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ON EDGE-TRANSITIVE CUBIC GRAPHS OF SQUARE-FREE ORDER

GUI XIAN LIU AND ZAI PING LU

Abstract. A regular graph is said to be semisymmetric if it is edge-transitive but not vertex-transitive. In this paper, we give a complete list of connected semisymmetric cubic graph of square-free order, which consists of one single graph of order 210 and four infinite families of such graphs.

1. INTRODUCTION

All graphs in this paper are assumed to be simple and finite.

A graph $\Gamma = (V, E)$ with vertex set V and edge set E is said to be *vertex-transitive* or *edge-transitive* if its automorphism group $\mathsf{Aut}\Gamma$ acts transitively on V and E, respectively. A regular graph is said to be *semisymmetric* if it is edge-transitive but not vertex-transitive. Recall that an *arc* in a graph Γ is an ordered pair of adjacent vertices. Then a graph Γ is said to be *arc-transitive* if Γ is vertex-transitive and $\mathsf{Aut}\Gamma$ acts transitively on the set of all arcs in Γ.

The class of semisymmetric graphs was introduced by Folkman [7] who constructed several infinite families of such graphs and posed eight open problems which spurred the interest in this topic, see [5, 6, 10, 12, 15, 16, 17, 18] for example. This paper deals with semisymmetric cubic graphs of square-free order.

It is well-known that a vertex- and edge-transitive graph of odd valency must be arc-transitive. Thus an edge-transitive cubic graph is either arc-transitive or semisymmetric. In a recent paper [14], the arc-transitive cubic graphs of square-free order were classified. This motivates us to classify the semisymmetric cubic graphs of square-free order, and thus we can get a complete list of edge-transitive cubic graphs of square-free order. Our main result is stated as follows.

Theorem 1.1. *Let $\Gamma = (V, E)$ be a connected semisymmetric cubic graph of square-free order. Then Γ is described in Table 1, where $\{u, w\} \in E$ and p is a prime.*

$A := \mathsf{Aut}\Gamma$	A_u	A_w	Γ	Remark
S_7	$\mathrm{S}_4{\times}\mathbb{Z}_2$	$\mathrm{D}_8{\times}\mathrm{S}_3$	Example 3.1	$\lvert V\rvert = 210$
$\mathbb{Z}_m{:}\mathbb{Z}_3^2$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	Example 3.2	$\lvert V\rvert = 6m$, $m \geq 91$
$\mathrm{PGL}(2, p)$	S_4	D_{24}	Example 3.3	$p \equiv \pm 11 \pmod{24}$
$\mathrm{PSL}(2, p)$	S_4	D_{24}	Example 3.4	$p \equiv \pm 23 \pmod{48}$
$\mathbb{Z}_3{\times}\mathrm{PSL}(2, p)$	A_4	D_{12}	Example 3.5	$p \equiv \pm 11 \pmod{24}$

Table 1. Semisymmetric cubic graphs of square-free order.

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2. Preliminaries

For a graph $\Gamma = (V, E)$ and a subgroup $G \leq \mathsf{Aut}\Gamma$, we call Γ a *G-edge-transitive* or a *G-vertex-transitive* graph if G acts transitively on the edge set E or the vertex set V, respectively. The graph Γ is said to be *G-semisymmetric* if it is regular and G-edge-transitive but not G-vertex-transitive.

Let $\Gamma = (V, E)$ be a G-semisymmetric. Then G is a bipartite graph with bipartition subsets, say U and W, being the G-orbits on V. For $v \in V$, the vertex-stabilizer G_v acts transitively on the set $\Gamma(v)$ of neighbors of v in Γ. Take an edge $\{u, w\} \in E$ with $u \in U$ and $w \in W$. Then each vertex of Γ can be written as u^x or w^y for some $x, y \in G$. Thus two vertices u^x and w^y are adjacent in Γ if and only if u and $w^{yx^{-1}}$ are adjacent, i.e., $yx^{-1} \in G_wG_u$. Moreover, it is well-known and easily shown that Γ is connected if and only if $\langle G_u, G_w\rangle = G$. Define a map by $u^x \mapsto G_ux$ and $w^y \mapsto G_wy$. Then it is easily shown that this map is an isomorphism from Γ to a bipartite graph defined as follows.

