The 3-rainbow index and connected dominating sets*

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Abstract

A tree in an edge-colored graph is said to be rainbow if no two edges on the tree share the same color. An edge-coloring of G is called a 3-rainbow coloring if for any three vertices in G, there exists a rainbow tree connecting them. The 3-rainbow index $rx_3(G)$ of G is defined as the minimum number of colors that are needed in a 3-rainbow coloring of G. This concept, introduced by Chartrand et al., can be viewed as a generalization of the rainbow connection. In this paper, we study the 3-rainbow index by using connected 3-way dominating sets and 3-dominating sets. We show that for every connected graph G on n vertices with minimum degree at least δ $(3 \leq \delta \leq 5)$, $rx_3(G) \leq \frac{3n}{\delta+1} + 4$, and the bound is tight up to an additive constant; whereas for every connected graph G on n vertices with minimum degree at least δ $(\delta \geq 3)$, we get that $rx_3(G) \leq \frac{\ln(\delta+1)}{\delta+1}(1 + o_\delta(1))n + 5$. In addition, we obtain some tight upper bounds of the 3-rainbow index for some special graph classes, including threshold graphs, chain graphs and interval graphs.

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1 Introduction

All graphs in this paper are undirected, finite and simple. We follow [1] for graph theoretical notation and terminology not described here. Let G be a nontrivial connected

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graph with an *edge-coloring* $c : E(G) \to \{1, 2, \ldots, t\}, t \in \mathbb{N}$, where adjacent edges may be colored the same. A path is said to be a *rainbow path* if no two edges on the path have the same color. An edge-colored graph G is called *rainbow connected* if for every pair of distinct vertices of G there exists a rainbow path connecting them. The *rainbow connection number* of G, denoted by $rc(G)$, is defined as the minimum number of colors that are needed in order to make G rainbow connected. The *rainbow k-connectivity* of G, denoted by $rc_k(G)$, is defined as the minimum number of colors in an edge-coloring of G such that every two distinct vertices of G are connected by k internally disjoint rainbow paths. These concepts were introduced by Chartrand et al. in [9,10]. Recently, there have been lots of results published on the rainbow connection subject. The interested reader can refer to [16, 17] for a survey on this topic.

The (k, ℓ)-rainbow index was also introduced by Chartrand et al. in [11], which can be viewed as a generalization of the rainbow connection and rainbow connectivity. We call a tree T in an edge-colored graph G a *rainbow tree* if no two edges of T have the same color. For $S \subseteq V(G)$, a *rainbow S-tree* is a rainbow tree connecting all vertices of S. Suppose that $\{T_1, T_2, \ldots, T_\ell\}$ is a set of rainbow S-trees. They are called *internally disjoint* if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$ for every pair of distinct integers i, j with $1 \le i, j \le \ell$ (note that these trees are vertex-disjoint in $G \setminus S$). Given two positive integers k, ℓ with $k \ge 2$, the *(k, ℓ)-rainbow index $rx_{k,\ell}(G)$* of G is the minimum number of colors needed in an edge-coloring of G such that for any set S of k vertices of G, there exist ℓ internally disjoint rainbow S-trees. In particular, for $\ell = 1$, we often write $rx_k(G)$ rather than $rx_{k,1}(G)$ and call it the *k-rainbow index*. An edge-coloring of G is called a *k-rainbow coloring* if for any set S of k vertices of G, there exists a rainbow S-tree. A simple result for the k-rainbow index [11] is that $k - 1 \le rx_k(G) \le n - 1$. It is easy to see that $rx_{2,\ell}(G) = rc_\ell(G)$. In the sequel, we always assume $k \ge 3$. We refer to [2–4, 12, 15, 18] for more details about the (k, ℓ)-rainbow index.

Computing the rainbow connection number of a graph is NP-hard [7], and the same is true for the computation of the (k, ℓ)-rainbow index. For this reason, one of the most important goals for studying rainbow connection number and rainbow index is to obtain good upper and lower bounds. In the search towards good upper bounds, an idea that turned out to be successful is considering the "strengthened" connected dominating set: find a suitable edge-coloring of the induced graph on such a set, and then extend it to the whole graph using a constant number of additional colors.

Given a graph G, a set $D \subseteq V(G)$ is called a *dominating set* if every vertex of $V \setminus D$ is adjacent to at least one vertex of D. Further, if the subgraph $G[D]$ of G induced by D is connected, we call D a *connected dominating set* of G. The *domination number* $\gamma(G)$ is the number of vertices in a minimum dominating set for G. Similarly, the *connected domination number* $\gamma_c(G)$ is the number of vertices in a minimum connected dominating set for G.

Let k be a positive integer. A dominating set D of G is called a *k-way dominating set* if $d(v) \geq k$ for every vertex $v \in V \setminus D$. In addition, if $G[D]$ is connected, we call D a *connected k-way dominating set*. A set $D \subseteq V(G)$ is called a *k-dominating set* of G if every vertex of $V \backslash D$ is adjacent to at least k distinct vertices of D. Furthermore, if $G[D]$ is connected, we call D a *connected k-dominating set*. Obviously, a (connected) k-dominating set is also a (connected) k-way dominating set, but the converse is not true.

There have been several results revealing the close relation between the dominating sets and the rainbow connection number or rainbow index.

Theorem 1. *[8] If D is a connected two-way dominating set of a connected graph G, then $rc(G) \leq rc(G[D]) + 3$.*

In [8], the authors employed Theorem 1 to get some tight upper bounds for the rainbow connection number of many special graph classes, which were otherwise difficult to obtain.

Theorem 2. *[18] Let G be a connected graph with minimal degree $\delta(G) \geq 3$. If D is a connected 2-dominating set of G, then $rx_3(G) \leq rx_3(G[D]) + 4$ and the bound is tight.*

From Theorem 2, the authors determined a tight upper bound for the 3-rainbow index of the complete bipartite graphs $K_{s,t}$ $(3 \leq s \leq t)$.

The proofs of the above two theorems are similar. First color the edges in $G[D]$ using k different colors ($k = rc(G[D])$ or $rx_3(G[D])$). Then select a spanning tree in every connected component of $H = G - D$. So we construct a spanning forest F of H and choose X and Y as any one of the bipartitions defined by the forest F. Color the edges between X and D and the edges between Y and D as well as the edges between X and Y with suitable colors, which gives an edge-coloring we want. Note that in the process all the edges in $E(H) - E(F)$ are ignored.

In this paper, we will take the edges in $E(H) - E(F)$ into consideration to get a more subtle coloring strategy. We show that for a connected graph G, $rx_3(G) \leq rx_3(G[D]) + 6$, where D is a connected 3-way dominating set of G. Moreover, this bound is tight. By using the results on spanning trees with many leaves, we obtain that $rx_3(G) \leq \frac{3n}{\delta+1} + 4$ for every connected graph G on n vertices with minimum degree at least δ $(3 \leq \delta \leq 5)$, and the bound is tight up to an additive constant; whereas for every connected graph G on n vertices with minimum degree at least δ $(\delta \geq 3)$, we get that $rx_3(G) \leq \frac{\ln(\delta+1)}{\delta+1}(1 + o_\delta(1))n + 5$. In addition, when considering a connected 3-dominating set D of G, we prove that $rx_3(G) \leq rx_3(G[D]) + 3$, and the bound is tight. The farthest we can get with this idea is some tight upper bounds for some special graph classes, including threshold graphs, chain graphs and interval graphs.

2 Preliminaries

For a graph G, we use $V(G)$, $E(G)$, $|G|$, $\delta(G)$, and $diam(G)$ to denote its vertex set, edge set, order (number of vertices), minimum degree and the diameter (maximum distance between every pair of vertices) of G, respectively. For $D \subseteq V(G)$, let $\overline{D} = V(G) \setminus D$, and $G[D]$ be the subgraph of G induced on D. For $v \in V(G)$, let $N(v)$ denote the set of neighbors of v. For two disjoint subsets X and Y of $V(G)$, $E[X,Y]$ denotes the set of edges of G between X and Y. The *join* of two graphs G and H, denoted by $G \vee H$, is the graph obtained from a disjoint union of G and H by adding edges joining every vertex of G to every vertex of H.

