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On locally primitive graphs of order $18p$*

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Abstract In this paper, we investigate locally primitive bipartite regular connected graphs of order $18p$. It is shown that such a graph is either arc-transitive or isomorphic to one of the Gray graph and the Tutte 12-cage.

Keywords Locally primitive graph, arc-transitive graph, normal cover, quasiprimitive group
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1 Introduction

All graphs in this paper are assumed to be finite and simple.

Let Γ be a graph. We use $V\Gamma$, $E\Gamma$ and $\mathsf{Aut}\Gamma$ to denote the vertex set, edge set and automorphism group of Γ, respectively. Then the graph Γ is said to be *vertex-transitive* or *edge-transitive* if some subgroup G of $\mathsf{Aut}\Gamma$ (denoted by $G \leq \mathsf{Aut}\Gamma$) acts transitively on $V\Gamma$ or $E\Gamma$, respectively. Recall that an arc in Γ is an ordered pair of adjacent vertices. Then the graph Γ is called *arc-transitive* if some $G \leq \mathsf{Aut}\Gamma$ acts transitively on the set of arcs of Γ. The graph Γ is said to be *locally primitive* if, for some subgroup $G \leq \mathsf{Aut}\Gamma$ and each $v \in V\Gamma$, the stabilizer G_v induces a primitive permutation group $G_v^{\Gamma(v)}$ on the neighborhood $\Gamma(v)$, the set of neighbors, of v in Γ. For convenience, such subgroups G are called *vertex-transitive*, *edge-transitive*, *arc-transitive* and *locally primitive* groups of Γ, respectively.

Studying of locally primitive graphs is one of the main themes in algebraic graph theory, which stems from a conjecture on bonding the stabilizers of locally primitive arc-transitive graphs [32, Conjecture 12]. The reader may consult [4, 9, 10, 11, 12, 14, 21, 22, 23, 24, 28, 29, 31] for some known results in this area.

In this paper, we aim at determining the arc-transitivity of certain locally primitive graphs. Let Γ be a connected graph and G be a locally primitive group on Γ. It is easily shown that G acts transitively on $E\Gamma$, and so Γ is edge-transitive. If G is vertex-transitive then Γ is necessarily an arc-transitive graph. Thus, for our purpose, we always assume that Γ is regular but G is not vertex-transitive. Then Γ is a bipartite graphs with two bipartition subsets being the G-orbits on $V\Gamma$. Giudici et al. [14] established a reduction for studying locally primitive bipartite graphs, which was successfully applied in [23] to the characterization of locally primitive graphs of order twice a prime power. In this paper we concentrate our

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attention on analyzing the locally primitive graphs of order $18p$. Our main result is stated as follows.

Theorem 1.1 *Let Γ be a connected regular graph of order $18p$, where p is a prime. Assume that Γ is locally primitive. Then Γ is either arc-transitive or isomorphic to one of the Gray graph and the Tutte 12-cage.*

2 Preliminaries

Let Γ be a graph and let $G \leq \mathsf{Aut}\Gamma$. Assume that G is edge-transitive but not vertex-transitive; in this case, we call G *semisymmetric* if Γ is regular. Then Γ is a bipartite graph with two bipartition subsets being the G-orbits on $V\Gamma$. Moreover, Γ is arc-transitive provided that Γ has an automorphism interchanging two of its bipartition subsets. For a given vertex $u \in V\Gamma$, the stabilizer G_u acts transitively on $\Gamma(u)$. Take $w \in \Gamma(u)$. Then each vertex of Γ can be written as u^g or w^g for some $g \in G$. Then two vertices u^g and w^h are adjacent in Γ if and only if u and $w^{hg^{-1}}$ are adjacent, i.e., $hg^{-1} \in G_wG_u$. Moreover, it is well-known and easily shown that Γ is connected if and only if $\langle G_u, G_w\rangle = G$. In particular, the next simple fact follows.

