of a 2-graph. It shows that the Lagrangian of a graph is determined by the order of its maximum clique.

Theorem 1 *[20] If G is a 2-graph in which a largest clique has order t, then,*

$$\lambda(G) = \lambda\left({K_t}^{\{2\}}\right) = \lambda\left([t]^{(2)}\right) = \frac{1}{2}\left(1 - \frac{1}{t}\right).$$

In 1941, Turán [37] provided an answer to the following question: What is the maximum number of edges in a graph on n vertices without containing a complete graph of order k, for a given k? This is the well-known Turán's theorem. Theorem 1 provided another proof of Turán's theorem, which was given by Motzkin and Straus in [20]. The Motzkin-Straus result and its extension were successfully employed in optimization to provide heuristics for the maximum clique problem [1, 2, 6, 12]. It has been also generalized to vertex-weighted graphs [12] and edge-weighted graphs with applications to pattern recognition in image analysis [1, 2, 6, 12, 24, 23, 5]. On the other hand, the Lagrangian of a hypergraph has been a useful tool in hypergraph extremal problems. More generally, researchers are interested in the following Turán type problem, which was also proposed in [37]. For an r-uniform hypergraph F and an integer n, what is the maximum number of edges an r-uniform hypergraph with n vertices can have without containing F as a subgraph? This number is denoted by $ex_r(n, F)$. For example, the well-known Turán theorem implies $ex_2(n, K_3^{\{2\}}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$. With the concept of $ex_r(n, F)$, we state another important definition in extremal hypergraph theory, which is called the Turán density. For an r-uniform hypergraph F, by a standard averaging argument of Katona, Nemetz, and Simonovits [17], the limit $\lim\limits_{n\to\infty} \frac{ex_r(n,F)}{\binom{n}{r}}$ exists. This limit is called the *Turán density* of F. The connection between Lagrangians and Turán densities can be used to give another proof of the fundamental result of Erdös-Stone-Simonovits on Turán densities, see Keevash's survey paper [18]. Several results about determining hypergraph Turán densities were obtained through applications of Lagrangians, e.g., see [7, 8, 21, 32, 11, 15].

Lagrangians are also applied in spectral graph theory [38]. After decades of works, spectral methods for 2-graphs reside on a solid ground, with traditions settled both in tools and problems. Naturally we want similar comfort and convenience for spectra of hypergraphs. Recently, several researchers have contributed to this goal, and their studies naturally bring together other fundamental parameters, like the Lagrangians and the number of edges, see [19] and [22]. Another fascinating application of Lagrangians was first established by Frankl and Rödl [10] in disproving Erdös' long standing jumping constant conjecture. Applications of Lagrangians in determining non-jumping densities for hypergraphs can also be found in [9, 26, 25, 30, 27].

However, the obvious generalization of Motzkin and Straus' result to hypergraphs is false, i.e., the Lagrangian of a hypergraph is not always the same as the Lagrangian of its maximum cliques. In fact, there are many examples of hypergraphs that do not achieve their Lagrangian on any proper subhypergraph. Indeed, estimating Lagrangians of hypergraphs are much more harder. Talbot and Tang et al. made some progress on estimating Lagrangians of uniform hypergraphs in [34, 35, 36]. An attempt to generalize the Motzkin-Straus theorem to hypergraphs is due to Sós and Straus [33]. Recently, in [3, 4], Buló and Pelillo generalized the Motzkin and Straus' result to r-graphs in some way by using a continuous characterization of maximal cliques other than Lagrangians of hypergraphs.

In [31], the authors proved the following Motzkin-Straus type result for 3-graphs.

Theorem 2 *[31] Let m and t be positive integers satisfying $\binom{t}{3} \le m \le \binom{t}{3} + \binom{t-1}{2}$. Let G be a 3-graph with m edges and contain a clique of order t. Then,*

$$\lambda(G) = \lambda\left([t]^{(3)}\right).$$

They pointed out that the upper bound $\binom{t}{3}+\binom{t-1}{2}$ in this theorem is the best possible. When $m = \binom{t}{3} + \binom{t-1}{2} + 1$, let H be a 3-graph with the vertex set $[t+1]$ and the edge set $[t]^{(3)} \cup \left\{i_1 i_2 (t+1) : i_1 i_2 \in [t-1]^{(2)}\right\} \cup \{1t(t+1)\}$. Take a legal weighting $\vec{x} = (x_1, \cdots, x_n)$, where $x_1 = x_2 = \cdots = x_{t-1} = \frac{1}{t}$ and $x_t = x_{t+1} = \frac{1}{2t}$. Then $\lambda(H) \ge \lambda(H, \vec{x}) > \lambda\left([t]^{(3)}\right)$.

Recently, a generalization of the concept of Turán density of a non-uniform hypergraph was given in [16]. The study of Turán densities of non-uniform hypergraphs has been motivated by the study of extremal poset problems [13, 14]. In [28], Peng et al. introduced the Lagrangian of a non-uniform hypergraph. Applying Lagrangians, the authors gave an extension of Erdös-Stone-Simonovits theorem to non-uniform hypergraphs whose edges contain 1 or 2 vertices in [28].

Definition 3 *[28] For a hypergraph H_n^R and a vector $\vec{x} = (x_1, \ldots, x_n) \in R^n$, define*

$$\lambda'(H_n^R, \vec{x}) = \sum_{j \in R} \left(j! \sum_{i_1 i_2 \ldots i_j \in H^j} x_{i_1} x_{i_2} \ldots x_{i_j} \right).$$

Definition 4 *[28] Let* $S = \{\vec{x} = (x_1, \cdots, x_n) : \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \; for \; i = 1, 2, \cdots, n\}$. *The Lagrangian of* H_n^R*, denoted by* $\lambda'(H_n^R)$*, is defined as*

$$\lambda'(H_n^R) = \max\{\lambda'(H_n^R, \vec{x}) : \vec{x} \in S\}.$$

The value x_i *is called the weight of the vertex* i *and any vector* $\vec{x} \in S$ *is called a legal weighting. A weighting* $\vec{y} \in S$ *is called an optimal weighting for* H *if* $\lambda'(H, \vec{y}) = \lambda'(H)$.

