Degree powers in C_5-free graphs*

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Abstract

Let G be a graph with degree sequence $d_1, d_2, \ldots, d_n$. Given a positive integer p, denote by $e_p(G) = \sum_{i=1}^{n} d_i^p$. Caro and Yuster introduced a Turán-type problem for $e_p(G)$: given an integer p, how large can $e_p(G)$ be if G has no subgraph of a particular type. They got some results for the subgraph of particular type to be a clique of order $r+1$ and a cycle of even length, respectively. Denote by $ex_p(n, H)$ the maximum value of $e_p(G)$ taken over all graphs with n vertices that do not contain H as a subgraph. Clearly, $ex_1(n, H) = 2ex(n, H)$, where $ex(n, H)$ denotes the classical Turán number. In this paper, we consider $ex_p(n, C_5)$ and prove that for any positive integer p and sufficiently large n, there exists a constant $c = c(p)$ such that the following holds: if $ex_p(n, C_5) = e_p(G)$ for some C_5-free graph G of order n, then G is a complete bipartite graph having one vertex class of size $cn + o(n)$ and the other $(1-c)n + o(n)$.

Keywords: degree power; Turán-type problem; H-free

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1 Introduction

All graphs considered here are finite, undirected, and have no loops or multiple edges. For standard graph-theoretic notation and terminology, the reader is referred to [1]. Denote by $ex(n, H)$ the classical Turán number, i.e., the maximum number of

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edges among all graphs with n vertices that do not contain H as a subgraph. Denote by $T_r(n)$ the r-partite Turán graph of order n, namely, $ex(n, K_{r+1}) = e(T_r(n))$. Given a graph G whose degree sequence is $d_1, \ldots, d_n$, and for a positive integer p, let $e_p(G) = \sum_{i=1}^{n} {d_i}^p$. Caro and Yuster [4] introduced a Turán-type problem for $e_p(G)$: given an integer, how large can $e_p(G)$ be if G has no subgraph of a particular type. Denote by $ex_p(n, H)$ the maximum value of $e_p(G)$ taken over all graphs with n vertices that do not contain H as a subgraph. Clearly, $ex_1(n, H) = 2ex(n, H)$. It is interesting to determine the value of $ex_p(n, H)$ and the corresponding extremal graphs. In [4], Caro and Yuster considered K_{r+1}-free graphs and proved that

$$ex_p(n, K_{r+1}) = e_p(T_r(n)) \tag{1}$$

for $1 \leq p \leq 3$.

Therefore, it is interesting to find the values of p for which equality (1) holds and determine the asymptotic value of $ex_p(n, K_{r+1})$ for large n. In [2], Bollobás and Nikiforov showed that for every real p $(1 \leq p < r)$ and sufficiently large n, if G is a graph of order n and has no clique of order $r+1$, then $ex_p(n, K_{r+1}) = e_p(T_r(n))$, and for every $p \geq r + \lceil \sqrt{2r} \rceil$ and sufficiently large n, $ex_p(n, K_{r+1}) > (1+\epsilon)e_p(T_r(n))$ for some positive $\epsilon = \epsilon(r)$. In [3], Bollobás and Nikiforov proved that if $e_p(G) > (1 - 1/r)^p n^{p+1} + C$, then G contains more than $\frac{Cn^{r-p}}{p2^{6r(r+1)+1}r^r}$ cliques of order $r+1$. Using this statement, they strengthened the Erdös–Stone theorem by using $e_p(G)$ instead of the number of edges.

When considering cycles as the forbidden subgraphs, Caro and Yuster [4] determined the value of $ex_2(n, C^*)$ for sufficiently large n, where C^* denotes the family of cycles with even length. And they also characterized the unique extremal graphs. In [7], Nikiforov proved that for any graph G with n vertices, if G does not contain C_{2k+2}, then for every $p \geq 2$, $e_p(G) \leq kn^p + O(n^{p-1/2})$. Since the graph $K_k + \overline{K}_{n-k}$, i.e., the join of K_k and $\overline{K}_{n-k}$ contains no C_{2k+2}, that gives $ex_p(n, C_{2k+2})$, hence $ex_p(n, C_{2k+2}) = kn^p(1 + o(1))$, which settles a conjecture of Caro and Yuster. Cheng et al. [5] determined the extremal values of the sum of degree squares of bipartite graphs, i.e., $p = 2$.