Definition 1. *BFS (breadth-first search) is a strategy for searching in a graph. It begins at a root and inspects all its neighbors. Then for each of those neighbors in turn, it inspects their neighbors which were unvisited, and so on until all the vertices in the graph are visited.*

Definition 2. *A BFS-tree (breadth-first search tree) is a spanning rooted tree returned by BFS. Let T be a BFS-tree with r as its root. For a vertex v, the height of v is the distance between v and r. All the vertices of height k form the kth level of T. The ancestors of v are the vertices on the unique $\{v,r\}$-path in T. The parent of v is its neighbor on the unique $\{v,r\}$-path in T. Its other neighbors are called the children of v. The siblings of v are the vertices in the same level as v. The left (resp. right) siblings of v are the siblings of v visited before (resp. after) v in BFS.*

Remark: BFS-trees have a nice property: every edge of the graph joins vertices on the same or consecutive levels. It is not possible for an edge to skip a level. Thus a neighbor of a vertex v has three possibilities: (1) a sibling of v; (2) the parent of v or a right sibling of the parent of v; (3) a child of v or a left sibling of the children of v; see Figure 1.

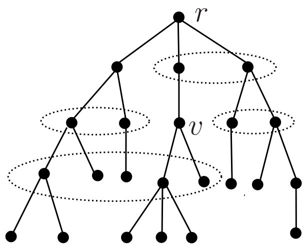

Figure 1: The vertices in the dotted circles are the potential neighbors of v.

Definition 3. *The Steiner distance $d(S)$ of a set S of vertices in a graph G is the minimum size of a tree in G containing S. The k-Steiner diameter $sdiam_k(G)$ of G is the maximum Steiner distance of S among all the sets S of k vertices in G. Obviously, $sdiam_2(G) = diam(G)$ and $sdiam_k(G) \leq sdiam_{k+1}(G)$.*

Definition 4. *Let G be a graph, $D \subseteq V(G)$ and $v \in V(G) \setminus D$. We call a path $P = v_0v_1 \cdots v_k$ a v-D path if $v_0 = v$ and $V(P) \cap D = \{v_k\}$. Two or more paths are called internally disjoint if none of them contains an inner vertex of another.*

Definition 5. *An edge-colored graph is rainbow if no two edges in the graph share the same color.*

Definition 6. *Let D be a dominating set of a graph G. For $v \in \overline{D}$, its neighbors in D are called the feet of v, and the corresponding edges are called the legs of v.*

Definition 7. *A graph G is called a threshold graph, if there exists a weight function $w : V(G) \to \mathbb{R}$ and a real constant t such that two vertices $u, v \in V(G)$ are adjacent if and only if $w(u) + w(v) \geq t$. We call t the threshold for G.*

Definition 8. *A bipartite graph $G = G[A, B]$ is called a chain graph, if the vertices of A can be ordered as $A = (a_1, a_2, \ldots, a_k)$ such that $N(a_1) \subseteq N(a_2) \subseteq \ldots \subseteq N(a_k)$.*

Definition 9. *An intersection graph of a family $\mathcal{F}$ of sets is a graph whose vertices can be mapped to the sets in $\mathcal{F}$ such that there is an edge between two vertices in the graph if and only if the corresponding two sets in $\mathcal{F}$ have a non-empty intersection. An interval graph is an intersection graph of intervals on the real line.*

3 Main results

Theorem 3. *If D is a connected 3-way dominating set of a connected graph G, then $rx_3(G) \leq rx_3(G[D]) + 6$. Moreover, the bound is tight.*

The proof of Theorem 3 is given in Section 4. Let us first show how this implies the following results.

Corollary 4. *For every connected graph G with $\delta(G) \geq 3$, $rx_3(G) \leq \gamma_c(G) + 5$.*

Proof. In this case, every connected dominating set of G is a connected 3-way dominating set. Now take a minimum connected dominating set D in G. Then $rx_3(G[D]) \leq |D| - 1 = \gamma_c(G) - 1$. It follows from Theorem 3 that $rx_3(G) \leq rx_3(G[D]) + 6 \leq \gamma_c(G) + 5$. □

From the following lemma, we can get the next corollary.

Lemma 5. *(1) [14] Every connected graph on n vertices with minimum degree $\delta \geq 3$ has a spanning tree with at least $\frac{1}{4}n + 2$ leaves.*

(2) [13] Every connected graph on n vertices with minimum degree $\delta \geq 4$ has a spanning tree with at least $\frac{2}{5}n + \frac{8}{5}$ leaves.

(3) [13] Every connected graph on n vertices with minimum degree $\delta \geq 5$ has a spanning tree with at least $\frac{1}{2}n + 2$ leaves.

Corollary 6. *(1) For every connected graph G on n vertices with $\delta(G) \geq 3$, $rx_3(G) \leq \frac{3}{4}n + 3$.*

(2) For every connected graph G on n vertices with $\delta(G) \geq 4$, $rx_3(G) \leq \frac{3}{5}n + \frac{17}{5}$.

(3) For every connected graph G on n vertices with $\delta(G) \geq 5$, $rx_3(G) \leq \frac{1}{2}n + 3$.

Moreover, these bounds are tight up to an additive constant.

Proof. We only prove (1); (2) and (3) can be derived by the same arguments.

Clearly, we can take a connected dominating set consisting of all the non-leaves in the spanning tree. Thus by Lemma 5, for every connected graph G on n vertices with minimum degree $\delta(G) \geq 3$, $\gamma_c(G) \leq n - (\frac{1}{4}n + 2) = \frac{3}{4}n - 2$. Then it follows from Corollary 4 that $rx_3(G) \leq \frac{3}{4}n + 3$.

On the other hand, the factors in these bounds cannot be improved, since there exist infinitely many graphs G^* such that $rx_3(G^*) \geq \frac{3}{\delta+1}n - \frac{\delta+7}{\delta+1}$. We construct the graphs as follows (the construction was also mentioned in [5]): first take m copies of $K_{\delta+1}$, denoted by $X_1, X_2, \ldots, X_m$ and label the vertices of X_i with $x_{i,1}, \ldots, x_{i,\delta+1}$. Then take two copies of $K_{\delta+2}$, denoted by X_0 and X_{m+1} and similarly label their vertices. Now join $x_{i,2}$ and $x_{i+1,1}$ for $i \in \{0, 1, \ldots, m\}$ with an edge and delete the edges $x_{i,1}x_{i,2}$ for $i \in \{0, 1, \ldots, m+1\}$. It is easy to see that $diam(G^*) = \frac{3}{\delta+1}n - \frac{\delta+7}{\delta+1}$. The k-Steiner diameter of a graph is a trivial lower bound for its k-rainbow index [11], and so $rx_3(G^*) \geq sdiam_3(G^*) \geq diam(G^*) = \frac{3}{\delta+1}n - \frac{\delta+7}{\delta+1}$. For $\delta = 3$, $rx_3(G^*) \geq \frac{3}{4}n - \frac{5}{2}$; for $\delta = 4$, $rx_3(G^*) \geq \frac{3}{5}n - \frac{11}{5}$; for $\delta = 5$, $rx_3(G^*) \geq \frac{1}{2}n - 2$. Therefore, all these upper bounds are tight up to an additive constant. □

As to general δ, Caro et. al. [6] proved that for every connected graph G on n vertices with minimum degree δ, $\gamma_c(G) = \frac{\ln(\delta+1)}{\delta+1}(1 + o_\delta(1))n$. Combining with Corollary 4, we get the following result.

Corollary 7. *For every connected graph G on n vertices with minimum degree $\delta \geq 3$, $rx_3(G) \leq \frac{\ln(\delta+1)}{\delta+1}(1 + o_\delta(1))n + 5$.*

The above bound is not believed to be optimal for $rx_3(G)$ in terms of δ. We pose the following conjecture, which has already been proved for $\delta \in \{3, 4, 5\}$ in Corollary 6. Note that if the conjecture is true, it gives an upper bound tight up to an additive constant by the construction of the graph G^*.