Remark 1 *Consider the connection between Definition 2 and Definition 4. If* G *is an* r*-uniform graph, then*

$$\lambda'(G) = r!\lambda(G).$$

In this paper, we give some Motzkin-Straus type results for non-uniform hypergraphs. Our results provide a solution to the maximum value of a class of nonhomogeneous multilinear functions over the standard simplex of the Euclidean space. The main results in this paper will be stated in the next section.

2 Main results and implications

In [28], the authors proved the following generalization of Motzkin-Straus result to $\{1, 2\}$-graphs.

Theorem 3 *[28] If* H *is a* $\{1, 2\}$*-graph and the order of its maximum complete* $\{1, 2\}$*-subgraph is* t *(where* $t \geq 2$*), then,*

$$\lambda'(H) = \lambda'\left(K_t^{\{1,2\}}\right) = 2 - \frac{1}{t}.$$

In this paper, we attempt to explore the relationship between the Lagrangian of a non-uniform hypergraph and the order of its maximum cliques, and we give a Motzkin-Straus type result for $\{1, r\}$-graphs. For any hypergragh (graph) G, denote the number of its edges by $e(G)$.

Theorem 4 *Let* H *be a* $\{1, r\}$*-graph. If both the order of its maximum complete* $\{1, r\}$*-subgraph and the order of its maximum complete* $\{1\}$*-subgraph are* t*, where* $t \geq \lceil \frac{[r(r-1)-1]^{r-2}}{[r(r-1)]^{r-3}} \rceil$*, then,*

$$\lambda'(H) = \lambda'\left(K_t^{\{1,r\}}\right) = 1 + \frac{\prod_{i=1}^{r-1}(t-i)}{t^{r-1}}.$$

Furthermore, for $\{1,3\}$-graph, we give a result as follows.

Theorem 5 *Let H be a $\{1,3\}$-graph. If the order of its maximum complete $\{1,3\}$-subgraph is t, where $t \geq 5$, H^3 contains a maximum complete 3-graph of order s, where $s \geq t$, and the number of edges in H^3 satisfies $\binom{s}{3} \leq e(H^3) \leq \binom{s}{3} + \binom{t-1}{2}$, then,*

$$\lambda'(H) = \lambda'\left({K_t}^{\{1,3\}}\right) = 1 + \frac{(t-1)(t-2)}{t^2}.$$

Notice that, if $r = 3$, we require $t \geq 5$ in Theorems 4 and 5. In fact, for the case $t = 3$ or 4, it follows from the proof of Theorem 5, Theorem 5 holds when $s = t$. However, Theorem 5 fails to hold when $t = 3$ or 4 and $s \geq t+1$. For $t = 3$, $s \geq t+1$, let G be the $\{1,3\}$-graph with the vertex set $V(G) = [n]$ for some integer $n \geq s$, and the edge set $E(G) = E^1 \cup E^3$, where $E^1 = \{\{1\},\{2\},\{3\}\}$, $[s]^{(3)} \subseteq E^3$ and $\binom{s}{3} \leq |E^3| \leq \binom{s}{3} + \binom{t-1}{2}$. Take a legal weighting $\vec{x} = (x_1, \cdots, x_n)$, where $x_1 = x_2 = x_3 = 0.333$, $x_4 = \cdots = x_s = \frac{0.001}{s-3}$, $x_{s+1} = \cdots = x_n = 0$, then $\lambda'(G) \geq \lambda'(G, \vec{x}) > 1 + \frac{(3-1)(3-2)}{3^2} = \lambda'\left({K_3}^{\{1,3\}}\right)$. This example also shows that Theorem 4 fails to hold when $t = 3$ and $r = 3$. For $t = 4$, $s \geq t+1$, let G be a $\{1,3\}$-graph with the vertex set $V(G) = [n]$ for some integer $n \geq s$, and the edge set $E(G) = E^1 \cup E^3$, where $E^1 = \{\{1\},\{2\},\{3\},\{4\}\}$, $[s]^{(3)} \subseteq E^3$ and $\binom{s}{3} \leq |E^3| \leq \binom{s}{3} + \binom{t-1}{2}$. Take a legal weighting $\vec{x} = (x_1, \cdots, x_n)$, where $x_1 = x_2 = x_3 = x_4 = 0.2498$, $x_5 = \cdots = x_s = \frac{0.0008}{s-4}$, $x_{s+1} = \cdots = x_n = 0$, then $\lambda'(G) \geq \lambda'(G, \vec{x}) > 1 + \frac{(4-1)(4-2)}{4^2} = \lambda'\left({K_4}^{\{1,3\}}\right)$. This example also shows that Theorem 4 fails to hold when $t = 4$ and $r = 3$.

The bound of $e(H^3)$ in Theorem 5 is necessary, and it is also the best possible. When $e(H^3) = \binom{s}{3} + \binom{t-1}{2} + 1$, let H be a $\{1,3\}$-graph with the vertex set $[n]$ for some integer $n \geq s+1$, and the edge set $E(H) = E^1 \cup E^3$, where $E^1 = \{\{1\}, \cdots, \{t\}, \{s+1\}\}$, $E^3 = \{[s]^{(3)} \cup \{1t\,(s+1)\} \cup \{i_1 i_2 (s+1) : i_1 i_2 \in [t-1]^{(2)}\}\}$. Then $[s+1]^{(3)} \nsubseteq E^3$, $|E^3| = \binom{s}{3} + \binom{t-1}{2} + 1$. Take a legal weighting $\vec{x} = (x_1, \cdots, x_n)$, where $x_1 = x_2 = \cdots = x_{t-1} = \frac{1}{t}$, $x_t = x_{s+1} = \frac{1}{2t}$ and the remaining coordinates of $\vec{x}$ are equal to zero. Then $\lambda'(H) \geq \lambda'(H, \vec{x}) > \lambda'\left([t]^{(3)}\right)$.

Applying similar method used in the proof of Theorem 4, we will obtain the following result similar to Theorem 4 for $\{1, r_2, \cdots, r_l\}$-graphs, where $l \geq 2$.