Let G be a finite group and L, $R \leq G$. The bi-coset graph $B(G, L, R)$ is defined with bipartition subsets $[G : L] = \{Lx \mid x \in G\}$ and $[G : R] = \{Ry \mid y \in G\}$ such that Lx and Ry are adjacent if and only if $yx^{-1} \in RL$. Then, considering the right multiplication on $[G : L]$ and $[G : R]$, the group G induces an edge-transitive subgroup $\widehat{G}$ of $\mathsf{Aut}B(G, L, R)$. The following facts on bi-coset graphs are well-known, see [5] for example.

Lemma 2.1. *Let $\Gamma = B(G, L, R)$ be the bi-coset graph defined as above. Then*

(1) *Γ is $\widehat{G}$-edge-transitive;*
(2) *Γ is connected if and only if $\langle L, R\rangle = G$, in this case, $G \cong \widehat{G}$ when $L \cap R$ contains no non-trivial normal subgroups of G;*
(3) *Γ is regular if and only if $|R| = |L|$, and so Γ has valency $|L : (L \cap R)|$.*

Let $\Gamma = (V, E)$ be a connected G-semisymmetric cubic graph. Then, for an edge $\{u, w\} \in E$, the pair (G_u, G_w) is known by [9]. In particular, setting $G_{uw} = G_u \cap G_w$, the following result holds.

Theorem 2.2. *Let $\Gamma = (V, E)$ be a connected cubic G-semisymmetric graph of order $2n$. Then $|G_u| = |G_w| = 3 \cdot 2^i$ and $|G| = 3n \cdot 2^i$, where $\{u, w\} \in E$ and $0 \leq i \leq 7$. In particular, G_{uw} is a Sylow 2-subgroup of G if further n is odd.*

Let U and V be the G-orbits on V, and take a normal subgroup $N \lhd G$. Suppose that N is intransitive on both U and W. For $v \in V$, we denote by $\bar{v}$ the N-orbit containing v. Set $\overline{U} = \{\bar{u} \mid u \in U\}$ and $\overline{W} = \{\bar{w} \mid w \in W\}$. The *normal quotient* Γ_N is defined as the graph on $\overline{U} \cup \overline{W}$ with edge set $\{\{\bar{u}, \bar{w}\} \mid \{u, w\} \in E\}$. Then the following lemma holds, see [8] or [16].

Lemma 2.3. *Let $\Gamma = (V, E)$ be a connected G-semisymmetric cubic graph with bipartition subsets U and W. Let $N \lhd G$. Then one of the following statements holds.*

(1) *N is semiregular on both U and W, G induces a subgroup X of $\mathsf{Aut}\Gamma_N$ such that $X \cong G/N$ and Γ_N is a connected X-semisymmetric cubic graph.*
(2) *N acts transitively on at least one of U and W.*

3. Examples

In this section we construct the graphs involved in Theorem 1.1.

By [3], there is a unique semisymmetric cubic graph **S210** of order 210, which has automorphism group S_7. We next construct this graph as the incidence graph of an incidence structure.

Example 3.1. Consider the complete graph K_7. A k-matching in K_7 is a set of k edges such that no two have a vertex in common. Let U and W be the sets of 3- and 2-matchings in K_7, respectively. Then $|U| = |W| = 105$. Define a bipartite graph $\mathit{\Gamma}$ on $U \cup W$ such $u \in U$ and $w \in W$ are adjacent if and only if w is contained in u. Then $\mathit{\Gamma}$ is an S_7-edge-transitive cubic graph of order 210, and so $\mathit{\Gamma} \cong \mathbf{S210}$ by [2, 3].