Conjecture 1. *For every connected graph G on n vertices with minimum degree $\delta \geq 3$, $rx_3(G) \leq \frac{3n}{\delta+1} + C$, where C is a positive constant.*

With regard to the graphs possessing vertices of degree 1 or 2, we obtain the following result.

Corollary 8. *For every connected graph G, $rx_3(G) \leq \gamma_c(G) + n_1 + n_2 + 5$, where n_1 and n_2 denote the number of vertices of degrees 1 and 2 in G, respectively.*

Proof. Obviously, adding all the vertices of degrees 1 and 2 into a minimum connected dominating set gives rise to a connected 3-way dominating set in G of size no more than $\gamma_c(G) + n_1 + n_2$. Consequently, by Theorem 3, $rx_3(G) \leq \gamma_c(G) + n_1 + n_2 + 5$. □

We proceed with another upper bound for the 3-rainbow index of graphs concerning the connected 3-dominating set.

Theorem 9. *If D is a connected 3-dominating set of a connected graph G with $\delta(G) \geq 3$, then $rx_3(G) \leq rx_3(G[D]) + 3$. Moreover, the bound is tight.*

Proof. Since D is a connected 3-dominating set, every vertex in $\overline{D}$ has at least three legs. Color one of them with 1, one of them with 2, and all the others with 3. Let $k = rx_3(G[D])$. Then we can color the edges in $G[D]$ with k different colors from $\{4, 5, \ldots, k+3\}$ such that for every triple of vertices in D, there exists a rainbow tree in $G[D]$ connecting them. If there remain uncolored edges in G, we color them with 1.

Next we will show that this edge-coloring is a 3-rainbow coloring of G. For any triple $\{u, v, w\}$ of vertices in G, if $(u, v, w) \in D \times D \times D$, then there is already a rainbow tree connecting them in $G[D]$. If one of them is in $\overline{D}$, say $(u, v, w) \in \overline{D} \times D \times D$, join any leg of u (colored by 1, 2, or 3) with the rainbow tree connecting v, w and the corresponding foot of u in $G[D]$. If two of them are in $\overline{D}$, say $(u, v, w) \in \overline{D} \times \overline{D} \times D$, join one leg of u colored by 1, one leg of v colored by 2 with the rainbow tree connecting w and the corresponding feet of u, v in $G[D]$. If $(u, v, w) \in \overline{D} \times \overline{D} \times \overline{D}$, join one leg of u colored by 1, one leg of v colored by 2, one leg of w colored by 3 with the rainbow tree connecting the corresponding feet of u, v, w in $G[D]$.

The tightness of the bound can be seen from the next Corollary. □

As immediate consequences of Theorem 3 and Theorem 9, we get the following:

Corollary 10. *Let G be a connected graph with $\delta(G) \geq 3$.*

(1) If G is a threshold graph, then $rx_3(G) \leq 5$;

(2) If G is a chain graph, then $rx_3(G) \leq 6$;

(3) If G is an interval graph, then $rx_3(G) \leq diam(G) + 4$. Thus $diam(G) \leq rx_3(G) \leq diam(G) + 4$;

Moreover, all these upper bounds are tight.

Proof. (1) Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$ where $w(v_1) \geq w(v_2) \geq \ldots \geq w(v_n)$. Since the minimum degree of G is at least three, v_i $(1 \leq i \leq 3)$ is adjacent to all the other vertices in G. Thus $D = \{v_1, v_2, v_3\}$ is a connected 3-dominating set of G. Note that D

induces a K_3, so $rx_3(G[D]) = 2$. It follows from Theorem 9 that $rx_3(G) \le rx_3(G[D])+3 = 5$.

(2) Suppose that $G = G[A, B]$ and the vertices of A can be ordered as $A = (a_1, a_2, \dots, a_k)$ such that $N(a_1) \subseteq N(a_2) \subseteq \dots \subseteq N(a_k)$. Since the minimum degree of G is at least three, a_i $(k-2 \le i \le k)$ is adjacent to all the vertices in B, and $N(a_1)$ has at least three vertices, say $\{b_1, b_2, b_3\}$. Clearly b_i $(1 \le i \le 3)$ is adjacent to all the vertices in A. Thus $D = \{a_{k-2}, a_{k-1}, a_k, b_1, b_2, b_3\}$ is a connected 3-dominating set of G. Note that D induces a $K_{3,3}$, so $rx_3(G[D]) = 3$ (see [12]). It follows from Theorem 9 that $rx_3(G) \le rx_3(G[D]) + 3 = 6$.

(3) If G is isomorphic to a complete graph, then $rx_3(G) = 2$ or 3 (see [11]), the assertion holds trivially. Otherwise, it was shown in [8] that every interval graph G which is not isomorphic to a complete graph has a dominating path P of length at most $diam(G) - 2$. Since $\delta(G) \ge 3$, P is a connected 3-way dominating set of G. It follows from Theorem 3 that $rx_3(G) \le rx_3(P) + 6 \le diam(G) + 4$. On the other hand, $rx_3(G) \ge sdiam_3(G) \ge diam_(G)$. We conclude that for a connected interval graph G with $\delta(G) \ge 3$, $diam(G) \le rx_3(G) \le diam(G) + 4$.

Here we give examples to show the tightness of these upper bounds.

Example 1: A threshold graph G with $\delta(G) \ge 3$ and $rx_3(G) = 5$.

Consider the graph $G = tK_1 \vee K_3$, where $t \ge 2 \times 4^3 + 1$. The vertices in tK_1 are labeled by $x_1, x_2, \dots, x_t$, and the vertices in K_3 are labeled by y_1, y_2, y_3. It is easy to see that it is a threshold graph (y_1, y_2, y_3 can be given a weight 1, others a weight 0 and the threshold 1). By contradiction, we assume that G can be colored with four colors. Let S_i denote the star with x_i as its center and $E(S_i) = \{x_iy_1, x_iy_2, x_iy_3\}$. Every S_i can be colored in 4^3 different ways. Since $t \ge 2 \times 4^3 + 1$, there exist three completely identical edge-colored stars, say S_1, S_2 and S_3. If two of the three edges in S_i $(1 \le i \le 3)$ receive the same color, then there are no rainbow trees connecting x_1, x_2, x_3, a contradiction. If the three edges in S_i $(1 \le i \le 3)$ receive distinct colors, then the rainbow tree connecting x_1, x_2, x_3 must contain the vertices y_1, y_2, y_3. Thus the tree has at least five edges, but only four different colors, a contradiction.

Example 2: A chain graph G with $\delta(G) \ge 3$ and $rx_3(G) = 6$.

Consider the chain graph $G = G[A, B]$, where $A = \{a_1, a_2, \dots, a_k\}$, $B = \{b_1, b_2, \dots, b_t\}$, $N(a_1) = N(a_2) = \cdots = N(a_{k-3}) = \{b_1, b_2, b_3\}$, $N(a_{k-2}) = N(a_{k-1}) = N(a_k) = \{b_1, b_2, \dots, b_t\} = B$, and $t \ge 2 \times 5^3 + 4$. By contradiction, we assume that G can be colored with five colors. Let S_i $(4 \le i \le t)$ denote the star with b_i as its center and $E(S_i) = \{b_ia_{k-2}, b_ia_{k-1}, b_ia_k\}$. Every S_i can be colored in 5^3 different ways. Since $t - 3 \ge 2 \times 5^3 + 1$, among the $t - 3$ $S_i's$ there exist three completely identical edge-colored stars, say S_4, S_5 and S_6. If two of the three edges in S_i $(4 \le i \le 6)$ receive the same color, then there are no rainbow trees connecting b_4, b_5, b_6, a contradiction. If the three edges in S_i $(4 \le i \le 6)$ receive distinct colors, then the rainbow tree connecting b_4, b_5, b_6 must contain a_{k-2}, a_{k-1}, a_k and at least one vertex in $B \setminus \{b_4, b_5, b_6\}$ to connect a_{k-2}, a_{k-1}, a_k. Thus the tree has at least six edges, but only five different colors, a contradiction.

Example 3: An interval graph G with $\delta(G) \geq 3$ and $rx_3(G) = diam(G) + 4$.