Theorem 6 *Let H be a $\{1, r_2, \cdots, r_l\}$-graph. If both the order of its maximum complete $\{1, r_2, \cdots, r_l\}$-subgraph and the order of its maximum complete $\{1\}$-subgraph are t, where $t \geq f(r_2, \cdots, r_l)$ for some function $f(r_2, \cdots, r_l)$, then,*

$$\lambda'(H) = \lambda'\left({K_t}^{\{1, r_2, \cdots, r_l\}}\right).$$

A formula for function $f(r_2, \cdots, r_l)$ in Theorem 6 is given directly in the proof of this theorem. For example, applying Theorem 6 to $\{1,2,3\}$-graphs by using the formula for $f(r_2, \cdots, r_l)$ as given in the proof, we obtain

Corollary 1 *Let H be a $\{1,2,3\}$-graph. If both the order of its maximum complete $\{1,2,3\}$-subgraph and the order of its maximum complete $\{1\}$-subgraph are t, where $t \geq 8$, then,*

$$\lambda'(H) = \lambda'\left(K_t^{\{1,2,3\}}\right) = 1 + \frac{t-1}{t} + \frac{(t-1)(t-2)}{t^2}.$$

Clearly, each of Theorems 4-6 provides a solution to the maximum value of a class of nonhomogeneous multilinear functions over the standard simplex of the Euclidean space.

3 Some preliminaries

We will impose two additional conditions on any optimal legal weighting $\vec{x} = (x_1, \cdots, x_n)$ for an $R(H)$-graph H:

(i) $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$,

(ii) $|\{j : x_j > 0\}|$ is minimal, i.e., if $\vec{y}$ is a legal weighting for H satisfying $|\{j : y_j > 0\}| < |\{j : x_j > 0\}|$, then $\lambda'(H, \vec{y}) < \lambda'(H)$.

Let $H = (V, E)$ be an $R(H)$-graph. For $r \in R(H)$, we will denote the $(r-1)$-neighborhood of a vertex $i \in V$ by $E_i^r = \{A \in V^{(r-1)} : A \cup \{i\} \in E^r\}$. Similarly, we denote the $(r-2)$-neighborhood of a pair of vertices $i, j \in V$ by $E_{ij}^r = \{B \in V^{(r-2)} : B \cup \{i,j\} \in E^r\}$. We also denote the complement of E_i^r by $\overline{E}_i^r = \{A \in V^{(r-1)} : A \cup \{i\} \in V^{(r)} \setminus E^r\}$, and define $\overline{E}_{ij}^r = \{B \in V^{(r-2)} : B \cup \{i,j\} \in V^{(r)} \setminus E^r\}$. For ease of notation, define $E_{i \setminus j}^r = E_i^r \cap \overline{E}_j^r$. The following lemma gives some necessary conditions of an optimal weighting for an r-graph G.

Lemma 1 *[10] Let $G = (V, E)$ be an r-graph and $\vec{x} = (x_1, \cdots, x_n)$ be an optimal legal weighting for G with $k (\leq n)$ positive weights $x_1, \cdots, x_k$. Then for every $\{i,j\} \in [k]^{(2)}$, (a) $\lambda(E_i^r, \vec{x}) = \lambda(E_j^r, \vec{x}) = r\lambda(G)$, (b) there is an edge in E containing both i and j.*

A similar result for a non-uniform hypergraph is given in [28]. For completeness, we give the proof as follows.

Lemma 2 *If $\vec{x} = (x_1, \cdots, x_n)$ is an optimal legal weighting of a hypergraph H and $x_1 \geq x_2 \geq \ldots \geq x_k > x_{k+1} = x_{k+2} = \ldots = x_n = 0$, then, $\frac{\partial \lambda'(H,\vec{x})}{\partial x_1} = \frac{\partial \lambda'(H,\vec{x})}{\partial x_2} = \cdots = \frac{\partial \lambda'(H,\vec{x})}{\partial x_k}$, and for every $\{i,j\} \in [k]^{(2)}$, there is an edge in E containing both i and j.*

Proof. First, we prove $\frac{\partial \lambda'(H,\vec{x})}{\partial x_1} = \frac{\partial \lambda'(H,\vec{x})}{\partial x_2} = \cdots = \frac{\partial \lambda'(H,\vec{x})}{\partial x_k}$. By contradiction. Suppose there exist i and j ($1 \leq i < j \leq k$) such that $\frac{\partial \lambda'(H,\vec{x})}{\partial x_i} > \frac{\partial \lambda'(H,\vec{x})}{\partial x_j}$. We define a new legal weighting $\vec{y}$ for H as follows. Let $y_\ell = x_\ell$ for $\ell \neq i, j$, $y_i = x_i + \delta$ and $y_j = x_j - \delta \geq 0$, then for some small enough δ, we have

$$\begin{aligned}
\lambda'(H,\vec{y}) - \lambda'(H,\vec{x}) &= \delta(\frac{\partial \lambda'(H,\vec{x})}{\partial x_i} - x_j \frac{\partial^2 \lambda'(H,\vec{x})}{\partial x_i \partial x_j}) - \delta(\frac{\partial \lambda'(H,\vec{x})}{\partial x_j} - x_i \frac{\partial^2 \lambda'(H,\vec{x})}{\partial x_i \partial x_j}) \\
&\quad + (\delta x_j - \delta x_i - \delta^2) \frac{\partial^2 \lambda'(H,\vec{x})}{\partial x_i \partial x_j} \\
&= \delta(\frac{\partial \lambda'(H,\vec{x})}{\partial x_i} - \frac{\partial \lambda'(H,\vec{x})}{\partial x_j}) - \delta^2 \frac{\partial^2 \lambda'(H,\vec{x})}{\partial x_i \partial x_j} > 0,
\end{aligned}$$

contradicting to that $\vec{x}$ is an optimal vector. Now we prove the second part. By contradiction. Suppose there exist i and j ($1 \leq i < j \leq k$) such that $\{i,j\} \not\subseteq e$ for any edge $e \in E$. We define a new weighting $\vec{y}$ for H as follows. Let $y_\ell = x_\ell$ for $\ell \neq i, j$, $y_i = x_i + x_j$ and $y_j = x_j - x_j = 0$, then $\vec{y}$ is clearly a legal weighting for H, and

$$\lambda'(H,\vec{y}) - \lambda'(H,\vec{x}) = x_j(\frac{\partial \lambda'(H,\vec{x})}{\partial x_i} - \frac{\partial \lambda'(H,\vec{x})}{\partial x_j}) - x_j^2 \frac{\partial^2 \lambda'(H,\vec{x})}{\partial x_i \partial x_j} = 0.$$

So $\vec{y}$ is an optimal vector and $|\{i : y_i > 0\}| = k - 1$, contradicting to the minimality of k. $\square$

In [34], Talbot introduced the definition of a left-compressed r-uniform hypergraph. Let us generalize this concept to non-uniform hypergraphs.

