Characterization of graphs with rainbow connection number $m-2$ and $m-3$*

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Abstract

A path in an edge-colored graph, where adjacent edges may be colored the same, is a rainbow path if no two edges of it are colored the same. A nontrivial connected graph G is rainbow connected if there is a rainbow path connecting any two vertices, and the rainbow connection number of G, denoted by $rc(G)$, is the minimum number of colors that are needed in order to make G rainbow connected. Chartrand et al. obtained that G is a tree if and only if $rc(G) = m$, and it is easy to see that G is not a tree if and only if $rc(G) \leq m-2$, where m is the number of edges of G. So there is an interesting problem: Characterize the graphs G with $rc(G) = m-2$. In this paper, we resolve down this problem. Furthermore, we also characterize the graphs G with $rc(G) = m-3$.

Keywords: edge-colored graph, rainbow path, rainbow connected, rainbow connection number

AMS Subject Classification 2010: 05C15, 05C40

1 Introduction

All graphs in this paper are finite, undirected and simple. We follow the terminology and notation of Bondy and Murty [1]. Let G be a nontrivial connected graph on which is defined a coloring $c : E(G) \to \{1, 2, \cdots, \ell\}$, $\ell \in \mathbb{N}$, of the edges of G, where adjacent edges may be colored the same. A path is a *rainbow path* if no two edges of it are colored the same. An edge-colored graph G is *rainbow connected* if any two vertices are connected by a rainbow path. Clearly, if a graph is rainbow connected, it must

*Supported by NSFC No.11371205 and PCSIRT.

be connected. Conversely, any connected graph has a trivial edge-coloring that makes it rainbow connected; just color each edge with a distinct color. Thus, we define the *rainbow connection number* of a connected graph G, denoted by $rc(G)$, as the smallest number of colors that are needed in order to make G rainbow connected. If G_1 is a connected spanning subgraph of G, then $rc(G) \leq rc(G_1)$. Chartrand et al. [3] obtained that $rc(G) = 1$ if and only if G is complete, and that $rc(G) = m$ if and only if G is a tree, as well as that a cycle with $k > 3$ vertices has rainbow connection number $\lceil \frac{k}{2} \rceil$, and a triangle has rainbow connection number 1. Also notice that, clearly, $rc(G) \geq diam(G)$, where $diam(G)$ denotes the diameter of G. For more information on rainbow connections, we refer to [4, 6]. In an edge-colored graph G, we use $c(e)$ to denote the color of edge e and for a subgraph G_2 of G, $c(G_2)$ denotes the set of colors of edges in G_2.

Since $rc(G) = m$ if and only if G is a tree, $rc(G) \neq m - 1$ and G is not a tree if and only if $rc(G) \leq m - 2$ (Observation 3 below), then there is an interesting problem: Characterize the graphs with $rc(G) = m - 2$. In this paper, we resolve this problem. Furthermore, we also characterize the graphs G with $rc(G) = m - 3$.

We use $V(G)$, $E(G)$ for the set of vertices and edges of G, respectively. A *pendant edge* of G is an edge incident to a vertex of degree 1. The girth of G, denoted by $g(G)$, is the length of a smallest cycle in G. A block of G is a maximal connected subgraph of G that does not have any cut vertex. So every block of a nontrivial connected graph is either a K_2 or a 2-connected subgraph. All the blocks of a graph G form a block decomposition of G. A *rooted tree* $T(x)$ is a tree T with a specified vertex x, called the *root* of T. Let $L(x)$ denote the set of leaves of $T(x)$ and $|L(x)| = l(x)$. If $T(x)$ is a trivial tree, then $l(x) = 0$. We let P_n and C_n be the path and cycle with n vertices, respectively. And xPy denotes a path from x to y. Let $[t] = \{1, \cdots, t\}$ denote the set of the first t natural numbers. For a set S, $|S|$ denotes the cardinality of S.

2 Some basic results

We first give an observation which will be useful in the sequel.