Lemma 2.1 *Let Γ be a connected graph and $G \leq \mathsf{Aut}\Gamma$. Assume that G is edge-transitive but not vertex-transitive. Let $\{u, w\}$ be an edge of Γ. Then*

(1) G_u and G_w contain no nontrivial normal subgroups in common; and

(2) $r \leq \max\{|\Gamma(u)|, |\Gamma(w)|\}$ for each prime divisor r of $|G_u|$.

Moreover, Γ is arc-transitive if one of the following conditions holds:

(3) G has an automorphism σ of order 2 with $G_u^\sigma = G_w$.

(4) G has an abelian subgroup acting regularly on both bipartition subsets of Γ.

Proof. Since Γ is connected, $\langle G_u, G_w\rangle = G \leq \mathsf{Aut}\Gamma$. Then part (1) follows.

Let r be a prime divisor of $|G_u|$ with $r > \max\{|\Gamma(u)|, |\Gamma(w)|\}$, and let R be a Sylow r-subgroup of G_u. Then R fixes $\Gamma(u)$ point-wise, and so $R \leq G_{w'}$ for each $w' \in \Gamma(u)$. Take Q be a Sylow r-subgroup of G_w with $Q \geq R$. Then Q fixes $\Gamma(w)$ point-wise, hence $Q \leq G_u$. Thus $R = Q$. By the connectedness of Γ, for each $v \in V\Gamma$, it is easily shown that R is a Sylow r-subgroup of G_v. Thus R fixes $V\Gamma$ point-wise, and so R=1 as $R \leq \mathsf{Aut}\Gamma$. Then part (2) follows.

Suppose that G has an automorphism σ of order 2 with $G_u^\sigma = G_w$. Define a bijection $\iota : V\Gamma \to V\Gamma$ by $(u^g)^\iota = w^{g^\sigma}$ and $(w^h)^\iota = u^{h^\sigma}$. It is easy to check that $\iota \in \mathsf{Aut}\Gamma$ and ι interchanges two bipartition subsets of Γ. This implies that Γ is arc-transitive.

Suppose that G has a subgroup R which is regular on both bipartition subsets of Γ. Then each vertex in $V\Gamma$ can be written uniquely as u^x or w^y for some $x, y \in R$. Set $S = \{s \in R \mid w^s \in \Gamma(u)\}$. Then u^x and w^y are adjacent if and only if $yx^{-1} \in S$. If R is abelian, then it is easily shown that $u^x \mapsto w^{x^{-1}}, w^x \mapsto u^{x^{-1}}$, $\forall x \in R$ is an automorphism of Γ, which leads to the arc-transitivity of Γ. $\square$

Let G be a finite transitive permutation group on a set Ω. The orbits of G on the cartesian product $\Omega \times \Omega$ are the *orbitals* of G, and the diagonal orbital $\{(\alpha, \alpha)^g \mid g \in G\}$ is said to be trivial. For a G-orbital Δ and $\alpha \in \Omega$, the set $\Delta(\alpha) = \{\beta \mid (\alpha, \beta) \in \Delta\}$ is a G_α-orbit on Ω and called a *suborbit* of G at α. The *rank* of G on Ω is the number of G-orbitals, which equals to the number of G_α-orbits on Ω for any given $\alpha \in \Omega$. A G-orbital Δ is called *self-paired* if $(\beta, \alpha) \in \Delta$ for some $(\alpha, \beta) \in \Delta$, while the suborbit $\Delta(\alpha)$ is said to be *self-paired.* For a G-orbital Δ, the *paired orbital* Δ^* is defined as $\{(\beta, \alpha) \mid (\alpha, \beta) \in \Delta\}$. Then a G-orbital Δ is self-paired if and only if $\Delta^* = \Delta$. For a non-trivial G-orbital Δ, the *orbital bipartite graph* $B(G, \Omega, \Delta)$ is the graph on two copies of Ω, say $\Omega{\times}\{1, 2\}$, such that $\{(\alpha, 1), (\beta, 2)\}$ is an edge if and only if $(\alpha, \beta) \in \Delta$. Then $B(G, \Omega, \Delta)$ is G-semisymmetric, where G acts on $\Omega{\times}\{1, 2\}$ as follows:

$$(\alpha, i)^g = (\alpha^g, i),\ g \in G,\ i = 1,\ 2.$$

If Δ is self-paired, then $(\alpha, 1) \leftrightarrow (\alpha, 2)$, $\alpha \in \Omega$ gives an automorphism of $B(G, \Omega, \Delta)$, which yields that $B(G, \Omega, \Delta)$ is G-arc-transitive. The next lemma indicates it is possible that $B(G, \Omega, \Delta)$ is arc-transitive even if Δ is not self-paired.

Lemma 2.2 *Let X be a permutation group on Ω and G is a transitive subgroup of X with index $|X : G| = 2$. Let Δ be a G-orbital. If $\Delta \cup \Delta^*$ is an X-orbital, then $B(G, \Omega, \Delta)$ is arc-transitive.*

Proof. Assume that $\Delta{\cup}\Delta^*$ is an X-orbital. To show $\Gamma := B(G, \Omega, \Delta)$ is arc-transitive, it suffices to find an automorphism of Γ which interchanges two bipartition subsets of Γ. Take $x \in X \setminus G$. It is easily shown that $\Delta^x = \Delta^*$ and $(\Delta^*)^x = \Delta$. Define $\hat{x} : \Omega{\times}\{0, 1\} \to \Omega{\times}\{0, 1\}$; $(\alpha, 0) \mapsto (\alpha^x, 1)$, $(\beta, 1) \mapsto (\beta^x, 0)$. It is easy to check $\hat{x} \in \mathsf{Aut}\Gamma$, and so the lemma follows. $\square$

Moreover, the next lemma is easily shown, see also [14].

Lemma 2.3 *Assume that Γ is a connected G-semisymmetric graph of valency at least 2 with bipartition subsets U and W, and that, for an edge $\{u, w\} \in E\Gamma$, two stabilizers G_u and G_w are conjugate in G. Then there is a bijection $\iota : U \leftrightarrow W$ such that $G_u = G_{\iota(u)}$ and $\{u, \iota(u)\} \notin E\Gamma$ for all $u \in U$. Moreover, $\Delta = \{(u, \iota^{-1}(w)) \mid \{u, w\} \in E\Gamma, u \in U, w \in W\}$ is a G-orbital on U. In particular, $\Gamma \cong B(G, U, \Delta)$, and ι extends to an automorphism of Γ if and only if Δ is self-paired.*

Remark on Lemma 2.3. Let Γ and $G \leq \mathsf{Aut}\Gamma$ be as in Lemma 2.3. Then $\{G_u \mid u \in U\} = \{G_w \mid w \in W\}$, and so $\cap_{u \in U} G_u = \cap_{w \in W} G_w = 1$ as $G \leq \mathsf{Aut}\Gamma$. Thus G is faithful on both parts of Γ. Take $u \in U$ and $w \in W$ with $G_u = G_w$. Then $u^g \leftrightarrow w^g$, $g \in G$ gives a bijection meeting the requirement of Lemma 2.3. Thus one can define l^2 bijections ι, where l is the number of the points in U fixed by a stabilizer G_u. By [7, Theorem 4.2A], $l = |\mathrm{N}_G(G_u) : G_u|$.