Let $H = ([n], E)$ be an $R(H)$-graph, where n is a positive integer. For $e \in E$, and $i, j \in [n]$ with $i < j$, then, define

$$L_{ij}(e) = \begin{cases} (e \backslash \{j\}) \cup \{i\} & \text{if } i \notin e \text{ and } j \in e, \\ e & \text{otherwise.} \end{cases}$$

and

$$\mathcal{L}_{ij}(E) = \{L_{ij}(e) : e \in E\} \cup \{e : e, L_{ij}(e) \in E\}. \tag{1}$$

Note that $|\mathcal{L}_{ij}(E)| = |E|$ from the definition of $\mathcal{L}_{ij}(E)$.

We say that E (or H) is *left-compressed* if $\mathcal{L}_{ij}(E) = E$ for every $1 \leq i < j$.

When we consider the uniform hypergraph, i.e., the hypergraph has only one edge type, the above definition of left-compressed is just the same as Talbot's. And if an $R(H)$-graph H is left-compressed, then for every $r \in R(H)$, the r-level hypergraph H^r is left-compressed. An equivalent perhaps more intuitive definition of left-compressed hypergraph is that an $R(H)$-graph $H = ([n], E)$ is *left-compressed* if and only if for any $r \in R(H)$, $j_1 j_2 \cdots j_r \in E$ implies r-subset $i_1 i_2 \cdots i_r \in E$ provided $i_p \le j_p$ for every p, $1 \le p \le r$. Moreover, if H is a left-compressed $R(H)$-graph and $i < j$, then for every $r \in R(H)$, $E^r_{j\setminus i} = \emptyset$. And for a left-compressed $R(H)$-graph H, if $r \in R(H)$, and if there exists an r-edge $[k-(r-1)]\cdots(k-1)k$, then from the definition of left-compressed, $[k]^{(r)}$ is a subgraph of H^r.

Lemma 3 *Let $H = ([n], E)$ be an $R(H)$-graph, $i, j \in [n]$ with $i < j$ and $\vec{x} = (x_1, \cdots, x_n)$ be an optimal legal weighting of H. Write $H_{ij} = ([n], \mathcal{L}_{ij}(E))$. Then,*

$$\lambda'(H, \vec{x}) \le \lambda'(H_{ij}, \vec{x}).$$

Proof. If $1 \notin R(H)$, then,

$$\lambda'(H_{ij}, \vec{x}) - \lambda'(H, \vec{x}) = \sum_{r \in R(H)} \sum_{\substack{e \in E^r, L_{ij}(e) \notin E^r \\ i \notin e, j \in e}} \lambda'(e \backslash \{j\}, \vec{x})\,(x_i - x_j),$$

and if $1 \in R(H)$, then,

$$\lambda'(H_{ij}, \vec{x}) - \lambda'(H, \vec{x}) = \sum_{\substack{r \in R(H) \\ r \ge 2}} \sum_{\substack{e \in E^r, L_{ij}(e) \notin E^r \\ i \notin e, j \in e}} \lambda'(e \backslash \{j\}, \vec{x})\,(x_i - x_j) + (x_i - x_j)\, I,$$

where I satisfies that $I = 1$, if $i \notin E^1$ $j \in E^1$, and otherwise $I = 0$. Hence $\lambda'(H_{ij}, \vec{x}) - \lambda'(H, \vec{x})$ is nonnegative in any case, since $i < j$ implies that $x_i \ge x_j$. So this lemma holds. □

If there is exactly one element in $R(H)$, then H is a uniform hypergaph, which implies that Lemma 3 also holds for uniform hypergaphs. In [34], Talbot gave several interesting results on the issue about how large the Lagrangian of an r-uniform hypergraph with m edges can be, for given integer $r \ge 3$ and positive integer m. From Lemma 3 for uniform hypergraphs, Talbot assumed G, which has the largest Lagrangian among all m edges r-uniform hypergraphs, is left-compressed.

In the next section, we will use Lemma 3 to prove Theorem 4 by assuming the "extremal hypergraph" is left-compressed. Unfortunately, we can not use the same method to prove Theorem 5. That because for H satisfying the requirement of Theorem 5, the hypergraph H_{ij} described in Lemma 3 may not satisfy the requirement of Theorem 5. For example, let $s \geq t+1$, and let H be the $\{1,3\}$-graph with the vertex set $V(H)=[n]$ for some integer $n \geq s+1$, and the edge set $E(H)=E^1 \cup E^3$, where $E^1=\{\{1\},\cdots,\{t+1\}\}$, E^3 consisting all 3-subsets of $\{2,\cdots,s+1\}$. It is easy to see that H satisfies the requirement of Theorem 5, but the hypergraph $H_{1(t+2)}$ does not satisfy the requirement of Theorem 5, since there is a $K_{t+1}^{\{1,3\}}$ in $H_{1(t+2)}$. Thus, in Section 4, we use a different approach to prove Theorem 5 instead of using the concept of left-compressed. The proofs of Theorem 6 and Corollary 1 will be given in Section 6.