Consider the graph $G = tK_3 \vee K_1$, where $t \geq 2 \times 5^6 + 1$. The vertices in the ith K_3 ($1 \leq i \leq t$) are labeled by u_i, v_i, w_i, and the vertex in K_1 is labeled by v_0. It is easy to see that it is an interval graph with diameter 2. It follows from (3) that $rx_3(G) \leq diam(G) + 4 = 6$. We will show that $rx_3(G) = 6$. By contradiction, we assume that G can be colored with five colors. Let B_i denote the K_4 induced by v_0, u_i, v_i, w_i. Obviously, each B_i can be colored in at most 5^6 different ways. Since $t \geq 2 \times 5^6 + 1$, there exist three completely identical edge-colored subgraphs, say B_1, B_2, B_3. If two of the three edges incident with v_0 in B_i ($1 \leq i \leq 3$) receive the same color, say $c(v_0u_i) = c(v_0v_i) = 1$, then there are no rainbow trees connecting u_1, u_2, u_3, a contradiction. If the three edges incident with v_0 in B_i ($1 \leq i \leq 3$) receive distinct colors, say $c(v_0u_i) = 1, c(v_0v_i) = 2, c(v_0w_i) = 3$, then $c(u_iv_i) \neq c(u_iw_i)$ and $c(u_iv_i), c(u_iw_i) \in \{4, 5\}$ because there exists a rainbow tree connecting $\{u_1, u_2, u_3\}$. Without loss of generality, suppose $c(u_iv_i) = 4$ and $c(u_iw_i) = 5$. Since there exists a rainbow tree connecting $\{v_1, v_2, v_3\}$, then $c(v_iw_i) = 5$. But then there exist no rainbow trees connecting $\{w_1, w_2, w_3\}$ in G, a contradiction. □

4 Proof of Theorem 3

Let D be a connected 3-way dominating set of a connected graph G. We want to show that $rx_3(G) \leq rx_3(G[D]) + 6$.

To start with, we introduce some definitions and notation that are used in the sequel. A set of rainbow paths $\{P_1, P_2, \ldots, P_\ell\}$ is called *super-rainbow* if their union $\cup_{i=1}^{\ell} P_i$ is also rainbow. For a vertex v in $\overline{D}$, we call it *safe* if there are three internally disjoint super-rainbow $v - D$ paths. Otherwise, we call v *dangerous*. An edge coloring is called *nice* if all the vertices in $\overline{D}$ are safe under this coloring. Let $c(e)$ be the color of an edge e, $c(H)$ the set of colors appearing on the edges in a graph H. For a vertex v in a BFS-tree, we denote the height of v by $h(v)$, the parent of v by $p(v)$, the child of v by $ch(v)$, the ancestor of v in the first level by $\pi(v)$.

Claim 1. *Let D be a connected 3-way dominating set of a connected graph G, and c an edge coloring of G. If for any three vertices u, v, w in $\overline{D}$, there exists a rainbow $u - D$ path P^u, a rainbow $v - D$ path P^v and a rainbow $w - D$ path P^w such that $P^u \cup P^v \cup P^w$ is also rainbow under this coloring, then c is a 3-rainbow coloring.*

The proof of Claim 1 is given in section 4.3.

Let c be a nice edge coloring of G. Then for any three vertices u, v, w in $\overline{D}$, u, v, w are safe under this coloring. That is, there exist three internally-disjoint super-rainbow

$u-D$ paths P_1^u, P_2^u, P_3^u, three internally-disjoint super-rainbow $v-D$ paths P_1^v, P_2^v, P_3^v and three internally-disjoint super-rainbow $w-D$ paths P_1^w, P_2^w, P_3^w. If we can pick out P_i^u, P_j^v and P_k^w $(1 \leq i,j,k \leq 3)$ from these paths satisfying $P_i^u \cup P_j^v \cup P_k^w$ is also rainbow, then c is a 3-rainbow coloring from Claim 1. But unfortunately in some cases, we can not do that. For example, if $c(P_1^u) \cup c(P_2^u) \cup c(P_3^u) = \{1\} \cup \{2,4\} \cup \{5,6\}$, $c(P_1^v) \cup c(P_2^v) \cup c(P_3^v) = \{1\} \cup \{2,5\} \cup \{4,6\}$, $c(P_1^w) \cup c(P_2^w) \cup c(P_3^w) = \{1\} \cup \{2,6\} \cup \{4,5\}$, then one can check that $P_i^u \cup P_j^v \cup P_k^w$ is not rainbow for each $1 \leq i,j,k \leq 3$. Here we show a sufficient and necessary condition for the situation in which we can pick out suitable P_i^u, P_j^v and P_k^w. Note that P_1^u, P_1^v, P_1^w are of length one.

Claim 2. *Let D be a connected 3-way dominating set of a connected graph G, and c a nice edge coloring of G. For any three vertices u,v,w in $\overline{D}$, there exist $i,j,k \in \{1,2,3\}$ satisfying $P_i^u \cup P_j^v \cup P_k^w$ is rainbow if and only if*

(M1) *$c(P_1^u)$, $c(P_1^v)$, $c(P_1^w)$ are not the same, or*

(M2) *there exist two distinct vertices $x,y \in \{u,v,w\}$ and two integers $s,t \in \{2,3\}$ (s may equal to t) such that $c(P_s^x) \cap c(P_t^y) = \emptyset$.*

The proof of Claim 2 is also given in section 4.3.

From Claim 1 and 2, it suffices to find a nice edge coloring satisfying M1 and M2 using at most $rx_3(G[D]) + 6$ colors.

Let us give an overview of our idea. Firstly, we aim to color the edges in $E[D,\overline{D}]$ and $E(G[\overline{D}])$ with six different colors. Our coloring strategy has two steps: in the first step, we give a periodical coloring on some edges in $E[D,\overline{D}]$ and $E(G[\overline{D}])$. Then most vertices in $\overline{D}$ become safe. In the second step, we color the carefully chosen uncolored edges and recolor some colored edges intelligently to ensure that all the vertices in $\overline{D}$ are safe and the conditions M1 and M2 are satisfied. Then we extend the coloring to the whole graph by coloring the edges in $G[D]$ with $rx_3(G[D])$ fresh colors.

4.1 Coloring the edges in $E[D,\overline{D}]$ and $E(G[\overline{D}])$

4.1.1 First step: a periodical coloring

Assume that $C_1, C_2, \ldots, C_q$ are the connected components of the subgraph $G-D$.

If C_i $(1 \leq i \leq q)$ consists of an isolated vertex v, then v has at least three legs. We color one of them with 1, one of them with 2, and all the others with 3. Now $c(P_1^v) = \{1\}$, $c(P_2^v) = \{2\}$, $c(P_3^v) = \{3\}$. So v is safe.

If C_i $(1 \leq i \leq q)$ consists of an isolated edge uv, then u has at least two legs. We color one of them with 1, and all the others with 2. Similarly, v has at least two legs. We color one of them with 2, and all the others with 3. Color uv with 4. Now $c(P_1^u) = \{1\}$, $c(P_2^u) = \{2\}$, $c(P_3^u) = \{3,4\}$, and $c(P_1^v) = \{2\}$, $c(P_2^v) = \{3\}$, $c(P_3^v) = \{1,4\}$. So both u and v are safe.

If C_i $(1 \le i \le q)$ consists of at least three vertices, then there exists a vertex v_0 in C_i possessing at least two neighbors in C_i. Starting from v_0, we construct a BFS-tree T of C_i. Suppose the neighbors of v_0 in C_i are $\{v_1, v_2, \ldots, v_k\}$ $(k \ge 2)$, which forms the first level of T. For each vertex v in C_i, let e_v be one leg of v (if there are many legs, we pick one arbitrarily), $t(v)$ the corresponding foot of v, f_v the unique edge joining v and its parent in T.