In this paper, we will study $ex_p(n, C_5)$. For a fixed $(r+1)$-chromatic graph H, Bollobás and Nikiforov [3] showed that for every $r \geq 2$ and $p > 0$, $ex_p(n, H) = ex_p(n, K_{r+1}) + o(n^{p+1})$. This gives us that $ex_p(n, C_5) = ex_p(n, K_3) + o(n^{p+1})$. Our main result is the following theorem.

Theorem 1 *For any positive integer p and sufficiently large n, there exists a constant $c = c(p)$ such that the following holds: if $ex_p(n, C_5) = e_p(G)$ for some C_5-free graph*

G of order n, then G is a complete bipartite graph having one vertex class of size $cn+o(n)$ and the other of size $(1-c)n+o(n)$.

2 Proof of Theorem 1

When $p=1$, it is a well-known result of the classical Turán problem. So in the following we assume $p \geq 2$. Throughout the paper, let G be the extremal graph satisfying that $ex_p(n, C_5) = e_p(G)$. Observe that $T_2(n)$ contains no C_5, and we have

$$e_p(T_2(n)) = \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lceil \frac{n}{2} \right\rceil\right)^p + \left\lceil \frac{n}{2} \right\rceil \left(\left\lfloor \frac{n}{2} \right\rfloor\right)^p = \left(\frac{1}{2}\right)^p n^{p+1} + o\left(n^{p+1}\right).$$

By the definition of $ex_p(n, C_5)$, we have $e_p(G) \geq e_p(T_2(n))$. Hence, the coefficient of n^{p+1} in $e_p(G)$ must be at least $\left(\frac{1}{2}\right)^p$.

Lemma 1 *For every integer p and sufficiently large n, if $e_p(G) = ex_p(n, C_5)$, then $\Delta(G) = an + o(n)$, where the constant $a = a(p) \geq \frac{1}{2}$.*

Proof. Suppose $\Delta(G) = o(n)$, we then have $e_p(G) \leq n \cdot [\Delta(G)]^p = n \cdot o(n^p) = o(n^{p+1})$, a contradiction. Let $\Delta(G) = an + o(n)$. Then we have $e_p(G) \leq n \cdot (an)^p + o(n^{p+1}) = a^p n^{p+1} + o(n^{p+1})$, which implies $a \geq \frac{1}{2}$. ∎

In order to describe the structure of the extremal graph G, we introduce some classes of graphs and a graph operation on two or more graphs. Let S^k denote the set of graphs of order k as shown in Figure 1. And graphs S_1, S_2, S_3 are also shown in Figure 1. Each of these graphs has a labeled vertex, i.e., the cross vertex as shown in Figure 1.

Let $\mathcal{S} = \{S_1, S_2, S_3\}$, $\mathcal{S}^* = \mathcal{S} \cup S^4 \cup S^6 \cup \cdots$, for all possible integer k. When we say "*attaching*" two graphs in $\mathcal{S}^*$, it means that we identify the labeled vertices in each graph. Note that this attaching operation could be applied on more than two graphs. Before the proof, we recall a classical result of Erdös and Gallai [6].

Lemma 2 *If a graph of order n has more than $kn/2$ edges, then it contains a path of order $k+2$.* ∎

Proof of Theorem 1: We will consider the following two cases.

Case 1. For any vertex u with maximum degree in G, there is no edge in $G[N_G(u)]$.