Malnič et al. [17] constructed an infinite family of semisymmetric cubic graph from the $\mathbb{Z}_n$-cover of $K_{3,3}$. Using bi-coset graphs, we reconstruct here some members appearing in our main result.

Example 3.2. Let $F = \langle a_1\rangle\times\langle a_2\rangle\times\cdots\times\langle a_t\rangle$ with $t \geq 2$ and $\langle a_i\rangle \cong \mathbb{Z}_{p_i}$ for $1 \leq i \leq t$, where p_i's are distinct primes with $p_i \equiv 1 \pmod 3$. Then $\mathsf{Aut}(F)$ is abelian and has a subgroup isomorphic to $\mathbb{Z}_3^t$. For each i, take an integer r_i such that $r_i^2 + r_i + 1 \equiv 0 \pmod{p_i}$. Then $\mathsf{Aut}(F)$ has a subgroup $\langle \sigma, \tau\rangle \cong \mathbb{Z}_3^2$ such that $a_i^\sigma = a_i^{r_i}$ and $a_i^\tau = a_i^{r_i^{e_i}}$ for $1 \leq i \leq t$, where $e_i = 1$ or 2.

Let $G = F{:}\langle\sigma,\tau\rangle$, the semidirect product of F and $\langle\sigma,\tau\rangle$. Then $G \cong \mathbb{Z}_m{:}\mathbb{Z}_3^2$, where $m = p_1p_2\cdots p_t$. Take $L = \langle\sigma\rangle$ and $R = \langle a\tau\rangle$, where $a = a_1a_2\cdots a_t$. Then the bi-coset graph $B(G, L, R)$ is a connected G-edge-transitive cubic graph of order $6m$.

Lipschutz and Xu [15] constructed two infinite families of semisymmetric cubic graphs from groups $\mathrm{PSL}(2, p)$ and $\mathrm{PGL}(2, p)$.

Example 3.3. Let $G = \mathrm{PGL}(2, p)$ for prime p with $p \equiv \pm 11 \pmod{24}$. Then G has a Sylow 2-subgroup isomorphic to D_8, a subgroup isomorphic to S_4 and a subgroup isomorphic to D_{24}, see [1]. Take $\mathrm{S}_4 \cong L < G$ and $\mathrm{D}_{24} \cong R < G$. Then, by [15], the bi-coset graph $B(G, L, R)$ is a connected semisymmetric cubic graph of order $\frac{p(p^2-1)}{12}$.

Example 3.4. Let $G = \mathrm{PSL}(2, p)$ for a prime p with $p \equiv \pm 23 \pmod{48}$. Then G has a Sylow 2-subgroup isomorphic to D_8, a subgroup isomorphic to S_4 and a subgroup isomorphic to D_{24}, see [11, II.8.27]. Take $\mathrm{S}_4 \cong L < G$ and $\mathrm{D}_{24} \cong R < G$. Then, by [15], $B(G, L, R)$ is a connected semisymmetric cubic graph of order $\frac{p(p^2-1)}{24}$.

Finally, we constructed a new family of edge-transitive graphs which are covers of the graphs constructed as in Example 3.3.

Example 3.5. Let $G = M\times T$, where $M = \langle c\rangle \cong \mathbb{Z}_3$ and $T = \mathrm{PSL}(2, p)$ for prime p with $p \equiv \pm 11 \pmod{24}$. Take $\mathrm{A}_4 \cong L_1 < G$ and $\mathrm{D}_{12} \cong R < G$. Set $L_1 = P{:}\langle d\rangle$ for $P \cong \mathbb{Z}_2^2$ and $\langle d\rangle \cong \mathbb{Z}_3$. Let $L = P{:}\langle cd\rangle$. Then $B(G, L, R)$ is a connected G-edge-transitive cubic graph of order $\frac{p(p^2-1)}{4}$.

Remark The graphs in Examples 3.2 and 3.5 are semisymmetric, which will be proved at the end of Section 4.