Let G be a finite transitive permutation group on Ω. Let $N = \{x_1 = 1, x_2, \cdots, x_n\}$ be a group of order n lying in the center $\mathbf{Z}(G)$ of G. Then N is normal in G, and N is semiregular on Ω, that is, $N_\alpha = 1$ for all $\alpha \in \Omega$. Denote by $\bar{\alpha}$ the N-orbit containing $\alpha \in \Omega$ and by $\bar{\Omega}$ the set of all N-orbits. Then G induces a transitive permutation group $\bar{G}$ on $\bar{\Omega}$. Take a $\bar{G}$-orbital $\bar{\Delta}$ and $(\bar{\alpha}, \bar{\beta}) \in \bar{\Delta}$. Noting that $G_{\bar{\alpha}} = N{\times}G_\alpha$, it follows that $\bar{\Delta}(\bar{\alpha}) = \{(\bar{\beta})^h \mid h \in G_\alpha\}$. Set

$$\Delta_i(\alpha) = \{\beta^{x_i h} \mid h \in G_\alpha\},\ 1 \leq i \leq n.$$

Then all $\Delta_i(\alpha)$ are suborbits of G at α, which are not necessarily distinct. It is easily shown that $N{\times}G_\alpha$ acts transitively on $\Omega_1 := \{\beta^{x_i h} \mid 1 \leq i \leq n,\ h \in G_\alpha\}$. It follows that all G_α-orbits on Ω_1 have the same length divided by $|\bar{\Delta}(\bar{\alpha})|$. For each i, let Δ_i be the G-orbital corresponding to $\Delta_i(\alpha)$.

Lemma 2.4 *Let G, N, $\bar{\Delta}$ and Δ_i be as above.*

(1) All $\Delta_i(\alpha)$ are suborbits of G of the same length divisible by $|\bar{\Delta}(\bar{\alpha})|$.

(2) If $\bar{\Delta}$ is self-paired then, for each i, there is some j such that $\Delta_i(\alpha) = \Delta_j^*(\alpha)$.

(3) $B(G, \Omega, \Delta_i) \cong B(G, \Omega, \Delta_j)$ for $1 \leq i, j \leq n$.

Proof. Part (1) of this lemma follows from the argument above the lemma.

Assume that $\bar{\Delta}$ is self-paired. Then there is some $g \in G$ such that $(\bar{\alpha}, \bar{\beta})^g = (\bar{\beta}, \bar{\alpha})$. Thus, for each i, there are some i' and j' such that $(\alpha, \beta^{x_i})^g = (\beta^{x_{j'}}, \alpha^{x_{i'}}) = (\beta^{x_{i'}^{-1} x_{j'}}, \alpha)^{x_{i'}}$. Setting $x_{i'}^{-1} x_{j'} = x_j$, we have $(\alpha, \beta^{x_i})^g = (\beta^{x_j}, \alpha)^{x_{i'}}$. Then $\Delta_i = \Delta_j^*$.

For each i, define $f_i : \Omega \times \{1, 2\} \to \Omega \times \{1, 2\}$ by $f_i(\delta, 1) = (\delta, 1)$ and $f_i(\delta, 2) = (\delta^{x_i}, 2)$, where $\delta \in \Omega$. It is easily shown that f_i is an isomorphism from $B(G, \Omega, \Delta_1)$ to $B(G, \Omega, \Delta_i)$. Thus part (3) of this lemma follows. □

Let Γ be a G-semisymmetric graph. Suppose that G has a normal subgroup N which acts intransitively on at least one of the bipartition subsets of Γ. Then we define the *quotient graph* Γ_N to have vertices the N-orbits on $V\Gamma$, and two N-orbits B and B' are adjacent in Γ_N if and only if some $v \in B$ and some $v' \in B'$ are adjacent in Γ. It is easy to see that the quotient Γ_N is a regular graph if and only if all N-orbits have the same length. Moreover, if Γ_N is regular then its valency is a divisor of that of Γ. The graph Γ is called a *normal cover* of Γ_N (with respect to G and N) if Γ_N and Γ have the same valency, which yields that N is the kernel of G acting the N-orbits (vertices of Γ_N). Thus, if Γ is a normal cover of Γ_N then the quotient group G/N can be identified with a subgroup of $\mathsf{Aut}\Gamma_N$, and so Γ_N is G/N-semisymmetric.