In this case, we can construct a complete bipartite graph H, which satisfies that $e_p(H) \geq e_p(G)$. The complete bipartite graph $H = (X, Y)$ can be constructed as

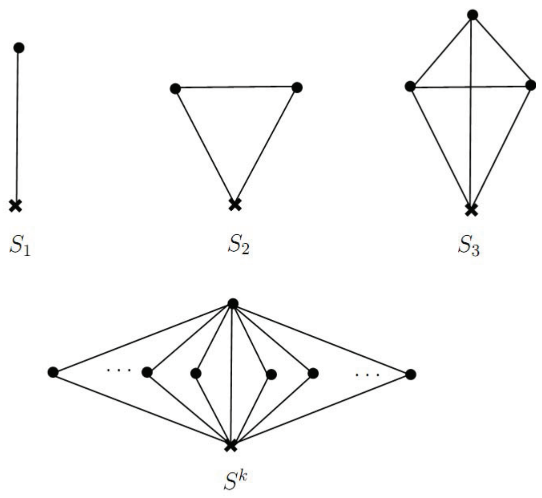

Figure 1: The illustration of S^k and S_i.

follows: $X = (V(G) \setminus N_G(u)) \cup \{u\}$ and $Y = N_G(u)$. It is easy to check that $d_H(v) \geq d_G(v)$ for any vertex $v \in V(G)$, hence $e_p(H) \geq e_p(G)$. Since G is the extremal graph, we can deduce that G itself is isomorphic to H.

Case 2. There exists a vertex v with maximum degree in G, such that there is at least one edge in $G[N_G(v)]$.

Let u be such a vertex with maximum degree. By Lemma 1, we assume that $d_G(u) = an + o(n)$, where $a \geq \frac{1}{2}$. Let A denote the set $\{u\} \bigcup N_G(u)$, and B denote $V(G) \setminus (\{u\} \bigcup N_G(u))$, respectively. Since G is C_5-free, we have that $G[A]$ is also C_5-free. Then we can get that $G[A]$ must be constructed by attaching some graphs in $\mathcal{S}^*$, and moreover, u is just the vertex identified by labeled vertices. For example, $G[A]$ may be isomorphic to the graph as shown in Figure 2.

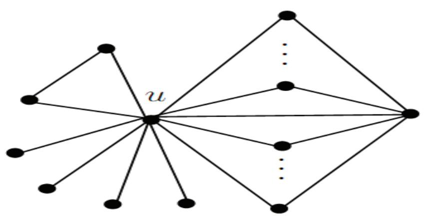

Figure 2: An example of $G[A]$.

In fact, considering the edges between A and B, we can obtain the following two observations. Note that the vertices in B can only be adjacent to the unlabeled vertices, since all of neighbors of u are in A. Without loss of generality, suppose $G[A]$

is constructed by attaching t_i S_i's, $i = 1, 2, 3$ and r_k S^k's, for possible k. Observe that if $w \in B$, then w cannot be adjacent to two graphs among all t_i S_i's, and r_k S^k's, except one case that the two graphs are S_1 and S_1.

Observation 1 *For any vertex w in B, the edges between w and A can only be one of the following four cases:*

(a) w is not adjacent to any vertex in A.

(b) w is adjacent to some unlabeled vertices of S_1's;

(c) w is adjacent to one or two unlabeled vertices of exactly one S_2;

(d) w is adjacent to only one graph F among all t_i S_i's, and r_k S^k's. Moreover, w is adjacent to exactly one unlabeled vertex in F.

Observation 2 *For any edge w_1w_2 in $G[B]$, the edges between w_1 (w_2) and A can only be one of the following two cases:*

(a) w_1 and w_2 are adjacent to the same unlabeled vertex in exactly one graph among all t_i S_i's and r_k S^k's;

(b) one of w_1 and w_2, say w_1, is adjacent to no vertices in A, w_2 is adjacent to vertices in A as described of Observation 1.

With the aid of the above two observations and the assumption of G, we can prove the following claim.