Observation 1. [5] If G is a connected graph and $\{E_i\}_{i\in[t]}$ is a partition of the edge set of G into connected subgraphs $G_i = G[E_i]$, then

$$rc(G) \leq \sum_{i=1}^{t} rc(G_i).$$

□

We now give a necessary condition for an edge-colored graph to be rainbow connected. If G is rainbow connected under some edge-coloring, then for any two cut edges (if they exist) $e_1 = u_1u_2$ and $e_2 = v_1v_2$, there must exist some $1 \leq i, j \leq 2$, such that any $u_i - v_j$ path must contain edge e_1, e_2. So we have:

Observation 2. If G is rainbow connected under some edge-coloring c where e_1 and e_2 are any two cut edges, then $c(e_1) \neq c(e_2)$.

For a connected graph G, if it is a tree, then $rc(G) = m$; if it contains a unique cycle of length k, then we give the cycle a rainbow coloring using $\lceil \frac{k}{2} \rceil$ colors (if the cycle is a triangle, we just need one color) and color each other edge with a fresh color. Then by Observation 1, we have $rc(G) \leq (m-k) + \lceil \frac{k}{2} \rceil \leq m-2$. So we have the following observation.

Observation 3. Let G be a connected graph with m edges. Then $rc(G) \neq m-1$ and G is not a tree if and only if $rc(G) \leq m-2$. Moreover, if G contains a cycle of length $k (k \geq 4)$, then $rc(G) \leq m - \lfloor \frac{k}{2} \rfloor$.

For a connected graph G, if it contains two edge-disjoint 2-connected subgraphs B_1 and B_2, then by Observation 3, we give B_1 and B_2 a rainbow coloring using $|E(B_1)|-2$ and $|E(B_2)|-2$ colors, respectively, and color each other edge with a fresh color. Then by Observation 1, we have $rc(G) \leq m-4$. So the following lemma holds.

Lemma 1. *Let G be a connected graph with m edges. If it contains two edge-disjoint 2-connected subgraphs, then $rc(G) \leq m-4$.*

To *subdivide* an edge e is to delete e, add a new vertex x, and join x to the ends of e. Any graph derived from a graph G by a sequence of edge subdivisions is called a *subdivision* of G. Given a rainbow coloring of G, if we subdivide an edge $e = uv$ of G by xu and xv, then we assign xu the same color as e and assign xv a new color, which also make the subdivision of G rainbow connected. Hence, the following lemma holds.

Lemma 2. *Let G be a connected graph, and H be a subdivision of G. Then $rc(H) \leq rc(G) + |E(H)| - |E(G)|$.*

The Θ-*graph* is a graph consisting of three internally disjoint paths with common end vertices and of lengths a, b, and c, respectively, such that $a \leq b \leq c$. Then $a+b+c=m$.

Lemma 3. *Let G be a Θ-graph with m edges. If $m=5$, then $rc(G) = m-3$; otherwise, $rc(G) \leq m-4$.*

Proof. Let the three internally disjoint paths be P_1, P_2, P_3 with the common end vertices u and v, and the lengths of P_1, P_2, P_3 be a, b, c, respectively, where $a \leq b \leq c$. If $m=5$, we color uP_1v with color 1, uP_2v with colors $1, 2$, and uP_3v with colors $2, 1$. The resulting coloring makes G rainbow connected. Thus, $rc(G) \leq m-3$. Since $diam(G) = 2$, it follows that $rc(G) = m-3$. For $m \geq 6$, we first consider the graph Θ_1 with $a=1$, $b=2$ and $c=3$. We color uP_1v with color 1, uP_2v with colors $1, 1$, and uP_3v with colors $2, 1, 2$. Next we consider the graph Θ_2 with $a=2$, $b=2$ and $c=2$. We color uP_1v with colors 1,2, uP_2v with colors $2, 1$, and uP_3v with colors $2, 2$. The resulting colorings make Θ_1 and Θ_2 rainbow connected. For a general Θ-graph G with $m \geq 6$, it is a subdivision of Θ_1 or Θ_2, hence by Lemma 2, $rc(G) \leq m-4$. □