4. The proof of Theorem 1.1

In this section we always assume that $\Gamma = (V, E)$ is a connected G-semisymmetric cubic graph of square-free order $2n$. Then $|G| = 3n \cdot 2^i$ for $i \leq 7$; in particular, $|G|$ is not divisible by 3^3 or r^2, where r is a prime no less than 5.

Let U and W be the G-orbits on V. Then $|U| = |W| = n$ is odd and square-free. If G is unfaithful on one of U and W then it is easily shown that $\Gamma \cong \mathsf{K}_{3,3}$, and so Γ is arc-transitive. If $n = 3$ then $|G|$ is divisible by 9, and so G is unfaithful on both U and W, hence $\Gamma \cong \mathsf{K}_{3,3}$ is arc-transitive.

Therefore, in the following, we assume further that G is faithful on both U and W; in particular, $|U| = |W| > 3$. For a prime p, we use $\mathbf{O}_p(G)$ to denote the maximal normal p-subgroup of G. Then we have a simple observation as follows.

Lemma 4.1. $\mathbf{O}_2(G) = 1$ *and* $|\mathbf{O}_p(G)| = 1$ *or* p*, where* p *is an odd prime.*

Proof. This lemma is trivial for $p \geq 5$. Thus we let $p = 2$ or 3 in the following. Note that $\mathbf{O}_p(G)$ fixes both U and W set-wise. Since $\mathbf{O}_p(G) \lhd G$, all $\mathbf{O}_p(G)$-orbits on U has the same length which is a divisor of $|U| = n$. Thus each $\mathbf{O}_p(G)$-orbit on U has length 1 or 3, and so either $\mathbf{O}_p(G) = 1$, or $\mathbf{O}_3(G) \cong \mathbb{Z}_3$ or $\mathbb{Z}_3^2$. If $\mathbf{O}_3(G)$ is transitive on one of U and W, then $|U| = |W| = 3$, a contradiction. Then $\mathbf{O}_3(G)$ acts intransitively on both U and W. By Lemma 2.3, $\mathbf{O}_3(G)$ is semiregular on U, and hence $\mathbf{O}_3(G) \cong \mathbb{Z}_3$. □

Lemma 4.2. *Assume that* G *is soluble. Then one of the following holds.*

(1) *G has a cyclic normal subgroup of order n, and Γ is arc-transitive.*
(2) *$G \cong \mathbb{Z}_m{:}\mathbb{Z}_3^2$ and Γ is constructed as in Example* 3.2.

Proof. Let F be the Fitting subgroup of G. Then $F = \mathbf{O}_{p_1}(G) \times \cdots \times \mathbf{O}_{p_t}(G)$ for prime divisors $p_1, \cdots, p_t$ of $|G|$. By Lemma 4.1, F is cyclic, $|F|$ is odd and square-free. In particular, $\mathsf{Aut}(F)$ is abelian. Since G is soluble, $\mathbf{C}_G(F) \leq F$, and so $\mathbf{C}_G(F) = F$. Thus $G/F = \mathbf{N}_G(F)/\mathbf{C}_G(F) \lesssim \mathsf{Aut}(F)$. Then G/F ia abelian. It is easily shown that F is semiregular on both U and W. For $v \in U \cup W$, we have $FG_v/F = G_v/(F \cap G_v) \cong G_v$. Then G_v is abelian, and so $G_v \cong \mathbb{Z}_3$ by [9].

(1) Assume first that F is transitive on U. Then F is transitive on W. Thus F is regular on both U and W. Take an edge $\{u, w\} \in E$ with $u \in U$ and $w \in W$. Then each vertex of Γ can be written uniquely as u^x or w^y for $x, y \in F$. Define a map $\theta : V \to V$; $u^x \mapsto w^{x^{-1}}$, $w^y \mapsto u^{y^{-1}}$. Then θ is a bijection on V. Moreover, since F is abelian, $\{u^x, w^y\} \in E$ if and only if $\{u, w^{x^{-1}y}\} = \{u, w^{yx^{-1}}\} \in E$, i.e., $\{u^{y^{-1}}, w^{x^{-1}}\} \in E$. This says that θ is an automorphism of Γ interchanging U and W. Thus Γ is arc-transitive, and part (1) of this lemma follows.