4 Proof of Theorem 4

Applying the theory of Lagrangian multipliers, it is easy to get that an optimal weighting $\vec{x}$ for ${K_t}^{\{1,r\}}$ is given by $x_i = 1/t$ for each i, $1 \leq i \leq t$. So $\lambda'({K_t}^{\{1,r\}}) = 1 + \frac{\prod_{i=1}^{r-1}(t-i)}{t^{r-1}}$. So we only need to prove $\lambda'(H) = \lambda'\left({K_t}^{\{1,r\}}\right)$. Since ${K_t}^{\{1,r\}} \subseteq H$, clearly, $\lambda'(H) \geq \lambda'\left({K_t}^{\{1,r\}}\right)$. Thus, to prove Theorem 4, it suffices to prove that $\lambda'(H) \leq \lambda'\left({K_t}^{\{1,r\}}\right)$. Suppose H has n vertices, denote $\lambda'_{\{n,t,\{1,r\}\}} = max\{\lambda'(G) : G$ is a $\{1,r\}$-graph with n vertices, G contains a maximum complete subgraph $K_t^{\{1,r\}}$ and a maximum complete subgraph $K_t^{\{1\}}\}$. If $\lambda'_{\{n,t,\{1,r\}\}} \leq \lambda'\left({K_t}^{\{1,r\}}\right)$, then $\lambda'(H) \leq \lambda'\left({K_t}^{\{1,r\}}\right)$. Hence we can assume H is an extremal hypergraph, i.e., $\lambda'(H) = \lambda'_{\{n,t,\{1,r\}\}}$. If H is not left-compressed, performing a sequence of left-compressing operations (i.e. replace E by $\mathcal{L}_{ij}(E)$ if $\mathcal{L}_{ij}(E) \neq E$), we will get a left-compressed $\{1,r\}$-graph H' with the same number of edges. The condition that the order of a maximum complete $\{1\}$-subgraph of H is t guarantees that both the order of a maximum $\{1,r\}$ complete subgraph of H' and the order of a maximum $\{1\}$ complete subgraph of H' are still t. By Lemma 3, H' is an extremal graph as well. So we can assume that the edge set of H is left-compressed, $H^1=[t]$ and $[t]^{(r)} \subseteq H^r$. Let $\vec{x}=(x_1,\cdots,x_n)$ be an optimal legal weighting for H, where $x_1 \geq x_2 \geq \ldots \geq x_k > x_{k+1} = x_{k+2} = \ldots = x_n = 0$. If $k \leq t$, then $\lambda'(H) \leq \lambda'([k]^{\{1,r\}}) \leq \lambda'([t]^{\{1,r\}})$. So it suffices to show that $x_{t+1}=0$.

Let $1 \leq i \leq t$. If $x_{t+1} > 0$, then by Lemma 2, there exists $e \in H^r$ such that $\{i, t+1\} \subset e$ and $\frac{\partial \lambda'(H,\vec{x})}{\partial x_i} = \frac{\partial \lambda'(H,\vec{x})}{\partial x_{t+1}}$.

Recall that $i \in E^1$ and $t+1 \notin E^1$, then,

$$\begin{aligned}0 =& \frac{\partial \lambda'(H,\vec{x})}{\partial x_i} - \frac{\partial \lambda'(H,\vec{x})}{\partial x_{t+1}} \\ =& 1 + r!\lambda\left(E^r_{i\backslash(t+1)},\vec{x}\right) + r!x_{t+1}\lambda\left(E^r_{i(t+1)},\vec{x}\right) - r!x_i\lambda\left(E^r_{i(t+1)},\vec{x}\right).\end{aligned}$$

Let $C = r!\lambda\left(E^r_{i(t+1)}, \vec{x}\right)$. Then $0 < C \leq r!\frac{(1-x_i-x_{t+1})^{r-2}}{(r-2)!}$ since $E^r_{i(t+1)}$ is an $(r-2)$-uniform hypergraph on $[n] \setminus \{i, t+1\}$. Thus, $x_i \geq \frac{1}{C} + x_{t+1}$. So

$$x_i \geq \frac{1}{r(r-1)(1-x_i-x_{t+1})^{r-2}} + x_{t+1}. \tag{2}$$

The above inequality clearly implies that $x_i > \frac{1}{r(r-1)}$. Combining this with (2), we have

$$x_i > \frac{[r(r-1)]^{r-3}}{[r(r-1)-1]^{r-2}}. \tag{3}$$

Recall that $t \geq \lceil \frac{[r(r-1)-1]^{r-2}}{[r(r-1)]^{r-3}} \rceil$, with the aid of (3), $\sum\limits_{i=1}^{t} x_i > 1$, a contradiction to the definition of legal weighting vectors. So $x_{t+1} = 0$. The proof is thus complete. ∎

5 Proof of Theorem 5

As shown in Theorem 4, $\lambda'(K_t^{\{1,3\}}) = 1 + \frac{(t-1)(t-2)}{t^2}$. So we only need to prove $\lambda'(H) = \lambda'\left(K_t^{\{1,3\}}\right)$. Since $K_t^{\{1,3\}} \subseteq H$, clearly, $\lambda'(H) \geq \lambda'\left(K_t^{\{1,3\}}\right)$. Thus, to prove Theorem 5, it suffices to prove that $\lambda'(H) \leq \lambda'\left(K_t^{\{1,3\}}\right) = 1 + \frac{(t-1)(t-2)}{t^2}$. Suppose H has n vertices, this time we denote $\mu_{\{n,t,s,m,\{1,3\}\}} = max\{\lambda'(G) : G$ is a $\{1,3\}$-graph with n vertices, G contains a maximum complete subgraph $K_t^{\{1,3\}}$, G^3 contains a maximum clique of order s and $e(G^3) = m$, where $\binom{s}{3} \leq m \leq \binom{s}{3} + \binom{t-1}{2}\}$. If $\mu_{\{n,t,s,m,\{1,3\}\}} \leq 1 + \frac{(t-1)(t-2)}{t^2}$, then $\lambda'(H) \leq 1 + \frac{(t-1)(t-2)}{t^2}$. Hence we can assume H is an extremal hypergraph, i.e., $\lambda'(H) = \mu_{\{n,t,s,m,\{1,3\}\}}$. Let $\vec{x} = (x_1, \cdots, x_n)$ be an optimal legal weighting for H, where $x_1 \geq x_2 \geq \ldots \geq x_k > x_{k+1} = x_{k+2} = \ldots = x_n = 0$. Note that if $k \leq t$, then $\lambda'(H, \vec{x}) \leq \sum\limits_{i=1}^{k} x_i + \lambda'([k]^{(3)}, \vec{x}) \leq 1 + \lambda'([k]^{(3)}) = 1 + \frac{(k-1)(k-2)}{k^2} \leq 1 + \frac{(t-1)(t-2)}{t^2}$. Also, if $s = t$, then from Theorem 2 and Remark 1, $\lambda'(H, \vec{x}) \leq \sum\limits_{i=1}^{k} x_i + \lambda'(H^3, \vec{x}) \leq 1 + \lambda'(H^3) = 1 + \frac{(s-1)(s-2)}{s^2} = 1 + \frac{(t-1)(t-2)}{t^2}$. So in the sequel, we assume $k \geq t+1$ and $s \geq t+1$.