Claim 1 *$G[A]$ is isomorphic to the graph obtained by attaching one S_2 and $d_G(u)-2$ S_1's.*

Proof. Let $\mathcal{A} = \{S \in \{S_2, S_3, S^k\} : S \subseteq G[A]$ and some unlabeled vertex v in S has degree $d_G(v) = O(n)\}$. By the previous observations, we have $|\mathcal{A}| = o(n)$, since $|A| = an+o(n)$, $|B| = (1-a)n+o(n)$ and the number of edges between the vertices in $\mathcal{A}$ and B will be no more than $2|B|$. So $\sum\limits_{v\in V(S),\ S\in\mathcal{A}} d_G{}^p(v) = o(n^{p+1})$. Therefore, the vertices in $\mathcal{A}$ have no contribution to the value of the coefficient of n^{p+1} in $e_p(G)$. In order to maximize the value of $e_p(G)$, $G[A]$ must consist of as many S_1's as possible. Since we assume that there exists at least one edge in $G[A]$, $G[A]$ must be isomorphic to the graph obtained by attaching of one S_2 and $d_G(u) - 2$ S_1's. □

Let A_1 denote the set of all the unlabeled vertices in S_1 contained in $G[A]$.

From Claim 1, the extremal graph in Case 1 satisfies the description in Claim 1. We construct two graphs G' and G^* to characterize the extremal graph G in detail. Let $V(G') = V(G^*) = V(G)$, both G' and G^* satisfy the assumption of Case 2 and the description of Claim 1, i.e., in both G' and G^*, let u be the vertex with maximum degree $d_G(u)$, there exist edges in $G[N_{G'}(u)]$ and $G[N_{G^*}(u)]$. Without loss of generality, let $A = \{u\} \bigcup N_{G'}(u) = \{u\} \bigcup N_{G^*}(u)$, and let $B = V(G') \setminus (\{u\} \bigcup N_{G'}(u)) = V(G^*) \setminus (\{u\} \bigcup N_{G^*}(u))$. Observe that $G'[A]$ and $G^*[A]$ satisfy the description of Claim 1. Hence, we can still use notation A_1 to denote the set of all the unlabeled vertices in S_1 contained in $G'[A]$, and the same set in $G^*[A]$.

The difference between G' and G^* is as follows. For G', $G'[B]$ is empty, every vertex in B is adjacent to every vertex in A_1 and there is no edge between $A \setminus A_1$ and B. And for G^*, there are two vertices in B, say w_1, w_2, such that $G^*[B]$ is a complete bipartite graph with one class $\{w_1, w_2\}$, every vertex in $B \setminus \{w_1, w_2\}$ is adjacent to every vertex in A_1, there is no edge between $A \setminus A_1$ and B, and also no edge between A_1 and $\{w_1, w_2\}$.

The next claim characterizes the extremal graph G in Case 2. Since we only consider the case when n is sufficiently large, from the preceding discussions, we can assume that $d_G(u) = an$ instead of $an + o(n)$ to simplify the calculation.

Claim 2 *$e_p(G)$ is equal to either $e_p(G')$ or $e_p(G^*)$.*

Proof. Firstly, we calculate $e_p(G')$ and $e_p(G^*)$. For any vertex $v \in A_1$, $d_{G'}(v) = (1-a)n$; for any vertex $v \in A \setminus (A_1 \cup \{u\})$, $d_{G'}(v) = 2$; and for any vertex v in B, $d_{G'}(v) = an - 2$. Hence,

$$e_p(G') = (an)^p + 2 \times 2^p + (an-2)[(1-a)n]^p + [(1-a)n-1](an-2)^p.$$

Similarly, Observe that for any vertex $v \in A_1$, $d_{G^*}(v) = (1-a)n - 2$, and for any vertex $v \in A\setminus(A_1 \cup \{u\})$, $d_{G^*}(v) = 2$, also we have $d_{G^*}(u) = an$, $d_{G^*}(w_i) = (1-a)n-3$, $i = 1, 2$, and for any vertex w in $B \setminus \{w_1, w_2\}$, $d_{G^*}(w) = an$. It is easy to calculate that $e_p(G^*) = [(1-a)\,n-2]\,(an)^p + (an-2)\,[(1-a)\,n-2]^p + 2[(1-a)\,n-3]^p + 2 \times 2^p$.

We assume $e_p(G) > e_p(G')$. Then, there must exist some vertex v satisfying $d_G(v) > d_{G'}(v)$. Note that for each vertex $v \in A_1$, the degree of v is at most $(1-a)n$, we only need to consider such two cases.