(2) Assume that F is intransitive on U. Then F is intransitive on W. By Lemma 2.3, the quotient graph Γ_F is a X-semisymmetric cubic graph, where $X \cong G/F$ is abelian. Let $u \in U$. Then both $F{:}G_u$ and $F{:}G_w$ are normal in G.

Clearly, $F{:}G_u$ is not semiregular on U. By Lemma 2.3, $F{:}G_u$ is transitive on W. Then it follows that FG_u is regular on W. Then $|W| = |FG_u| = 3|F|$ and, since $|W|$ is square-free, $|F|$ is coprime to 3. Thus Γ_F has order $2\frac{|W|}{|F|} = 6$, and so $\Gamma_F \cong \mathsf{K}_{3,3}$. Since X is abelian and Γ_F is X-semisymmetric, we have $G/F \cong X \cong \mathbb{Z}_3^2$, and hence $G = F{:}Y$, where $Y \cong \mathbb{Z}_3^2$ with $G_u < Y$. Set $G_u = \langle \sigma \rangle$ and $Y = \langle \sigma, \tau \rangle$. Then

$G_w = \langle a\tau\rangle$ for some $a \in F$. Since Γ is connected, $F{:}\langle\sigma,\tau\rangle = F{:}Y = G = \langle G_u, G_w\rangle = \langle\sigma, a\tau\rangle \le \langle a,\sigma,\tau\rangle = \langle a\rangle\langle\sigma,\tau\rangle$. It follows that $F = \langle a\rangle$, and so $G = \langle a\rangle{:}\langle\sigma,\tau\rangle$.

Let M be the center of $F{:}G_u$. Then $M \lhd G$, and so $M \le F$ by the choice of F. Thus $FG_u = M{\times}(N{:}G_u)$ with $F = M{\times}N$. Note that NG_u is a Hall subgroup of FG_u. It follows that NG_u is a characteristic subgroup of FG_u, and so NG_u is normal in G as $FG_u \lhd G$. Clearly, NG_u is neither semiregular nor transitive on U. Again by Lemma 2.3, NG_u is transitive on W. Recall that FG_u is regular on W. Then $FG_u = NG_u$, and so $M = 1$, that is, $FG_u = \langle a\rangle{:}\langle\sigma\rangle$ has trivial center. Choose integers $k_1, k_2, \cdots, k_t$ such that $1 = \sum_{i=1}^{t} k_i \prod_{j\ne i} p_j$. Set $a_i = a^{k_i \prod_{j\ne i} p_j}$ for $1 \le i \le t$. Then a_i has order p_i, $a = a_1a_2\cdots a_t$ and $\langle a\rangle = \langle a_1\rangle{\times}\langle a_2\rangle{\times}\cdots{\times}\langle a_t\rangle$. Noting that each $\langle a_i\rangle$ is normal in $\langle a\rangle{:}\langle\sigma\rangle$, we have $a_i^\sigma = a_i^{r_i}$ for some integer r_i. Clearly, $r_i \not\equiv 1 \,(\mathsf{mod}\ p_i)$ as $\langle a\rangle{:}\langle\sigma\rangle$ has trivial center. Since σ has order 3, we have $a = a^{\sigma^3} = a_1^{r_1^3}a_2^{r_2^3}\cdots a_t^{r_t^3}$. It follows that $r_i^2 + r_i + 1 \equiv 0 \,(\mathsf{mod}\ p_i)$ for $1 \le i \le t$.

Note that the above argument is available for FG_w. Then FG_w has trivial center. Note that $FG_w = \langle a\rangle{:}\langle a\tau\rangle = \langle a\rangle{:}\langle\tau\rangle$. Arguing similarly as above, we have $a_i^\sigma = a_i^{s_i}$ some integer s_i with $s_i^2 + s_i + 1 \equiv 0 \,(\mathsf{mod}\ p_i)$, where $1 \le i \le t$.