Now we color the edges e_v and f_v as follows: $c(e_{v_0}) = 2$; $c(f_{v_i}) = 4$ and $c(e_{v_i}) = 1$ for $1 \le i \le k-1$; $c(f_{v_k}) = 5$ and $c(e_{v_k}) = 3$; for each vertex v in $V(C_i) \setminus \{v_0, v_1, \ldots, v_k\}$, if $\pi(v) = v_k$, then set $c(f_v) = 4$ and $c(e_v) = 2$ when $h(v) \equiv 0 \pmod 3$, $c(f_v) = 5$ and $c(e_v) = 3$ when $h(v) \equiv 1 \pmod 3$, $c(f_v) = 6$ and $c(e_v) = 1$ when $h(v) \equiv 2 \pmod 3$; if $\pi(v) = v_i$ $(1 \le i \le k-1)$, then set $c(f_v) = 6$ and $c(e_v) = 2$ when $h(v) \equiv 0 \pmod 3$, $c(f_v) = 4$ and $c(e_v) = 1$ when $h(v) \equiv 1 \pmod 3$, $c(f_v) = 5$ and $c(e_v) = 3$ when $h(v) \equiv 2 \pmod 3$. In fact, this gives a periodical coloring depicted as Figure 2.

We call the subtree of T rooted at v_i $(1 \le i \le k-1)$ of type I and the subtree of T rooted at v_k of type II. There may be many subtrees of type I, but only one subtree of type II. The subtrees of the same type are colored in the same way. More precisely, if two vertices u, v lie in the same level and belong to subtrees of the same type, then $c(e_u) = c(e_v)$ and $c(f_u) = c(f_v)$ after the first step.

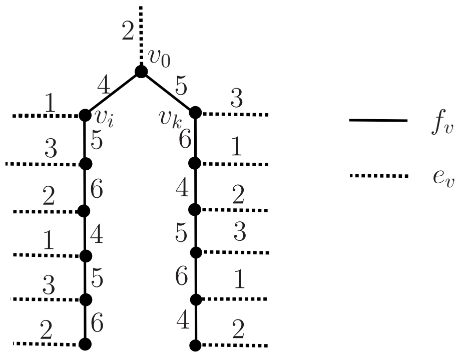

Figure 2: The left branch represents the coloring of subtrees of type I and the right branch represents the coloring of the subtree of type II.

Now each non-leaf vertex v in T has three internally disjoint super-rainbow $v - D$ paths. For the root v_0, $P_1^{v_0} = v_0 t(v_0)$; $P_2^{v_0} = v_0 v_1 t(v_1)$; $P_3^{v_0} = v_0 v_k t(v_k)$. For other non-leaf vertex v in T, $P_1^v = vt(v)$; $P_2^v = vp(v)t(p(v))$; $P_3^v = vch(v)t(ch(v))$. Note that v may have many children $u_1, u_2, \ldots, u_\ell$, but all the $e_{u_i}{}'s$ $(f_{u_i}{}'s)$ receive the same color. So they only contribute one path to the set of three internally disjoint super-rainbow $v - D$ paths. In other words, after the first step all the non-leaf vertices in T are safe.

As to each leaf w in T, since w has no children, it has exactly two internally disjoint super-rainbow $w - D$ paths: $P_1^w = wt(w)$, $P_2^w = wp(w)t(p(w))$. In other words, after the first step all the leaves in T are dangerous.

Example 4: The root v_0 is safe: $c(P_1^{v_0}) = \{2\}$, $c(P_2^{v_0}) = \{1, 4\}$, $c(P_3^{v_0}) = \{3, 5\}$.

If v_i $(1 \leq i \leq k-1)$ is not a leaf of T, then v_i is safe: $c(P_1^{v_i}) = \{1\}$, $c(P_2^{v_i}) = \{2, 4\}$, $c(P_3^{v_i}) = \{3, 5\}$.

If v_k is not a leaf of T, then v_k is safe: $c(P_1^{v_k}) = \{3\}$, $c(P_2^{v_k}) = \{2, 5\}$, $c(P_3^{v_k}) = \{1, 6\}$.

Example 5: If w is a leaf of T in the second level with parent v_k, then w is dangerous: $c(P_1^w) = \{1\}$, $c(P_2^w) = \{3, 6\}$.

For the sake of brevity, we write $\{1, 24, 35\}$ instead of $c(P_1) = \{1\}, c(P_2) = \{2, 4\}, c(P_3) = \{3, 5\}$ and $c(P_1) = \{1\}, c(P_2) = \{3, 5\}, c(P_3) = \{2, 4\}$. All the possible color sets of the three internally disjoint super-rainbow paths connecting a non-leaf vertex in T to D are:

$\{1, 24, 35\}$, $\{2, 36, 14\}$, $\{3, 15, 26\}$, $\{1, 36, 24\}$, $\{2, 14, 35\}$, $\{3, 25, 16\}$.

Bearing in mind that D is a connected 3-way dominating set, each leaf in T is incident with at least one uncolored edge. In the second step, we will color such edges and recolor some colored edges suitably to ensure that all the vertices in C_i are safe and the conditions M1 and M2 are satisfied.

4.1.2 Second step: more edges with a more intelligent coloring

Let w be a leaf in T and $g_w = wu$ one uncolored edge incident with w.

If g_w connects w to D, then we give g_w a smallest color from $\{1, 2, 3, 4, 5, 6\} \backslash (c(P_1^w) \cup c(P_2^w))$. For instance, $c(g_w) = 2$ for the vertex w in Example 5. Obviously, now w has three internally disjoint super-rainbow $w - D$ paths: $P_1^w = wt(w)$, $P_2^w = wu$, $P_3^w = wp(w)t(p(w))$. In other words, w is safe after the second step. All the possible color sets of the three internally disjoint super-rainbow $w - D$ paths are

$\{1, 2, 36\}$, $\{1, 3, 24\}$, $\{1, 3, 25\}$, $\{2, 3, 15\}$, $\{2, 3, 14\}$.

Now it remains to deal with the leaves in T whose incident uncolored edges all lie in C_i. Let A denote the set of such vertices. Firstly we flag all the vertices in $V(C_i) \setminus A$, which are already safe. Note that we only flag the safe vertices. Once one vertex gets flagged, it is always flagged. Secondly we arrange the vertices in A in a linear order by the following three rules:

(R1) for $w, w' \in A$, let $\pi(w) = v_i$ and $\pi(w') = v_j$, if $i > j$, then w is before w' in the ordering;

(R2) if $\pi(w) = \pi(w')$ and $h(w) < h(w')$, then w is before w' in the ordering;

(R3) if $\pi(w) = \pi(w')$, $h(w) = h(w')$ and w is reached earlier than w' in the BFS-algorithm, then w is before w' in the ordering.

Assume the vertices in A are ordered as $A = (w_1, w_2, \ldots, w_s)$. We will visit them one by one. Suppose that now we go to the vertex w_i $(w_1, w_2, \ldots, w_{i-1}$ have been visited). If w_i is flagged, then we go to the next vertex w_{i+1}; otherwise, we do two operations: (i) coloring g_{w_i} suitably; (ii) recoloring e_{w_i} suitably if necessary. Note that e_{w_i} is the only edge which may be recolored when dealing with w_i. Furthermore, we recolor it in such a way that the parent of w_i is still safe. In fact, for that sake, we have no choice but to recolor e_{w_i} with the unique color which is from $\{1,2,3,4,5,6\}$ but does not appear on the three super-rainbow paths of $p(w_i)$ after the first step. For example, in *Subcase* 1.2, the color set of the three super-rainbow paths of $p(w_i)$ after the first step is $\{2,1,4,5,3\}$, so we recolor e_{w_i} with 6. It is also worth mentioning that, the edge e_v of a vertex v in $V(C_i) \setminus A$ remains the color received in the first step all the time, whereas the edge e_w of a vertex w in A remains the color received in the first step before dealing with w. We distinguish the following four cases:

Case 1: $\pi(w_i) = v_k$ and there exists at least one uncolored edge connecting w_i to some subtree of type I. Then we choose one such edge $g_{w_i} = w_i u$ such that the height of u is as small as possible. Since T is a BFS-tree and the subtree of w_i is to the right of the subtree of u, we get $h(u) = h(w_i)$ or $h(u) = h(w_i) + 1$.

Fact 1. e_u is not recolored.