Since $e(H^3) \leq \binom{s}{3} + \binom{t-1}{2}$, there is a unique $K_s^{\{3\}}$ in H^3, otherwise, if H^3 contains two different $K_s^{\{3\}}$, then $e(H^3) \geq \binom{s}{3} + \binom{s-1}{2}$, a contradiction to the range of $e(H^3)$. Let $\{i_1, \ldots, i_s\}$ be the vertex set of that unique $K_s^{\{3\}}$ in H^3. We can assume there exists a unique vertex set $\{j_1, \ldots, j_t\} \subseteq \{i_1, \ldots, i_s\}$ such that $\{j_1, \ldots, j_t\}$ induces a $K_t^{\{1,3\}}$ in H. Otherwise, since $e(H^3) \leq \binom{s}{3} + \binom{t-1}{2}$, there is a $K_t^{\{1,3\}}$ whose vertex set consists of a vertex $a \notin \{i_1, \ldots, i_s\}$ and $t-1$ vertices from $\{i_1, \ldots, i_s\}$, denote these $t-1$ vertices by $b_1, \ldots, b_{t-1}$. Notice that this $K_t^{\{1,3\}}$ is the unique $K_t^{\{1,3\}}$ in H. Then we take one vertex b from $\{i_1, \ldots, i_s\} \setminus \{b_1, \ldots, b_{t-1}\}$, add a new 1-edge $\{b\}$ to H, we can see that the new $\{1,3\}$-graph H' satisfies the conditions of

Theorem 5, and $\lambda'(H') \geq \lambda'(H)$ since $H \subset H'$, which implies that H' is also an extremal hypergraph. Hence we can assume that there exists a unique vertex set $\{j_1, \ldots, j_t\} \subseteq \{i_1, \ldots, i_s\}$ such that $\{j_1, \ldots, j_t\}$ induces a $K_t^{\{1,3\}}$ in H. Note that any vertex in $\{i_1, \ldots, i_s\} \setminus \{j_1, \ldots, j_t\}$ is not a 1-edge in H.

Consider the relationship between the set $[k]$ and $\{i_1, \ldots, i_s\}$, we have three cases.

Case 1. $[k] \subseteq \{i_1, \ldots, i_s\}$.

Denote H_0 the $\{1,3\}$-subgraph induced by $[k]$ in H, then $\lambda'(H_0) = \lambda'(H_0, \vec{x}) = \lambda'(H)$. We can see that H_0 satisfies the conditions of Theorem 4 ($r = 3$), thus $\lambda'(H_0) \leq \lambda'\left(K_t{}^{\{1,3\}}\right) = 1 + \frac{(t-1)(t-2)}{t^2}$, so $\lambda'(H) \leq \lambda'\left(K_t{}^{\{1,3\}}\right) = 1 + \frac{(t-1)(t-2)}{t^2}$.

Case 2. $[k] \cap \{i_1, \ldots, i_s\} = \emptyset$.

In this case, there are at most $\binom{t-1}{2}$ 3-edges contributing nonzero value to $\lambda'(H, \vec{x})$. Let H_0^3 be the subgraph induced by $[k]$ in H^3, then $e(H_0^3) \leq \binom{t-1}{2}$. By adding some 3-edges to H_0^3, we can find a 3-graph G such that $H_0^3 \subset G$, $K_t^{\{3\}} \subset G$, and $e(G) \leq \binom{t}{3} + \binom{t-1}{2}$, by Theorem 2 and Remark 1, $\lambda'(H_0^3) \leq \lambda'(G) = \frac{(t-1)(t-2)}{t^2}$. Hence $\lambda'(H, \vec{x}) \leq 1 + \lambda'(H^3, \vec{x}) = 1 + \lambda'(H_0^3, \vec{x}) \leq 1 + \lambda'(H_0^3) \leq 1 + \lambda'(G) = 1 + \frac{(t-1)(t-2)}{t^2}$.

Case 3. $[k] \cap \{i_1, \ldots, i_s\} \neq \emptyset$, and $[k] \nsubseteq \{i_1, \ldots, i_s\}$.

Let $|[k] \cap \{i_1, \ldots, i_s\}| = p$, and we will prove the claim below.

Claim 1 $|\{j_1, \ldots, j_t\} \cap [k]| = min\{p, t\}$.

Proof. Clearly, $|\{j_1, \ldots, j_t\} \cap [k]| \leq min\{p, t\}$. If $|\{j_1, \ldots, j_t\} \cap [k]| < min\{p, t\}$, then there exist two vertices i, j such that $i \in \{i_1, \ldots, i_s\} \setminus \{j_1, \ldots, j_t\}$, $x_i > 0$ and $j \in \{j_1, \ldots, j_t\}$, $x_j = 0$. Denote E_S^3 the edge set of $K_s^{\{3\}}$ induced by $\{i_1, \ldots, i_s\}$ in H^3. We construct a new $\{1,3\}$-graph $H' = ([n], E')$, with $E' = (E \backslash A) \cup A'$, where A is the edge set of all 3-edges containing i but not j in $E \backslash E_S^3$, A' is the edge set obtained from A by replacing i with j for all 3-edges in A. It is obvious that $|E'^3| = |E^3|$, H' contains a $K_t^{\{1,3\}}$ and the order of maximum complete 3-subgraph in H' is still s, moreover, we say that there is no $K_{t+1}^{\{1,3\}}$ in H'. Otherwise, there is a $K_{t+1}^{\{1,3\}}$ in H', then the vertex set of $K_{t+1}^{\{1,3\}}$ can not include vertices in $\{i_1, \ldots, i_s\} \setminus \{j_1, \ldots, j_t\}$, which indicates that there are at least $\binom{s}{3} + \binom{t}{2}$ 3-edges in H'. Since $|E'^3| = |E^3| \leq \binom{s}{3} + \binom{t-1}{2}$, it is a contradiction. So the order of maximum complete $\{1,3\}$-subgraph in H' is still t. We define a legal weighting $\vec{x'}$ for H', such that $x'_l = x_l$, for $l \neq i, j$, and $x'_i = x_j = 0$, $x'_j = x_i$. Then we can derive that $\lambda'(H', \vec{x'}) - \lambda'(H, \vec{x}) \geq x_i > 0$. This implies that $\lambda'(H') > \lambda'(H)$, a contradiction to the assumption of H. $\square$

We still denote H_0^3 the subgraph induced by $[k]$ in H^3, and there are two subcases to consider.

Subcase 3.1. $p \le t$.