Case 1. There exists some vertex $v \in B$, such that $d_G(v) > d_{G'}(v)$, and for each vertex $v' \in A$, $d_G(v')$ is no larger than $d_{G'}(v')$.

Let $B_1 = N_G(v) \cap B$. In the following, we will consider the following two subcases.

Subcase 1.1. $|N_G(v) \cap A| = xn$ and $|B_1| = yn$, where $0 \le x < a$, $0 < y \le 1-a$, $x + y \le a$, and if $x = 0$, $|N_G(v) \cap A| \ge 2$.

Firstly, we know that $G[B_1]$ contains no path of order 4, since otherwise, there will exist one C_5 including v.

By Lemma 2, we have that the number of edges in $G[B_1]$, denote by $e(G[B_1])$, is no more than $2yn/2 = yn$. Hence, $e(G[B_1]) = \sum_{v\in B_1} d_{G[B_1]}(v) \le 2yn$. We will calculate the maximum possible value of $e_p(G)$. We assume that there is one vertex in B_1 with degree $d_{G[B_1]} = yn-1$ and the remaining vertices in B_1 with degree $d_{G[B_1]} = 1$. For each vertex in B_1, we can assume that it is adjacent to each vertex in $B \setminus B_1$. (Note that the vertices in B_1 cannot be adjacent to the vertices in A from the previous observations.) Suppose that all the vertices in $B \setminus B_1$ reach the maximum degree an in G. We can see that such a situation can maximize the value of $e_p(G)$, and it may be much larger than the exact value of $e_p(G)$. We then have

$$\begin{aligned} e_p(G) \le\ & (an)^p+2\times 2^p+(an)^p+[(1-a)\,n-2-yn]\,(an)^p \\ & +(an-2)\,[(1-a)\,n-yn]^p+(yn-1)\,[(1-a)\,n-yn]^p+[(1-a)\,n-2]^p. \end{aligned}$$

Expanding the right hand side of the inequality above, the coefficient of n^{p+1} is

$$(1-a-y)\,a^p+a(1-a-y)^p+y(1-a-y)^p=(y+a)\,(1-a-y)^p+(1-a-y)\,a^p.$$

Since from the previous calculation we know that the coefficient of n^{p+1} in $e_p(G')$ is $a(1-a)^p+a^p\,(1-a)$, to derive a construction to our assumption, it is sufficient to show that

$$(y+a)\,(1-a-y)^p+(1-a-y)\,a^p < a(1-a)^p+a^p\,(1-a)$$

for sufficiently large n. Let

$$f(a,y) = a(1-a)^p+a^p\,(1-a)-\left[(y+a)\,(1-a-y)^p+(1-a-y)\,a^p\right].$$

We will show that $f(a,y) > 0$. We first suppose that $\frac{1-a}{2} < y < 1-a$, i.e., $1-a-y < y < a$. Then, we have

$$\begin{aligned} f(a,y) &= a(1-a)^p-y(1-a-y)^p-a(1-a-y)^p+a^py \\ &> a(1-a)^p-y(1-a-y)^p-ay^p+a^py \\ &> ay^p-y(1-a-y)^p-ay^p+a^py = ya^p-y(1-a-y)^p > 0. \end{aligned}$$

Now we suppose $0 < y \leq \frac{1-a}{2}$. In this case, we have

$$\begin{aligned}
f(a,y) &= a(1-a)^p - (y+a)(1-a-y)^p + a^p y \\
&\geq a(1-a)^p + a^p y - \frac{1+a}{2}(1-a-y)^p \\
&= a(1-a)^p + a^p y - \frac{1+a}{2}(1-a)^p + \frac{1+a}{2}(1-a)^p - \frac{1+a}{2}(1-a-y)^p \\
&= a^p y + \frac{a-1}{2}(1-a)^p + \frac{1+a}{2}\left[(1-a)^p - (1-a-y)^p\right] \\
&> a^p y + \frac{a-1}{2}(1-a)^p + \frac{1+a}{2}y^p > a^p y + \frac{a-1}{2}(1-a)^p + \frac{1-a}{2}y^p \\
&= a^p y + \frac{1-a}{2}\left[y^p - (1-a)^p\right] \geq y\left[a^p + y^p - (1-a)^p\right] > 0.
\end{aligned}$$

Hence, we have proved that $f(a,y) > 0$.