3 Characterizing unicyclic graphs with $rc(G) = m - 2$ and $m - 3$

In this section we first give an observation about unicyclic graphs which will be used frequently. Let G be a connected unicyclic graph with the unique cycle $C = v_1v_2 \cdots v_sv_1$. For brevity, orient C clockwise. Then G has the structure as follows: a tree, denoted by $T(v_i)$, is attached at each vertex v_i of C. Note that, $T(v_i)$ may be trivial. Let $i \neq j$. If $e_i = x_iy_i(e_j = x_jy_j)$ is a pendant edge which belongs to a tree $T(v_i)(T(v_j))$. Then there is a unique path $x_iP_iv_i(x_jP_jv_j)$ from $x_i(x_j)$ to $v_i(v_j)$. Since v_i and v_j divide C into two segments v_iCv_j and v_jCv_i, there are exactly two paths between x_i and x_j in G. Let $c = \{1, 2, \cdots, \ell\}$ be an edge coloring of G. Since each edge in $G \setminus E(C)$ is a cut edge, by Observation 2, they must obtain distinct colors. It is easy to see that $|c(x_iP_iv_i) \cap c(C)| \leq 1$. In the process of coloring, we always first color $G \setminus E(C)$ with $[t]$ colors, then color C, where $t = |E(G) \setminus E(C)|$. Thus, after coloring $E(G) \setminus E(C)$, the unique path $x_iP_iv_i$ can be viewed as a pendant edge and every $T(v_i)$ will be a star with the center vertex v_i. Suppose $|c(x_iP_iv_i) \cap c(C)| = 1$ and $|c(x_jP_jv_j) \cap c(C)| = 1$, then we can adjust the colors of cut edges such that $c(e_i) = 1$ and $c(e_j) = 2$. Thus, $1, 2 \in v_iCv_j$ or $1, 2 \in v_jCv_i$, namely, 1,2 can only be assigned in the same path from v_i to v_j. Moreover, another path from v_i to v_j should be rainbow. We summarize the above argument into an observation.

Observation 4. Let G be a connected unicyclic graph with the unique cycle $C = v_1v_2 \cdots v_sv_1$, and let $c = \{1, 2, \cdots, \ell\}$ be an edge coloring of G. Let $p \in T(v_i)$ and $q, r \in T(v_j)$.

(i) If $p, q \in C$, then they are in the same path from v_i to v_j and the other path from v_i to v_j should be rainbow.

(ii) If q, r are in the unique path from a vertex x of $V(G) \setminus V(C)$ to v_j, then q and r can not both belong to C.

In this section we only deal with unicyclic graphs. According to the girth of G, we introduce some graph classes and discuss them by some lemmas. Note that, $l(v_i)$ is the number of leaves of the tree attached at the vertex v_i from the unique cycle of G.

Let i be an integer with $1 \leq i \leq 3$ and the addition is performed modulo 3. Let $\mathcal{G} = \{G : m = n, g(G) = 3\}$, $\mathcal{G}_1 = \{G : G \in \mathcal{G}, l(v_i) \geq 1, l(v_{i+1}) \geq 1, l(v_{i+2}) \geq 1, \textit{ or } l(v_i) \geq 3\}$, $\mathcal{G}_2 = \{G : G \in \mathcal{G}, l(v_i) = 0, l(v_{i+1}) \leq 2, l(v_{i+2}) \leq 2\}$. Obviously, $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$.