In this subcase, H_0^3 consists of a $K_p^{\{3\}}$ and at most $\binom{t-1}{2}$ other 3-edges. Similarly to Case 2, by adding some 3-edges, we can deduce that $\lambda(H_0^3) \le \lambda(K_t^{\{3\}})$, then, $\lambda'(H^3, \vec{x}) = \lambda'(H_0^3, \vec{x}) \le 3!\lambda(H_0^3) \le 3!\lambda(K_t^{\{3\}}) = \frac{(t-1)(t-2)}{t^2}$, so $\lambda'(H, \vec{x}) \le 1 + \lambda'(H^3, \vec{x}) \le 1 + \frac{(t-1)(t-2)}{t^2}$.

Subcase 3.2. $p \ge t+1$.

We prove that we may assume for any $j \in \{j_1, \dots, j_t\}$, $i \in \{i_1, \dots, i_s\} \backslash \{j_1, \dots, j_t\}$,

$$x_j \ge x_i, \tag{4}$$

and

$$\lambda\left(E^3_{j\backslash i}, \vec{x}\right) \ge \lambda\left(E^3_{i\backslash j}, \vec{x}\right) \tag{5}$$

hold.

In fact, if H dose not satisfy (4) and (5), through the following two steps, we will find a new $\{1,3\}$-graph H^* and a new legal weighting vector $\vec{z}$ satisfying (4) and (5), and H^* is an extremal hypergraph as well.

Step 1. For every $i \in \{i_1, \dots, i_s\} \setminus \{j_1, \dots, j_t\}$, search for every vertex $j \in \{j_1, \dots, j_t\}$ satisfying $E^3_{i\backslash j} \setminus E^3_{j\backslash i} \neq \emptyset$ (i.e., $E^3_{i\backslash j} \neq \emptyset$). If such a vertex exists, then for each $U \in (E^3_{i\backslash j} \setminus E^3_{j\backslash i}) = E^3_{i\backslash j}$, replace the 3-edge $\{U \cup \{i\}\}$ by $\{U \cup \{j\}\}$. Check the value of x_i and x_j, if $x_i > x_j$, then exchange the weight of these two vertices i, j.

Denote the new $\{1,3\}$-graph $H^* = ([n], E^*)$ and the new legal weighting vector $\vec{y}$ obtained from Step 1. We see that $|E^{*3}| = |E^3|$, the order of maximum complete 3-subgraph in H' is still s. Similar to the argument we used in Claim 1, there is no $K_{t+1}^{\{1,3\}}$ in H^*. Otherwise, there is a $K_{t+1}^{\{1,3\}}$ in H^*, then the vertex set of $K_{t+1}^{\{1,3\}}$ can not include vertices in $\{i_1, \dots, i_s\} \setminus \{j_1, \dots, j_t\}$, which indicates that there are at least $\binom{s}{3} + \binom{t}{2}$ 3-edges in H^*. Since $|E^{*3}| = |E^3| \le \binom{s}{3} + \binom{t-1}{2}$, it is a contradiction. So the order of maximum complete $\{1,3\}$-subgraph in H^* is still t. Moreover, H^* with the weighting vector $\vec{y}$ satisfies (5).

Step 2. For every $i \in \{i_1, \dots, i_s\} \setminus \{j_1, \dots, j_t\}$ in H^*, search for every vertex $j \in \{j_1, \dots, j_t\}$ satisfying $y_i > y_j$. Then exchange the weight of vertices i, j.

Denote the new legal weighting vector $\vec{z}$ for H^* obtained after Step 2, then, clearly, H^* with the weighting vector $\vec{z}$ satisfies (4) and (5), besides, one can easily get that $\lambda'(H^*) \ge \lambda'(H^*, \vec{z}) \ge \lambda'(H^*, \vec{y}) \ge \lambda'(H, \vec{x}) = \lambda'(H)$. That implies H^* is also an extremal hypergraph. Hence we can assume H and its optimal weighting vector $\vec{x}$ satisfy that for any $j \in \{j_1, \dots, j_t\}$, $i \in \{i_1, \dots, i_s\} \setminus \{j_1, \dots, j_t\}$, (4) and (5) hold.

For any pair $i, j \in [k]$, if $i \in \{i_1, \dots, i_s\} \setminus \{j_1, \dots, j_t\}$, $j \in \{j_1, \dots, j_t\}$, then, by

Lemma 2, there exists some edge e containing both i and j, and,

$$\begin{aligned}0 =& \frac{\partial \lambda'(H,\vec{x})}{\partial x_j} - \frac{\partial \lambda'(H,\vec{x})}{\partial x_i}\\ =& 1 + 3!\lambda\left(E^3_{j\setminus i},\vec{x}\right) + 3!x_i\lambda\left(E^3_{ij},\vec{x}\right) - 3!\lambda\left(E^3_{i\setminus j},\vec{x}\right) - 3!x_j\lambda\left(E^3_{ij},\vec{x}\right).\end{aligned}$$

Let $A = 3!\lambda\left(E^3_{j\setminus i},\vec{x}\right)$, $B = 3!\lambda\left(E^3_{i\setminus j},\vec{x}\right)$, $C = 3!\lambda\left(E^3_{ij},\vec{x}\right)$. With (5), we have $A \geq B$, thus, $x_j \geq \frac{1}{C} + x_i$, with $0 < C \leq 6(1-x_i-x_j)$. So

$$x_j \geq \frac{1}{6(1-x_i-x_j)} + x_i. \tag{6}$$

The above inequality clearly implies that $x_j > \frac{1}{6}$. Combining this with (6), we have

$$x_j > \frac{1}{5} + x_i. \tag{7}$$

Since $p \geq t+1$, there exists a vertex $b \in [k] \cap \{i_1,\ldots,i_s\} \setminus \{j_1,\ldots,j_t\}$. If $t \geq 5$, then $\sum\limits_{a\in E^1} x_a = \sum\limits_{a\in\{j_1,\cdots j_t\}} x_a > 1 + 5x_b > 1$, a contradiction to the definition of legal weighting vectors. Hence $t < 5$, which contradicts to the the condition $t \geq 5$ in Theorem 5.