Note that for each i the equation $x^2 + x + 1 \equiv 0 \,(\mathsf{mod}\ p_i)$ has exactly two solutions. Thus we may choose $s_i = r_i$ or r_i^2 with the restriction that $\tau \notin \{\sigma, \sigma^{-1}\}$. Then Γ is a G-edge-transitive graph constructed as in Example 3.2. □

For the case where G is insoluble, by [2, 3, 18], we can prove the following.

Lemma 4.3. *Assume that G is insoluble. Let $\{u, w\} \in E$. Then either*

(1) *G, G_u, G_w and $\mathsf{Aut}\Gamma$ are listed in Table 2, in particular, Γ is either arc-transitive or isomorphic one of the graphs given in Examples 3.1, 3.3 and 3.4, where p is a prime; or*

(2) *$G = \mathbb{Z}_3{\times}\mathrm{PSL}(2,p)$ for a prime p with $p \equiv \pm 11 \,(\mathsf{mod}\ 24)$, and Γ is a graph constructed in Example 3.5.*

G	G_u	G_w	$\mathsf{Aut}\Gamma$	Remark	Symmetric
A_6	S_4	S_4	$\mathrm{P\Gamma L}(2,9)$	Tutte's 8-cage	Yes
S_6	$\mathrm{S}_4{\times}\mathbb{Z}_2$	$\mathrm{S}_4{\times}\mathbb{Z}_2$	$\mathrm{P\Gamma L}(2,9)$	Tutte's 8-cage	Yes
$\mathrm{PSL}(2,p)$	D_{12}	D_{12}	$\mathrm{PGL}(2,p)$	$p \equiv \pm 11 \,(\mathsf{mod}\ 24)$	Yes
$\mathrm{PSL}(2,p)$	S_4	S_4	$\mathrm{PGL}(2,p)$	$p \equiv \pm 7 \,(\mathsf{mod}\ 16)$	Yes
A_7	S_4	$(\mathbb{Z}_2^2{\times}\mathbb{Z}_3).\mathbb{Z}_2$	S_7	**S210** (see [3])	No
S_7	$\mathrm{S}_4{\times}\mathbb{Z}_2$	$\mathrm{S}_3{\times}\mathrm{D}_8$	S_7	**S210**	No
$\mathrm{PSL}(2,p)$	A_4	D_{12}	$\mathrm{PGL}(2,p)$	$p \equiv \pm 11 \,(\mathsf{mod}\ 24)$	No
$\mathrm{PGL}(2,p)$	S_4	D_{24}	$\mathrm{PGL}(2,p)$	$p \equiv \pm 11 \,(\mathsf{mod}\ 24)$	No
$\mathrm{PSL}(2,p)$	S_4	D_{24}	$\mathrm{PSL}(2,p)$	$p \equiv \pm 23 \,(\mathsf{mod}\ 48)$	No

Table 2. Graphs having almost simple automorphism group.

Proof. Let M be the largest soluble normal subgroup of G. If M is transitive on one of U and W, then $G = MG_v$ for some $v \in U \cup W$, and so G is soluble, a contradiction. Thus M is intransitive on both U and W. By Lemma 2.3, M is semiregular on both U and W and Γ_M is a connected X-semisymmetric cubic graph, where $X \cong G/M$ is the subgroup of $\mathsf{Aut}\Gamma_M$ induced by G. By the choice of M, we know that X has no soluble normal subgroups. For $v \in V$, we denote by $\bar{v}$ the M-orbit containing v. Let $\{u, w\} \in E$ with $u \in U$ and $w \in W$. Then, by [18, Corollary 1.3], we have the following table, where p is a prime.