Lemma 4. *Let G be a graph belonging to $\mathcal{G}$. If $G \in \mathcal{G}_1$, then $rc(G) = m - 3$; otherwise $rc(G) = m - 2$.*

Proof. Let the unique cycle of G be $C = v_1v_2v_3v_1$. Suppose $G \in \mathcal{G}_1$, by Observation 2, each edge of $G \setminus E(C)$ must obtain a distinct color, color them with a set $[m - 3]$ of colors. We consider two cases. Without loss of generality, first suppose that $e_i = x_iy_i$

is a pendant edge in $T(v_i)$ that is assigned color i, where $1 \leq i \leq 3$. Set $c(v_1v_2) = 3$, $c(v_2v_3) = 1$, $c(v_3v_1) = 2$. Next suppose that $e_j = x_jy_j$ is a pendant edge of $T(v_1)$ that is assigned color j, where $1 \leq j \leq 3$. Color $E(C)$ with 1,2,3, respectively. It is easy to show that these two colorings are rainbow, and in these two cases, $rc(G) = m - 3$.

If $G \in \mathcal{G}_2$, by Observation 3, $rc(G) \leq m - 2$. By Observation 4, we know that at most two colors for $G \setminus E(C)$ can be assigned to C. Thus, we need a fresh color for C, and it follows that $rc(G) \geq m - 2$. Therefore, $rc(G) = m - 2$. □

Let i be an integer with $1 \leq i \leq 4$ and the addition is performed modulo 4. Set $\mathcal{H} = \{G : m = n, g(G) = 4\}$. Then $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$, where $\mathcal{H}_1 = \{G : G \in \mathcal{H}, l(v_i) = l(v_{i+2}) = 0, l(v_{i+1}) \leq 1, l(v_{i+3}) \leq 1\}$, $\mathcal{H}_2 = \{G : G \in \mathcal{H}, l(v_i) \geq 4, \mathit{or}\ l(v_i) \geq 1, l(v_{i+1}) \geq 2, l(v_{i+2}) \geq 1\}$, and $\mathcal{H}_3$ is the set of the rest unicyclic graphs with girth 4.

Lemma 5. *Let G be a graph belonging to $\mathcal{H}$. If $G \in \mathcal{H}_1$, then $rc(G) = m - 2$; if $G \in \mathcal{H}_2$, then $rc(G) = m - 4$; if $G \in \mathcal{H}_3$, then $rc(G) = m - 3$.*

Proof. Let the unique cycle of G be $C = v_1v_2v_3v_4v_1$. By Observation 2, each edge of $G \setminus E(C)$ must obtain a distinct color, this costs $m - 4$ colors, thus $rc(G) \geq m - 4$. Color $G \setminus E(C)$ with a set $[m-4]$ of colors. Suppose $G \in \mathcal{H}_1$. By Observation 3, $rc(G) \leq m-2$. By Observation 4, we know that at least two colors different from $c(G \setminus E(C))$ should be assigned to C, so it follows that $rc(G) \geq m - 2$. Hence, $rc(G) = m - 2$.

Suppose $G \in \mathcal{H}_2$. First let $e_i = x_iy_i$ be a pendant edge in $T(v_1)$ that is assigned color i, where $1 \leq i \leq 4$. Color $E(C)$ with 1,2,3,4, respectively. Next suppose that $e_j = x_jy_j$ is a pendant edge that is assigned color j such that $1 \in T(v_1)$, $2, 3 \in T(v_2)$ and $4 \in T(v_3)$, where $1 \leq j \leq 4$. Set $c(v_1v_2) = 4$, $c(v_2v_3) = 1$, $c(v_3v_4) = 3$, $c(v_1v_4) = 2$. It is easy to show that these two colorings are rainbow, and in these two cases, $rc(G) = m - 4$.

If $G \in \mathcal{H}_3$, by Observation 4, we check one by one that at least one color different from $c(G \setminus E(C))$ should be assigned to C, thus $rc(G) \geq m - 3$. If e_1 and e_2 are two pendant edges in a tree (say $T(v_1)$) that are assigned colors 1 and 2, respectively, then set $c(v_1v_2) = m - 3$, $c(v_2v_3) = 1$, $c(v_3v_4) = 2$, $c(v_1v_4) = m - 3$. By symmetry, it remains to consider the case that $l(v_1) = l(v_2) = l(v_3) = 1$. Suppose that $e_i = x_iy_i$ is a pendant edge in $T(v_i)$ that is assigned color i, where $1 \leq i \leq 3$. Set $c(v_1v_2) = 3$, $c(v_2v_3) = 1$, $c(v_3v_4) = m - 3$, $c(v_1v_4) = 2$. It is easy to show that these two colorings are rainbow, and in these two cases, $rc(G) = m - 3$. □