If $u \in V(C_i)\setminus A$, then e_u never gets recolored. If $u \in A$, since $\pi(w_i) = v_k$ and $\pi(u) = v_j$ $(1 \le j \le k-1)$, we have not dealt with u yet according to R1, thus e_u is not recolored.

We distinguish three subcases based on the height of w_i.

$*$ *Subcase 1.1*: $h(w_i) \equiv 0 (\text{mod } 3)$

If $h(u) = h(w_i)$, then color g_{w_i} with 5. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{1,4\} \cup \{3,5,6\}$ and $c(P_1^u) \cup c(P_2^u) \cup c(P_3^u) = \{2\} \cup \{3,6\} \cup \{1,4,5\}$.

If $h(u) = h(w_i) + 1$, then color g_{w_i} with 5. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{2\} \cup \{3,4,6\} \cup \{1,5\}$ and $c(P_1^u) \cup c(P_2^u) \cup c(P_3^u) = \{1\} \cup \{3,4,6\} \cup \{2,5\}$.

$*$ *Subcase 1.2*: $h(w_i) \equiv 1 (\text{mod } 3)$

If $h(u) = h(w_i)$, then color g_{w_i} with 6. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{3\} \cup \{2,5\} \cup \{1,6\}$ and $c(P_1^u) \cup c(P_2^u) \cup c(P_3^u) = \{1\} \cup \{2,4\} \cup \{3,6\}$.

If $h(u) = h(w_i) + 1$, then color g_{w_i} with 4 and recolor e_{w_i} with 6. In this way, we ensure that the parent of w_i is still safe. Now $c(P_1^{p(w_i)}) \cup c(P_2^{p(w_i)}) \cup c(P_3^{p(w_i)}) = \{2\} \cup \{1,4\} \cup \{5,6\}$. Moreover, we have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{6\} \cup \{2,5\} \cup \{3,4\}$ and $c(P_1^u) \cup c(P_2^u) \cup c(P_3^u) = \{3\} \cup \{1,5\} \cup \{4,6\}$.

$*$ *Subcase 1.3*: $h(w_i) \equiv 2 (\text{mod } 3)$

If $h(u) = h(w_i)$, then color g_{w_i} with 2 and recolor e_{w_i} with 4. In this way, we ensure that the parent of w_i is still safe. Now $c(P_1^{p(w_i)}) \cup c(P_2^{p(w_i)}) \cup c(P_3^{p(w_i)}) = \{3\} \cup \{2,5\} \cup \{4,6\}$. Moreover, we have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{4\} \cup \{3,6\} \cup \{1,2,5\}$ and $c(P_1^u) \cup c(P_2^u) \cup c(P_3^u) = \{3\} \cup \{1,5\} \cup \{2,4\}$.

If $h(u) = h(w_i) + 1$, then color g_{w_i} with 5. We have $c(P_1^{w_i}) \cup c(P_2^{w_i}) \cup c(P_3^{w_i}) = \{1\} \cup \{3,6\} \cup \{2,5\}$ and $c(P_1^u) \cup c(P_2^u) \cup c(P_3^u) = \{2\} \cup \{3,6\} \cup \{1,5\}$.

Now both w_i and u are safe after the process. We flag w_i and u (if u is not flagged).

For clarity, we present the above coloring in Table 1, where $j \equiv h(w_i)$ (mod 3) and

$k = h(u) - h(w_i)$.

	$k=0$	$k=1$
$j=0$	$c(g_{w_i})= 5$ w_i: $\{2, 14, 356\}$ u : $\{2, 36, 145\}$	$c(g_{w_i})= 5$ w_i: $\{2, 346, 15\}$ u : $\{1, 346, 25\}$
$j=1$	$c(g_{w_i}) = 6$ w_i: $\{3, 25, 16\}$ u : $\{1, 24, 36\}$	$c(g_{w_i}) = 4$ $c(e_{w_i})$: $3 \to 6$ $p(w_i)$: $\{2, 14, 56\}$ w_i: $\{6, 25, 34\}$ u : $\{3, 15, 46\}$
$j=2$	$c(g_{w_i})= 2$ $c(e_{w_i})$: $1 \to 4$ $p(w_i)$: $\{3, 25, 46\}$ w_i: $\{4, 36, 125\}$ u : $\{3, 15, 24\}$	$c(g_{w_i})= 5$ w_i: $\{1, 36, 25\}$ u : $\{2, 36, 15\}$

Table 1. The coloring in Case 1

Remarks: (1) One may wonder what is the effect of these operations. First of all, after the process, w_i becomes safe and gets flagged, and so does u if u is not flagged. In addition, the process guarantees that all the safe vertices remain safe. As mentioned above, $p(w_i)$ is still safe after this process. For any other safe vertex in $V(C_i) \setminus A$, obviously its three internally disjoint super-rainbow paths do not contain e_{w_i} or g_{w_i}, thus it is still safe after this process. For each safe vertex w_j in A, its three internally disjoint super-rainbow paths do not contain e_{w_i} or g_{w_i}. Otherwise, it implies that $j < i$ and w_i is already safe and flagged after dealing with w_j, so when visiting w_i we go to w_{i+1} directly without doing this process. Thus w_j is still safe after this process.

(2) The three internally disjoint super-rainbow paths of w_i is one of the following three cases; see Figure 3.

(i) $P_1^{w_i} = w_i t(w_i)$, $P_2^{w_i} = w_i p(w_i) t(p(w_i))$, $P_3^{w_i} = w_i u t(u)$;

(ii) $P_1^{w_i} = w_i t(w_i)$, $P_2^{w_i} = w_i p(w_i) t(p(w_i))$, $P_3^{w_i} = w_i u p(u) t(p(u))$;

(iii) $P_1^{w_i} = w_i t(w_i)$, $P_2^{w_i} = w_i p(w_i) p(p(w_i)) t(p(p(w_i)))$, $P_3^{w_i} = w_i u t(u)$.

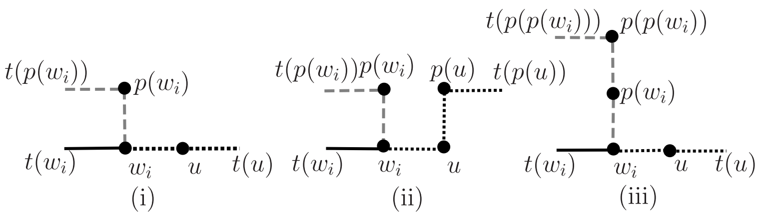

Figure 3: (2) of the Remarks.

Case 2: $\pi(w_i) = v_k$ and all the uncolored edges connect w_i to the subtree of type II. Then we choose one such edge $g_{w_i} = w_i u$ such that the height of u is as small as possible.

Since T is a $BFS-$tree, we get $h(u) = h(w_i) - 1$, $h(u) = h(w_i)$, or $h(u) = h(w_i) + 1$. The following two facts are easy to see:

Fact 2. If $h(u) = h(w_i) - 1$, then u is already flagged.

If $u \in V(C_i) \setminus A$, then u has been flagged at the very beginning. If $u \in A$, since $\pi(u) = \pi(w_i) = v_k$ and $h(u) < h(w_i)$, we have already dealt with u according to R2, thus u is flagged (note that e_u may be recolored).

Fact 3. If $h(u) = h(w_i) + 1$, then e_u is not recolored.

If $u \in V(C_i) \setminus A$, then e_u never gets recolored. If $u \in A$, since $\pi(u) = \pi(w_i) = v_k$ and $h(u) > h(w_i)$, we have not dealt with u yet according to R2, thus e_u is not recolored.

With similar arguments, we get the coloring presented in Table 2. Note that if e_u is recolored, then $u \in A$ has already been dealt with and gotten flagged. Recall that $j \equiv h(w_i) \pmod 3$ and $k = h(u) - h(w_i)$.