Subcase 1.2. $|N_G(v) \cap A| = an - o(n)$ and $|B_1| = o(n)$.

With similar methods, we have

$$\begin{aligned}
e_p(G) \leq\ & (an)^p + 2 \times 2^p + (an)^p + \left[(1-a)n - 2 - o(n)\right](an)^p \\
& + (an-2)\left[(1-a)n - o(n)\right]^p + (o(n)-1)\left[(1-a)n - o(n)\right]^p + \left[(1-a)n - 2\right]^p.
\end{aligned}$$

Similarly, there are two cases when we compare the values of $e_p(G)$ and $e_p(G')$.

- The $o(n)$ part of $|N_G(v) \cap A|$, denoted by ω, satisfies that $\omega \to +\infty$.

Observe that $n^p < \omega n^p < n^{p+1}$. So we need to consider the coefficient of ωn^p. By expanding the expression of $e_p(G)$, it is clear that the coefficient is $-a^p + (1-a)^p \leq 0$, which implies $e_p(G) \leq e_p(G')$, a contradiction.

- The $o(n)$ part of $|N_G(v) \cap A|$ is a constant.

Let $o(n) = c$, $c \geq 1$. We will prove in that subcase, G is isomorphic to G^*. Now we consider the structure of G. If a vertex in $B \setminus B_1$ has degree an, then at least two of its neighbors in B will be not adjacent to any vertices in A. So in order to maximize the number of vertices whose degree is $O(n)$, we suppose that as many as possible vertices in $B \setminus B_1$ have degree an, all of them have only two neighbors in B. It is not difficult to get that if they share two common neighbors in B, we will have a larger value of $e_p(G)$. Furthermore, let these two common neighbors be both in B_1, and there are no other vertices in B_1, we can get the maximum value of $e_p(G)$ in that situation. And we can see that c is equal to 2 in such case. Moreover, G is isomorphic to G^*.

Subcase 1.3. $|N_G(v) \cap A| = 1$.

Since $a \geq \frac{1}{2}$, $|B| = (1-a)n - 1$, and we assume that $d_G(v) > d_{G'}(v) = an - 2$, we have that $a = \frac{1}{2}$ and $|B_1| = (1-a)n - 2$, i.e., v is adjacent to every vertex in $B \setminus \{v\}$.

Let $N_G(v) \cap A = \{v'\}$, by Observation 2, the vertices in B can only be adjacent to v' in A. To maximize the value of $e_p(G)$, let all the vertices in B be adjacent to v' and $G[B]$ be a complete graph. Note that every vertex has its maximal possible degree. Hence, $e_p(G) \leq 2 \times 2^p + (an)^p + [(1-a)\,n]^p + (an-3) + [(1-a)\,n-1]\,[(1-a)\,n-1]^p = \left(\frac{1}{2}\right)^{p+1} n^{p+1} + o\left(n^{p+1}\right) < e_p\left(T_2\left(n\right)\right)$, a contradiction.

Case 2. There exists a vertex $v \in A \setminus (A_1 \cup \{u\})$ such that $d_G(v) > d_{G'}(v)$.

Let $A \setminus (A_1 \cup \{u\}) = \{v_1, v_2\}$. Without loss of generality, assume that $d_G(v_1) = 2 + x$, $d_G(v_2) = 2 + y$. Suppose that $w \in B$ is adjacent to v_1, from Observation 2, w can not be adjacent to any vertices in A_1, and to avoid 5-cycles, the neighbors of w in B can not be adjacent to any vertices in A_1. Just similar to Case 1, we can derive that there are two vertices w', w'' in B, such that all neighbors of v_1 in B is adjacent to w', and all neighbors of v_2 in B is adjacent to w'', the set of remaining vertices in B and $\{w', w''\}$ form a complete bipartite graph. Note that v_1 and v_2 have no common neighbors in B in order to avoid 5-cycles and maximize the value of $e_p(G)$. If either x or y is zero, then $w' = w''$. So, if $x \geq 1$, $y \geq 1$, then,

$$\begin{aligned} e_p(G) &= (2+x)^p + (2+y)^p + (an)^p + (x+y)\cdot 2^p + (an-2)\,[(1-a)\,n-2-x-y]^p \\ &\quad + [(1-a)\,n-3-x-y]\,(an)^p + [(1-a)\,n-3-y]^p + [(1-a)\,n-3-x]^p. \qquad (2) \end{aligned}$$