Combining all these cases, the proof is thus complete. ∎

6 Proof of Theorem 6

For any given $r_2,\cdots,r_l$, $\lambda'(H) \geq \lambda'\left({K_t}^{\{1,r_2,\cdots,r_l\}}\right)$, since ${K_t}^{\{1,r_2,\cdots,r_l\}} \subseteq H$. Thus, to prove Theorem 6, it suffices to prove that

$$\lambda'(H) \leq \lambda'\left({K_t}^{\{1,r_2,\cdots,r_l\}}\right).$$

Suppose H has n vertices, denote $\lambda'_{\{n,t,\{1,r_2,\cdots,r_l\}\}} = max\{\lambda'(G) : G$ is a $\{1,r_2,\cdots,r_l\}$-graph with n vertices, G contains a maximum complete subgraph $K_t^{\{1,r_2,\cdots,r_l\}}$ and a maximum complete subgraph $K_t^{\{1\}}\}$. If we have $\lambda'_{\{n,t,\{1,r_2,\cdots,r_l\}\}} \leq \lambda'\left({K_t}^{\{1,r_2,\cdots,r_l\}}\right)$, then $\lambda'(H) \leq \lambda'\left({K_t}^{\{1,r_2,\cdots,r_l\}}\right)$. Hence, we can assume H is an extremal hypergraph, i.e., $\lambda'(H) = \lambda'_{\{n,t,\{1,r_2,\cdots,r_l\}\}}$. If H is not left-compressed, performing a sequence of left-compressing operations (i.e. replace E by $\mathcal{L}_{ij}(E)$ if $\mathcal{L}_{ij}(E) \neq E$), we will get a left-compressed $\{1,r_2,\cdots,r_l\}$-graph H' with the same number of edges. The condition that the order of a maximum complete $\{1\}$-subgraph of H is t guarantees that both the order of a maximum $\{1,r_2,\cdots,r_l\}$ complete subgraph of H' and the order of a maximum $\{1\}$ complete subgraph of H' are still t. By Lemma 3, H' is an extremal graph as well. So we can assume that the edge set of H is left-compressed, $H^1 = [t]$, $[t]^{(r_j)} \subseteq H^{r_j}$ for all $2 \leq j \leq l$. Let $\vec{x} = (x_1, \cdots, x_n)$ be an optimal legal weighting for H, where $x_1 \geq x_2 \geq \ldots \geq x_k > x_{k+1} = x_{k+2} = \ldots = x_n = 0$. If $k \leq t$, then $\lambda'(H) \leq \lambda'(K_k^{\{1,r_2,\cdots,r_l\}}) \leq \lambda'(K_t^{\{1,r_2,\cdots,r_l\}})$. So it suffices to show that $x_{t+1} = 0$.

Let $1 \leq i \leq t$. If $x_{t+1} > 0$, then by Lemma 2, there exists $e \in E$ such that $\{i, t+1\} \subset e$ and $\frac{\partial \lambda'(H,\vec{x})}{\partial x_i} = \frac{\partial \lambda'(H,\vec{x})}{\partial x_{t+1}}$. If $2 \in \{r_2, \cdots, r_l\}$, let $\lambda(E^2_{i(t+1)}, \vec{x}) = 1$ when $i(t+1) \in E^2$, and let $\lambda(E^2_{i(t+1)}, \vec{x}) = 0$ when $i(t+1) \notin E^2$. Recall that $i \in E^1$ and $t+1 \notin E^1$, then by Lemma 2, we have

$$\begin{aligned} 0 &= \frac{\partial \lambda'(H,\vec{x})}{\partial x_i} - \frac{\partial \lambda'(H,\vec{x})}{\partial x_{t+1}} \\ &= 1 + r_1!\lambda\left(E^{r_1}_{i\backslash(t+1)}, \vec{x}\right) + r_1!x_{t+1}\lambda\left(E^{r_1}_{i(t+1)}, \vec{x}\right) - r_1!x_i\lambda\left(E^{r_1}_{i(t+1)}, \vec{x}\right) \\ &\quad + r_2!\lambda\left(E^{r_2}_{i\backslash(t+1)}, \vec{x}\right) + r_2!x_{t+1}\lambda\left(E^{r_2}_{i(t+1)}, \vec{x}\right) - r_2!x_i\lambda\left(E^{r_2}_{i(t+1)}, \vec{x}\right) \\ &\quad + \cdots + r_l!\lambda\left(E^{r_l}_{i\backslash(t+1)}, \vec{x}\right) + r_l!x_{t+1}\lambda\left(E^{r_l}_{i(t+1)}, \vec{x}\right) - r_l!x_i\lambda\left(E^{r_l}_{i(t+1)}, \vec{x}\right). \end{aligned}$$

Let $C = \sum_{j=2}^{l} r_j!\lambda\left(E^{r_j}_{i(t+1)}, \vec{x}\right)$, since there exists $e \in E$ such that $\{i, t+1\} \subset e$, then we have $C > 0$. Thus, $x_i \geq \frac{1}{C} + x_{t+1}$. Moreover, there must exist some function $g(r_2, \cdots, r_l)$, such that $C \leq g(r_2, \cdots, r_l)$. For example, we let $g(r_2, \cdots, r_l) = \sum_{j=2}^{l} r_j!\frac{(1-x_i-x_{t+1})^{r_j-2}}{(r_j-2)!} = \sum_{j=2}^{l} r_j(r_j-1)(1-x_i-x_{t+1})^{r_j-2}$ (if $r_j = 2$, set $\frac{(1-x_i-x_{t+1})^{r_j-2}}{(r_j-2)!} = 1$), then

$$x_i \geq \frac{1}{\sum_{j=2}^{l} r_j(r_j-1)(1-x_i-x_{t+1})^{r_j-2}} + x_{t+1}. \tag{8}$$

The above inequality clearly implies that $x_i > \frac{1}{\sum_{j=2}^{l} r_j(r_j-1)}$. Combining this with (8), let $h = \frac{1}{\sum_{j=2}^{l} r_j(r_j-1)}$, we have

$$x_i > \frac{1}{\sum_{j=2}^{l} r_j(r_j-1)(1-h)^{r_j-2}}. \tag{9}$$

Let $f(r_2, \cdots, r_l) = \lceil \sum_{j=2}^{l} r_j(r_j-1)(1-h)^{r_j-2} \rceil$, recall that we require $t \geq f(r_2, \cdots, r_l)$, with the aid of (9), $\sum_{i=1}^{t} x_i > 1$, a contradiction to the definition of legal weighting vectors. So $x_{t+1} = 0$.

The proof is thus complete. ∎

Applying Theorem 6 and the formula for $f(r_2, \cdots, r_l)$ as given in the above proof, we get Corollary 1.

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