		$k=0$	$k=1$	$k=-1$
$j=0$	e_u is not recolored	$c(g_{w_i})= 6$ $c(e_{w_i})$: $2 \to 5$ $p(w_i)$: $\{1, 36, 45\}$ w_i: $\{5, 14, 26\}$ u : $\{2, 14, 56\}$	$c(g_{w_i})= 6$ w_i: $\{2, 14, 36\}$ u : $\{3, 25, 146\}$	$c(g_{w_i})= 5$ w_i: $\{2, 14, 356\}$
	e_u is recolored	$c(g_{w_i})= 6$ w_i: $\{2, 14, 56\}$	impossible (by Fact 3)	$c(g_{w_i})= 5$ w_i: $\{2, 14, 356\}$
$j=1$	e_u is not recolored	$c(g_{w_i})= 4$ $c(e_{w_i})$: $3 \to 6$ $p(w_i)$: $\{2, 14, 56\}$ w_i: $\{6, 25, 34\}$ u : $\{3, 25, 46\}$	$c(g_{w_i})= 4$ w_i: $\{3, 25, 14\}$ u: $\{1, 36, 245\}$	$c(g_{w_i})= 6$ w_i: $\{3, 25, 146\}$
	e_u is recolored	$c(g_{w_i})= 4$ w_i: $\{3, 25, 46\}$	impossible (by Fact 3)	$c(g_{w_i})= 6$ w_i: $\{3, 25, 146\}$
$j=2$	e_u is not recolored	$c(g_{w_i})= 5$ $c(e_{w_i})$: $1 \to 4$ $p(w_i)$: $\{3, 25, 46\}$ w_i: $\{4, 36, 15\}$ u : $\{1, 36, 45\}$	$c(g_{w_i})= 5$ w_i: $\{1, 36, 25\}$ u: $\{2, 14, 356\}$	$c(g_{w_i})= 4$ w_i: $\{1, 36, 245\}$
	e_u is recolored	$c(g_{w_i})= 5$ w_i: $\{1, 36, 45\}$	impossible (by Fact 3)	$c(g_{w_i})= 4$ w_i: $\{1, 36, 245\}$

Table 2. The coloring in Case 2

Now both w_i and u are safe. We flag w_i and u (if u is not flagged).

Case 3: $\pi(w_i) = v_j (1 \le j \le k-1)$ and there exists at least one uncolored edge connecting w_i to some subtree of type I. Then we choose one such edge $g_{w_i} = w_i u$ such that the height of u is as small as possible. Since T is a BFS-tree, we get $h(u) = h(w_i) - 1$, $h(u) = h(w_i)$, or $h(u) = h(w_i) + 1$. We have the following two facts, which are similar to

Fact 2 and 3:

Fact $2'$**.** If $h(u) = h(w_i) - 1$, then u is already flagged.

If $u \in V(C_i)\backslash A$, then u has been flagged at the very beginning. If $u \in A$, let $\pi(u) = v_{j'}$, then $1 \leq j \leq j' \leq k-1$ and $h(u) < h(w_i)$. So we have already dealt with u according to R1 and R2, thus u is flagged (note that e_u may be recolored).

Fact $3'$**.** If $h(u) = h(w_i) + 1$, then e_u is not recolored.

If $u \in V(C_i) \setminus A$, then e_u never gets recolored. If $u \in A$, let $\pi(u) = v_{j'}$, then $1 \leq j' \leq j \leq k-1$ and $h(u) > h(w_i)$. So we have not dealt with u yet according to R1 and R2, thus e_u is not recolored.

The coloring in this case is illustrated in Table 3.

		$k=0$	$k=1$	$k=-1$
$j=0$	e_u is not recolored	$c(g_{w_i})$= 5 $c(e_{w_i})$: $2 \to 4$ $p(w_i)$: $\{3, 15, 46\}$ w_i: $\{4, 36, 25\}$ u : $\{2, 36, 45\}$	$c(g_{w_i})$= 5 w_i: $\{2, 36, 15\}$ u : $\{1, 24, 356\}$	$c(g_{w_i})$= 4 w_i: $\{2, 36, 145\}$
	e_u is recolored	$c(g_{w_i})$= 5 w_i: $\{2, 36, 45\}$	impossible (by Fact $3'$)	$c(g_{w_i})$= 4 w_i: $\{2, 36, 145\}$
$j=1$	e_u is not recolored	$c(g_{w_i})$= 6 $c(e_{w_i})$: $1 \to 5$ $p(w_i)$: $\{2, 14, 35\}$ or $\{2, 14, 56\}$ if $h(w_i) = 1$; $\{2, 36, 45\}$ if $h(w_i) \geq 4$ w_i: $\{5, 24, 16\}$ u : $\{1, 24, 56\}$	$c(g_{w_i})$= 6 w_i: $\{1, 24, 36\}$ u : $\{3, 15, 246\}$	$c(g_{w_i})$= 5 w_i: $\{1, 24, 356\}$
	e_u is recolored	$c(g_{w_i})$= 6 w_i: $\{1, 24, 56\}$	impossible (by Fact $3'$)	$c(g_{w_i})$= 5 w_i: $\{1, 24, 356\}$
$j=2$	e_u is not recolored	$c(g_{w_i})$= 4 $c(e_{w_i})$: $3 \to 6$ $p(w_i)$: $\{1, 24, 56\}$ w_i : $\{6, 15, 34\}$ u : $\{3, 15, 46\}$	$c(g_{w_i})$= 4 w_i: $\{3, 15, 24\}$ u: $\{2, 36, 145\}$	$c(g_{w_i})$= 6 w_i: $\{3, 15, 246\}$
	e_u is recolored	$c(g_{w_i})$= 4 w_i: $\{3, 15, 46\}$	impossible (by Fact $3'$)	$c(g_{w_i})$= 6 w_i: $\{3, 15, 246\}$

Table 3. The coloring in Case 3

Now both w_i and u are safe. We flag w_i and u (if u is not flagged).

Case 4: $\pi(w_i) = v_j (1 \leq j \leq k-1)$ and all the uncolored edges connect w_i to the subtree of type II. Then we choose one such edge $g_{w_i} = w_i u$ such that the height of u is as small as possible. Since T is a BFS-tree and the subtree of w_i is to the left of the subtree of u, we get $h(u) = h(w_i) - 1$ or $h(u) = h(w_i)$. We have the following fact:

Fact 4. u is already flagged.

If $u \in V(C_i) \setminus A$, then u has been flagged at the very beginning. If $u \in A$, since $\pi(u) = v_k$ and $\pi(w_i) = v_j$ $(1 \leq j \leq k-1)$, we have already dealt with u according to R1, thus u is flagged (note that e_u may be recolored).

The coloring in this case is illustrated in Table 4.

		$k=0$	$k=-1$
$j=0$	e_u is not recolored	$c(g_{w_i})= 5$ w_i: $\{2, 36, 145\}$	$c(g_{w_i})= 5$ w_i: $\{2, 36, 15\}$
	e_u is recolored	$c(g_{w_i})= 4$ w_i: $\{2, 36, 45\}$	$c(g_{w_i})= 5$ w_i: $\{2, 36, 45\}$
$j=1$	e_u is not recolored	$c(g_{w_i})= 6$ w_i: $\{1, 24, 36\}$	$c(g_{w_i})= 5$ w_i: $\{1, 346, 25\}$
	e_u is recolored	$c(g_{w_i})= 3$ w_i: $\{1, 24, 36\}$	$c(g_{w_i})= 6$ w_i: $\{1, 24, 56\}$
$j=2$	e_u is not recolored	$c(g_{w_i})= 3$ $c(e_{w_i})$: $3 \to 6$ $p(w_i)$: $\{1, 24, 56\}$ w_i: $\{6, 245, 13\}$	$c(g_{w_i})= 4$ $c(e_{w_i})$: $3 \to 6$ $p(w_i)$: $\{1, 24, 56\}$ w_i: $\{6, 15, 34\}$
	e_u is recolored	$c(g_{w_i})= 6$ w_i: $\{3, 15, 46\}$	$c(g_{w_i})= 4$ w_i: $\{3, 15, 46\}$

Table 4. The coloring in Case 4

Now both w_i and u are safe. We flag w_i and u (if u is not flagged).

Then we go to w_{i+1}, and repeat the process until all the vertices in A are visited. We do the same operation to all $C_i's$. If there still exist uncolored edges in $E[D, \overline{D}] \cup E(G[\overline{D}])$, then color them with 1. Now we have a coloring of all the edges in $E[D, \overline{D}] \cup E(G[\overline{D}])$ using six different colors from $\{1, 2, 3, 4, 5, 6\}$ such that all the vertices in $\overline{D}$ are safe.