Let i be an integer with $1 \leq i \leq 5$ and the addition is performed modulo 5. Set $\mathcal{J} = \{G : m = n, g(G) = 5\}$ and $\mathcal{J} = \mathcal{J}_1 \cup \{C_5\} \cup \mathcal{J}_2$, where $\mathcal{J}_1 = \{G : G \in \mathcal{J}, l(v_i) \leq 2, l(v_{i+2}) \leq 1, l(v_{i+1}) = l(v_{i+3}) = l(v_{i+4}) = 0\ \mathit{or}\ l(v_i) \leq 1, l(v_{i+1}) \leq 1, l(v_{i+2}) \leq 1, l(v_{i+3}) = l(v_{i+4}) = 0\}$, and $\mathcal{J}_2$ is the set of the rest unicyclic graphs with girth 5.

Lemma 6. *Let G be a graph belonging to $\mathcal{J}$. If G is isomorphic to a cycle C_5, then $rc(G) = m - 2$. If $G \in \mathcal{J}_1$, then $rc(G) = m - 3$. If $G \in \mathcal{J}_2$, then $rc(G) \leq m - 4$.*

Proof. Let the unique cycle of G be $C = v_1v_2v_3v_4v_5v_1$. If G is isomorphic to a cycle C_5, it is easy to see that $rc(G) = m - 2$. Suppose $G \in \mathcal{J}_1$. Suppose e_1 is a pendant edge of $T(v_1)$ that is assigned color 1. Set $c(v_1v_2) = m - 4$, $c(v_2v_3) = m - 3$, $c(v_3v_4) = 1$, $c(v_4v_5) = m - 4$, $c(v_1v_5) = m - 3$. Thus $rc(G) \leq m - 3$. On the other hand, since it costs $m-5$ colors for $G \setminus E(C)$, and by Observation 4, we know that at least two colors different from $c(G \setminus E(C))$ should be assigned to C, it follows that $rc(G) \geq m - 3$. Therefore, $rc(G) = m - 3$.

Suppose $G \in \mathcal{J}_2$. Without loss of generality, we consider the following three cases. If $l(v_i) \geq 3$ for some i with $1 \leq i \leq 5$, then we may suppose that e_1, e_2 and e_3 are the three pendant edges of $T(v_1)$ that are assigned colors 1,2,3, respectively. Set $c(v_1v_2) = m - 4$, $c(v_2v_3) = 3$, $c(v_3v_4) = 2$, $c(v_4v_5) = 1$, $c(v_1v_5) = m - 4$. If $l(v_i) = 2$, then we may suppose that e_1, e_2 are the two pendant edges of $T(v_1)$ that are assigned colors 1,2, respectively, and e_3 is a pendant edge of $T(v_2)$ that is assigned color 3. Set $c(v_1v_2) = m-4$, $c(v_2v_3) = 1$, $c(v_3v_4) = 2$, $c(v_4v_5) = m - 4$, $c(v_1v_5) = 3$. It remains to consider the case that $l(v_i) \leq 1$ for each i. Without loss of generality, let $l(v_1) = l(v_2) = l(v_4) = 1$. Suppose that e_i is a pendant edge that is assigned color i such that $e_1 \in T(v_1)$, $e_2 \in T(v_2)$ and $e_3 \in T(v_4)$, where $1 \leq i \leq 3$. Set $c(v_1v_2) = 3$, $c(v_2v_3) = m - 4$, $c(v_3v_4) = 1$, $c(v_4v_5) = 2$, $c(v_1v_5) = m - 4$. It is easy to show that these three colorings are rainbow, and in these three cases, $rc(G) \leq m - 4$. □

Let i be an integer with $1 \leq i \leq 6$ and the addition is performed modulo 6. Set $\mathcal{L} = \{G : m = n, g(G) = 6\}$ and $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$, where $\mathcal{L}_1 = \{G : G \in \mathcal{L}, l(v_i) \leq 1, l(v_{i+3}) \leq 1, l(v_{i+1}) = l(v_{i+2}) = l(v_{i+4}) = l(v_{i+5}) = 0\}$, $\mathcal{L}_2$ is the set of the rest unicyclic graphs with girth 6.