X	$X_{\bar{u}}$	$X_{\bar{w}}$	$\mathsf{Aut}\Gamma_M$	Remark	Symmetric
A_6	S_4	S_4	$\mathrm{P\Gamma L}(2,9)$	Tutte's 8-cage	Yes
S_6	$\mathrm{S}_4{\times}\mathbb{Z}_2$	$\mathrm{S}_4{\times}\mathbb{Z}_2$	$\mathrm{P\Gamma L}(2,9)$	Tutte's 8-cage	Yes
$\mathrm{PSL}(2,p)$	D_{12}	D_{12}	$\mathrm{PGL}(2,p)$	$p \equiv \pm 11 \pmod{24}$	Yes
$\mathrm{PSL}(2,p)$	S_4	S_4	$\mathrm{PGL}(2,p)$	$p \equiv \pm 7 \pmod{16}$	Yes
A_7	S_4	$(\mathbb{Z}_2^2{\times}\mathbb{Z}_3).\mathbb{Z}_2$	S_7	**S210**	No
S_7	$\mathrm{S}_4{\times}\mathbb{Z}_2$	$\mathrm{S}_3{\times}\mathrm{D}_8$	S_7	**S210**	No
$\mathrm{PSL}(2,p)$	A_4	D_{12}	$\mathrm{PGL}(2,p)$	$p \equiv \pm 11 \pmod{24}$	No
$\mathrm{PGL}(2,p)$	S_4	D_{24}	$\mathrm{PGL}(2,p)$	$p \equiv \pm 11 \pmod{24}$	No
$\mathrm{PSL}(2,p)$	S_4	D_{24}	$\mathrm{PSL}(2,p)$	$p \equiv \pm 23 \pmod{48}$	No

If $M = 1$ then part (1) of this lemma holds. Thus we assume next $M \neq 1$. Note that M is semiregular on both U and W. Then $|M|$ is square-free and odd. Let $\overline{U}$ and $\overline{W}$ be the sets of M-orbits on U and W, respectively. Then $n = |U| = |\overline{U}||M| = |\overline{W}||M| = |W|$; in particular, $|M|$ is coprime to $|\overline{U}| = |\overline{W}|$.

Take a normal subgroup $Y \lhd G$ of G such that $M \le Y$ and $Y/M \cong \mathrm{soc}(X)$. Then Y is transitive on both U and W. Since M has square-free order, $\mathsf{Aut}(M)$ is soluble. Since $Y/\mathbf{C}_Y(M) = \mathbf{N}_Y(M)/\mathbf{C}_Y(M) \lesssim \mathsf{Aut}(M)$, we know that $Y/\mathbf{C}_Y(M)$ is soluble. Since $\mathrm{soc}(X)$ is a nonabelian simple group, we have $M\mathbf{C}_Y(M)/M \cong \mathrm{soc}(X)$, and so $Y = M\mathbf{C}_Y(M)$ and $\mathbf{C}_Y(M)/(M \cap \mathbf{C}_Y(M)) \cong \mathrm{soc}(X)$. Note that $M \cap \mathbf{C}_Y(M)$ lies in the center of $\mathbf{C}_Y(M)$ and $M \cap \mathbf{C}_Y(M)$ has odd order. Checking the Schur multiplier of $\mathrm{soc}(X)$ (refer to [13, Theorem 5.14]), We conclude that either $M \cap \mathbf{C}_Y(M) = 1$, or $M \cap \mathbf{C}_Y(M) \cong \mathbb{Z}_3$ and $\mathrm{soc}(X) \cong \mathrm{A}_6$. The latter case yields that $|G| = 3n{\cdot}2^i$ is divisible 27, which is impossible as n is square-free. Then $Y = M\mathbf{C}_Y(M) = M{\times}\mathbf{C}_Y(M)$ and $T := \mathbf{C}_Y(M) \cong \mathrm{soc}(X)$. Moreover, T is transitive on both $\overline{U}$ and $\overline{W}$.