4.2 Coloring the edges in $E(G[D])$

Set $d := rx_3(G[D])$. Then we can color the edges in $G[D]$ with d fresh colors from $\{7, 8, \ldots, d+6\}$ such that for each triple of vertices in D, there exists a rainbow tree in $G[D]$ connecting them. Hereto we obtain a nice edge-coloring $c : E(G) \to \{1, 2, \ldots, d+6\}$.

4.3 Proof that c is a 3-rainbow coloring

First we give the proofs of Claim 1 and 2 as promised.

Proof of Claim 1: Let $S = \{u, v, w\} \subseteq V(G)$. If $|S \cap D| = 3$, i.e. $(u, v, w) \in D \times D \times D$, then there is already a rainbow S-tree in $G[D]$. If $|S \cap D| = 2$, say $(u, v, w) \in D \times D \times \overline{D}$, then let w' be the foot of w. The rainbow tree in $G[D]$ connecting u, v, w' together with the edge ww' forms a rainbow S-tree. If $|S \cap D| = 1$, say $(u, v, w) \in D \times \overline{D} \times \overline{D}$, then there exists a rainbow $v - D$ path P^v and a rainbow $w - D$ path P^w such that $P^v \cup P^w$ is also rainbow. Assume that the end-vertices of P^v, P^w in D are v', w' respectively. Then the rainbow tree in $G[D]$ connecting u, v', w' together with the paths P^v and P^w forms a connected rainbow subgraph of G, denoted by H. Obviously, a spanning tree of H is a rainbow S-tree. If $|S \cap D| = 0$, i.e. $(u, v, w) \in \overline{D} \times \overline{D} \times \overline{D}$, then there exists a rainbow $u - D$ path P^u, a rainbow $v - D$ path P^v and a rainbow $w - D$ path P^w such that $P^u \cup P^v \cup P^w$ is also rainbow. Assume that the end-vertices of P^u, P^v, P^w in D are u', v', w' respectively. Then the rainbow tree in $G[D]$ connecting u', v', w' together with the paths P^u, P^v and P^w forms a connected rainbow subgraph of G, denoted by H'. Obviously, a spanning tree of H' is a rainbow S-tree. So we come to the conclusion that the edge-coloring c is a 3-rainbow coloring. □

Proof of Claim 2: If M1 is true, without loss of generality, we assume $c(P_1^u) = 1$ and $c(P_1^v) = 2$. If $c(P_1^w) \in \{3, 4, 5, 6\}$, then $P_1^u \cup P_1^v \cup P_1^w$ is rainbow. If $c(P_1^w) \in \{1, 2\}$, without loss of generality, let $c(P_1^w) = 1$. Since $P_1^w \cup P_2^w \cup P_3^w$ is rainbow, we get that at least one of P_2^w and P_3^w, say P_2^w, contains no edges colored by 1 or 2. Then $P_1^u \cup P_1^v \cup P_2^w$ is rainbow. If M2 is true, without loss of generality, we assume that $c(P_2^u) \cap c(P_2^v) = \emptyset$. If $c(P_1^u)$, $c(P_1^v)$, $c(P_1^w)$ are not the same, then the assertion holds by M1; otherwise, without loss of generality, let $c(P_1^u) = c(P_1^v) = c(P_1^w) = 1$. Then P_2^u and P_2^v contain no edges colored by 1. Bearing in mind that $c(P_2^u) \cap c(P_2^v) = \emptyset$, we get that $P_1^w \cup P_2^u \cup P_2^v$ is rainbow. For the other direction, assume that M1 is not true, and we will show M2 holds by contradiction. Suppose that $c(P_1^u) = c(P_1^v) = c(P_1^w)$, and for any two distinct vertices $x, y \in \{u, v, w\}$ and any two integers $s, t \in \{2, 3\}$, $c(P_s^x) \cap c(P_t^y) \neq \emptyset$. Since $P_i^u \cup P_j^v \cup P_k^w$ is rainbow, we know that at most one of i, j, k is equal to 1, say $j, k \in \{2, 3\}$. Then by hypothesis, $c(P_j^v) \cap c(P_k^w) \neq \emptyset$, a contradiction to the fact that $P_i^u \cup P_j^v \cup P_k^w$ is rainbow.□

Next we will prove that this edge-coloring c is a 3-rainbow coloring, which yields that $rx_3(G) \leq rx_3(G[D]) + 6$. Since c is a nice edge coloring, it suffices to show that c satisfies conditions M1 and M2 by Claim 1 and 2. In order to verify this, we list out all the possible color sets of the three internally disjoint super-rainbow paths connecting a vertex in $\overline{D}$ to D under this coloring c. These color sets are divided into seven classes. Class 0 contains all the possible color sets of three internally disjoint super-rainbow paths, at least two of which have length one. Class i ($1 \leq i \leq 6$) contains all the possible color sets (not appearing in Class 0) of three internally disjoint super-rainbow paths, in which the path of length one has color i. Recall that we write $\{1, 24, 35\}$ instead of $c(P_1) = \{1\}, c(P_2) = \{2, 4\}, c(P_3) = \{3, 5\}$ and $c(P_1) = \{1\}, c(P_2) = \{3, 5\}, c(P_3) = \{2, 4\}$.

Class 0: $\{1, 2, 3\}$, $\{1, 2, 34\}$, $\{1, 2, 36\}$, $\{2, 3, 14\}$, $\{2, 3, 15\}$,
$\{1, 3, 24\}$, $\{1, 3, 25\}$

Class 1: $\{1, 24, 35\}$, $\{1, 36, 24\}$, $\{1, 36, 25\}$, $\{1, 24, 56\}$, $\{1, 36, 45\}$,

$\{1,36,245\}$, $\{1,24,356\}$, $\{1,346,25\}$.

Class 2: $\{2,36,14\}$, $\{2,14,35\}$, $\{2,14,56\}$, $\{2,36,15\}$, $\{2,36,45\}$, $\{2,36,145\}$, $\{2,14,356\}$, $\{2,346,15\}$.

Class 3: $\{3,15,26\}$, $\{3,25,16\}$, $\{3,15,46\}$, $\{3,25,46\}$, $\{3,15,24\}$, $\{3,25,14\}$, $\{3,25,146\}$, $\{3,15,246\}$.

Class 4: $\{4,36,15\}$, $\{4,36,25\}$, $\{4,36,125\}$.

Class 5: $\{5,14,26\}$, $\{5,24,16\}$.

Class 6: $\{6,25,34\}$, $\{6,15,34\}$, $\{6,245,13\}$.

For every triple $\{u,v,w\}$ of vertices in $\overline{D}$, if $c(P_1^u)$, $c(P_1^v)$ and $c(P_1^w)$ are not the same (i.e. M1 holds), we are done. Now suppose $c(P_1^u)=c(P_1^v)=c(P_1^w)$. If there exists one vertex in $\{u,v,w\}$ satisfying that at least two of its three paths are of length 1, without loss of generality, we assume that $c(P_1^u)=c(P_1^v)=c(P_1^w)=1$, P_2^u is of length 1, and $c(P_2^u)=2$. Since $P_1^v\cup P_2^v\cup P_3^v$ is rainbow, we can find out one path, say P_2^v, which contains no edges colored by 1 or 2. Then $c(P_2^u)\cap c(P_2^v)=\emptyset$ (i.e. M2 holds). Again we are done. Thus we only need to check whether M2 holds for every three color sets (not necessarily different) from one class except Class 0. Since the number of color sets in one class is no more than 8, the checking work can be done in a short time, and the answer in turn is affirmative. With this, we complete the proof of *Theorem* 3.

To end this section, we illustrate the tightness of the bound $rx_3(G)\leq rx_3(G[D])+6$ with the graph G in *Example* 3. Clearly, $D=\{v_0\}$ is a connected 3-way dominating set of G. It follows from Theorem 3 that $rx_3(G)\leq rx_3(G[D])+6=6$. On the other hand, we have already proved that $rx_3(G)=6$. So the bound is tight.

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