Suppose either x or y is zero, by symmetry, we need only consider the case when $y = 0$ and $x \geq 1$. In such case, we have

$$\begin{aligned} e_p\,(G) &= x \cdot 2^p + 2^p + (2+x)^p + (an)^p + (an-2)\,[(1-a)\,n-1-x]^p \\ &\quad + [(1-a)\,n-2-x]\,(an-1)^p + [(1-a)\,n-2]^p. \qquad (3) \end{aligned}$$

In equation (2), if x or y is $O(n)$, then the coefficient of n^{p+1} is strictly less than $a(1-a)^p + a^p\,(1-a)$. Since the coefficient of n^{p+1} in $e_p(G')$ is $a(1-a)^p + a^p\,(1-a)$, we have $e_p(G) < e_p(G')$, which contradicts to our assumption. Hence, x and y are both $o(n)$, and $(2+x)^p + (2+y)^p$ has no contribution to the coefficient of n^p. Thus, the coefficient of n^p in $e_p(G)$ is

$$\begin{aligned} & a^p - 2(1-a)^p - pa\,(2+x+y)\,(1-a)^{p-1} - a^p\,(3+x+y) + 2(1-a)^p \\ = \;& -pa\,(2+x+y)\,(1-a)^{p-1} - a^p\,(2+x+y)\,. \end{aligned}$$

From the expression of $e_p(G^*)$, the coefficient of n^p in $e_p(G^*)$ is $-2pa(1-a)^{p-1} - 2a^p$, which is larger than $-pa\,(2+x+y)\,(1-a)^{p-1} - a^p\,(2+x+y)$. Similarly, when $y = 0$, we can deduce that x is $o(n)$. With some calculations, one can see that the coefficient of n^p in (3) is less than that in $e_p(G^*)$. Hence, $e_p(G) < e_p(G^*)$ for sufficiently large n, i.e., G can not be the extremal graph, a contradiction.

Combining all cases above, we have proved this claim. □

In the sequel, we will prove that the extremal graph described in Case 2 will always have a smaller value of $e_p(\cdot)$ than the extremal graph in Case 1. Let G_1 and G_2 be the extremal graph in Case 1 and Case 2, respectively. So we have $e_p(G_2) = \max\{e_p(G'), e_p(G^*)\}$. It is easy to get that, the coefficient of n^{p+1} in the expression of $e_p(G_2)$ is $a(1-a)^p + a^p\,(1-a)$, which is equal to that of $e_p(G_1)$. The coefficient of n^p in the expression of $e_p(G')$ is

$$a^p - 2p\,(1-a)\,a^{p-1} - 2(1-a)^p - a^p = -2p\,(1-a)\,a^{p-1} - 2(1-a)^p < 0.$$

And the coefficient of n^p in the expression of $e_p(G^*)$ is $-2a^p - 2pa(1-a)^{p-1} < 0$.

Therefore, for sufficiently large n, $e_p(G_2) < [a(1-a)^p + a^p\,(1-a)]\,n^{p+1}$, i.e., $e_p(G_2) < e_p(G_1)$.

In conclusion, if $ex_p(n, C_5) = e_p(G)$ for some C_5-free graph G of order n, then G is isomorphic to G_1. Hence G is a complete bipartite graph. Moreover, the size of one class is $cn + o(n)$ and the other is $(1-c)n + o(n)$, where c maximizes the function $f(x) = x(1-x)^p + x^p\,(1-x)$ in $\left[\frac{1}{2}, 1\right]$. ∎

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