Lemma 7. *Let G be a graph belonging to $\mathcal{L}$. If $G \in \mathcal{L}_1$, then $rc(G) = m - 3$; otherwise $rc(G) \leq m - 4$.*

Proof. Let the unique cycle of G be $C = v_1v_2v_3v_4v_5v_6v_1$. By Observation 2, each edge of $G \setminus E(C)$ must obtain a distinct color, this costs $m - 6$ colors, thus $rc(G) \geq m - 6$. Color $G \setminus E(C)$ with a set $[m - 6]$ of colors. Suppose $G \in \mathcal{L}_1$. Set $c(v_1v_2) = m - 5$, $c(v_2v_3) = m - 4$, $c(v_3v_4) = m - 3$, $c(v_4v_5) = m - 5$, $c(v_5v_6) = m - 4$, $c(v_1v_6) = m - 3$. By Observation 2, $rc(G) \leq m - 3$. On the other hand, by Observation 4, we know that at least three colors different from $c(G \setminus E(C))$ should be assigned to C, it follows that $rc(G) \geq m - 3$. Therefore, $rc(G) = m - 3$.

Suppose $G \in \mathcal{L}_2$. If $l(v_i) \geq 2$, then we may suppose that e_1 and e_2 are the two pendant edges of $T(v_1)$ that are assigned colors 1,2, respectively. Set $c(v_1v_2) = m - 5$, $c(v_2v_3) = m - 4$, $c(v_3v_4) = 1$, $c(v_4v_5) = 2$, $c(v_5v_6) = m - 5$, $c(v_1v_6) = m - 4$. It remains to consider the case that $l(v_i) \leq 1$ for each i. Suppose $l(v_1) = l(v_2) = 1$. Let e_1 and e_2 be the two pendant edges that are assigned colors 1,2, respectively, such that $e_1 \in T(v_1)$ and $e_2 \in T(v_2)$. Set $c(v_1v_2) = m-5$, $c(v_2v_3) = m-4$, $c(v_3v_4) = 1$, $c(v_4v_5) = 2$, $c(v_5v_6) = m-5$, $c(v_1v_6) = m - 4$. Without loss of generality, let $l(v_1) = l(v_3) = 1$. Suppose that e_1 and e_2

are the two pendant edges that are assigned colors 1,2, respectively, such that $e_1 \in T(v_1)$ and $e_2 \in T(v_3)$. Set $c(v_1v_2) = m-5$, $c(v_2v_3) = m-4$, $c(v_3v_4) = 1$, $c(v_4v_5) = m-5$, $c(v_5v_6) = m-4$, $c(v_1v_6) = 2$. It is easy to show that these three colorings are rainbow, and in these three cases, $rc(G) \leq m-4$. $\square$

4 Characterizing graphs with $rc(G) = m-2$ and $m-3$

Now we are ready to characterize the graphs with $rc(G) = m-2$ and $rc(G) = m-3$.

Theorem 1. *$rc(G) = m-2$ if and only if G is isomorphic to a cycle C_5 or belongs to $\mathcal{G}_2 \cup \mathcal{H}_2$.*

Proof. Suppose that G is a graph with $rc(G) = m-2$. By Lemma 1, G contains a unique 2-connected subgraph. By Lemma 3, G contains no Θ-graph as a subgraph. It follows that G is a unicyclic graph. By Observation 3, the girth of G is at most 5. The cases that the girth of G is 3,4 and 5 have been discussed in Lemmas 4, 5 and 6, respectively. We conclude that G must be isomorphic to a graph shown in our theorem.