It is easy to see that T is a characteristic subgroup of Y, and hence T is normal in G. Clearly, T is not semiregular on both U and W. By Lemma 2.3, T is transitive on one of U and W. Then there is some $v \in V$ such that $T_{\bar{v}}$ is transitive on $\bar{v}$. Consider that action of $M{\times}T_{\bar{v}}$. Then both M and $T_{\bar{v}}$ act transitively on $\bar{v}$, and hence both of them induce regular permutation groups on $\bar{v}$, refer to [4, Theorem 4.2 A]. It follows that $T_{\bar{v}}$ has a normal subgroup of odd index $|\bar{v}| = |M|$. Noting that $T_{\bar{v}} \cong \mathrm{soc}(X)_{\bar{v}}$, we conclude that $X \cong \mathrm{PSL}(2,p)$, $M \cong \mathbb{Z}_3$, $v \in U$, $T_{\bar{v}} \cong X_{\bar{v}} \cong \mathrm{A}_4$, $T_v \cong \mathbb{Z}_2^2$ and $G = Y = M{\times}T$. Moreover, we have $\mathrm{D}_{12} \cong T_{\bar{w}} = T_w$ for $w \in W$. It follows that, for $\{u, w\} \in E$ with $u \in U$ and $w \in W$, $G_u = P{:}\langle cd\rangle \cong \mathrm{A}_4$ and $G_w = T_w \cong \mathrm{D}_{12}$, where $\mathbb{Z}_2^2 \cong P < T$, $c \in M$ and $d \in T$ have order 3. Thus Γ is isomorphic to a graph given in Example 3.5, and part (2) of this lemma follows. $\square$

Proof of Theorem 1.1. By the foregoing argument, it suffices to show that the graphs given in Examples 3.2 and 3.5 are semisymmetric.

Let Γ be a given in Examples 3.2 or 3.5. Let A^+ be the subgroup of $A := \mathsf{Aut}\Gamma$ which preserves the bipartition of Γ. Then $|A : A^+| \le 2$ and Γ is A^+-edge-transitive. Note that Γ has order more than 6. Then A^+ is faithful on both bipartition subsets of Γ. Let $\{u, w\}$ be an edge of Γ.

Suppose first that $\varGamma$ is given as in Example 3.5. Then $A^+ \gtrsim \mathbb{Z}_3\times\mathrm{PSL}(2,p)$. By Lemma 4.3, the only possibility is $A^+ \cong \mathbb{Z}_3\times\mathrm{PSL}(2,p)$. Then $A_u = A_u^+$ and $A_w = A_w^+$. Thus A_u and A_w are not conjugate in A, and so $\varGamma$ is not A-vertex-transitive. Then $A^+ = A = \mathsf{Aut}\varGamma$, and so $\varGamma$ is semisymmetric.

Now let $\varGamma$ be a graph given in Example 3.2. Then A^+ has a cyclic semiregular subgroup, whose order is $m := \frac{|U|}{3} = p_1p_2\cdots p_t$, where p_i's are distinct primes with $p_i \equiv 1\,(\mathsf{mod}\ 3)$. Checking the groups in Lemma 4.2 and 4.3, we conclude that either $A^+ = A \cong \mathbb{Z}_m{:}\mathbb{Z}_3^2$, or A is soluble and $\varGamma$ is arc-transitive. Suppose that the latter case occurs. By [14], we have $A \cong \mathrm{D}_{6m}{:}\mathbb{Z}_3$ and $A^+ \cong \mathbb{Z}_{3m}{:}\mathbb{Z}_3$. On other hand, by the construction of $\varGamma$, A^+ has a subgroup of order $9m$ which has trivial center. Thus $A^+ \cong \mathbb{Z}_{3m}{:}\mathbb{Z}_3$ has trivial center, which is impossible. Thus the former case holds, that is, $A = A^+ \cong \mathbb{Z}_m{:}\mathbb{Z}_3^2$. Then Theorem 1.1 follows.

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G. X. Liu, Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, P. R. China

E-mail address: lgxnkdx@mail.nankai.edu.cn

Z. P. Lu, Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, P. R. China

E-mail address: lu@nankai.edu.cn