Conversely, By Lemmas 4, 5 and 6, the result holds. $\square$

Let $\mathcal{M}$ be a class of graphs where in each graph a path is attached at each vertex of degree 2 of $K_4 - e$, respectively. Note that, the path may be trivial.

Theorem 2. *$rc(G) = m-3$ if and only if G is isomorphic to a cycle C_7 or belongs to $\mathcal{G}_1 \cup \mathcal{H}_3 \cup \mathcal{J}_1 \cup \mathcal{L}_1 \cup \mathcal{M}$.*

Proof. Suppose that G is a graph with $rc(G) = m-3$. By Lemma 1, G contains a unique 2-connected subgraph B. Set $V(B) = \{v_1, \cdots, v_s\}$, then G has the structure as follows: a tree, denoted by $T(v_i)$, is attached at each vertex v_i of B. If B is exactly a cycle, then by Observation 3, the girth of G is at most 7. The cases that the girth of G is 3,4,5 and 6 have been discussed in Lemmas 4, 5, 6 and 7, respectively. It remains to deal with the case that the girth of G is 7. If G is not isomorphic to a cycle C_7, then suppose that e_1 is a pendant edge of $T(v_1)$ that is assigned color 1. Color $G \setminus E(B)$ with a set $[m-7]$ of colors and set $c(v_1v_2) = m-6$, $c(v_2v_3) = m-5$, $c(v_3v_4) = m-4$, $c(v_4v_5) = 1$, $c(v_5v_6) = m-6$, $c(v_6v_7) = m-5$, $c(v_1v_7) = m-4$. By Observation 1, we have $rc(G) \leq m-4$.

So B is not a cycle. By Lemma 3, G contains no Θ-graph except a $K_4 - e$ as a subgraph. We first claim that B is isomorphic to a $K_4 - e$. If B is isomorphic to a K_4, we first color the edges of $G \setminus E(B)$ with $m-6$ colors, then give each edge of B the same new color, this costs $m-5$ colors totally, it is easy to check that this coloring is rainbow, and in this case, $rc(G) \leq m-5$, a contradiction. Set $V(K_4 - e) = \{v_1, v_2, v_3, v_4\}$, and $E(K_4 - e) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3\}$. If $G \notin \mathcal{M}$, then $l(v_i) \geq 1$ or $l(v_j) \geq 2$ where $i = 1$ *or* 3, $j = 2$ *or* 4. If $l(v_1) \geq 1$, suppose that e_1 is a pendant edge of $T(v_1)$ that is assigned color 1. Assign color 1 to v_2v_3 and $m-4$ to each other edge of K_4-e. If $l(v_2)\geq 2$, suppose that e_1 and e_2 are two pendant edges of $T(v_2)$ that are assigned colors 1 and 2, respectively. Set $c(v_1v_2)=c(v_2v_3)=c(v_1v_3)=m-4$, $c(v_3v_4)=1$, $c(v_1v_4)=2$. In both cases, $rc(G)\leq m-4$. We conclude that G must be isomorphic to a graph shown in our theorem.

Conversely, if G is isomorphic to a cycle C_7, then $rc(G)=m-3$. If $G\in\mathcal{M}$, it is easy to see that at least two new colors different from $c(G\setminus E(B))$ should be assigned to B. Since each edge of $G\setminus E(B)$ must obtain a distinct color, this costs $m-5$ colors, it follows that $rc(G)\geq m-3$. Set $c(v_1v_2)=c(v_3v_4)=c(v_1v_3)=m-4$, $c(v_2v_3)=c(v_1v_4)=m-3$, thus $rc(G)\leq m-3$. Therefore, $rc(G)=m-3$. By Lemmas 4, 5, 6 and 7, the result holds. $\square$

Remark: We have also characterized the graphs G with $rc(G)=m-4$. But, the proof is similar to the above ones, and very long and tedious, and therefore not written down here.

Acknowledgement. The authors would like to thank the referees for their helpful comments and suggestions.

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