Corollary 2.1 *Let Γ and $G \leq \mathsf{Aut}\Gamma$ be as in Lemma 2.3. Let $N \leq \mathbf{Z}(G)$. Then N is intransitive and semiregularly on both U and W. Assume further that $|N|$ is odd and that Γ_N is the orbital bipartite graph of some self-paired orbital of $\bar{G}$, where $\bar{G}$ is the subgroup of $\mathsf{Aut}\Gamma_N$ induced by G. Then Γ is arc-transitive.*

Proof. Recall that G is faithful on both U and W, see the remark after Lemma 2.3. Since $N \leq \mathbf{Z}(G)$, every subgroup of N is normal in G, so $N_v \leq G_v^g = G_{v^g}$ for $v \in V\Gamma$ and $g \in G$. It follows that $N_v = 1$, so N is semiregular on both U and W. Suppose that N is transitive on one of U and W, say U. Then $G = NG_u$ for $u \in U$, and so G_u is normal in G as $N \leq \mathbf{Z}(G)$. It follows that G_u fixes every vertex in U, so $G_u = 1$, which contradicts the transitivity of G_u on $\Gamma(u)$.

By Lemma 2.3, there is bijection $\iota : U \leftrightarrow W$ such that, for $u \in U$, the subset $\iota^{-1}(\Gamma(u))$ is a suborbit of G at u. By the remark after Lemma 2.3, we may choose ι such that it maps each N-orbit on U to some N-orbit on W. Thus ι induces a bijection $\bar{\iota}$ on $V\Gamma_N$ interchanging two bipartition subsets U_N and W_N of Γ_N, where U_N and W_N denote respectively the sets of N-orbits on U and W. Moreover, it is easily shown that $\bar{G}_{\bar{v}} = \bar{G}_{\bar{\iota}(\bar{v})}$ for any N-orbit $\bar{v}$, and that $\iota^{-1}(\Gamma(u)) = \{u'^h \mid h \in G_u\}$ for $u' \in U$ such that $\bar{u}' \in \bar{\iota}^{-1}(\Gamma_N(\bar{u}))$.

Assume Γ_N is the orbital bipartite graph of some self-paired orbital of $\bar{G}$. Then, again by Lemma 2.3, $\bar{\iota} \in \mathsf{Aut}\Gamma_N$ and $\bar{\iota}^{-1}(\Gamma_N(\bar{u}))$ is a self-paired suborbit of $\bar{G}$ at $\bar{u}$. If $|N|$ is odd then, by Lemma 2.4, Γ is isomorphic to the orbital bipartite graph of some self-paired orbital of G on U, and hence Γ is arc-transitive. $\Box$

Recall that, for a group G acts transitively on a set Ω, a block B is a non-empty subset of Ω such that $B = B^g$ or $B \cap B^g = \emptyset$ for every $g \in G$.

Lemma 2.5 *Let Γ be a connected graph, and let $G \leq \mathsf{Aut}\Gamma$ such that G is locally primitive but not vertex-transitive. Assume that U and W are G-orbits on $V\Gamma$ and that B is a block of G on W. Then either $B = W$, or $|\Gamma(u) \cap B| \leq 1$ for each $u \in U$.*

Proof. Note that for each $u \in U$ either $\Gamma(u) \cap B = \emptyset$ or $\Gamma(u) \cap B$ is a block of G_u on $\Gamma(u)$. Since G_u acts primitively on $\Gamma(u)$, we know that either $|\Gamma(u) \cap B| \leq 1$ or $\Gamma(u) \subseteq B$. Suppose that $\Gamma(u) \subseteq B$ for some $u \in U$. Take $w \in B$ and $v \in \Gamma(w)$. Since G is edge-transitive, there is $g \in G$ with $v^g = u$ and $w^g \in \Gamma(u) \subseteq B$. Then $w \in B^{g^{-1}} \cap B$, and so $B = B^{g^{-1}}$. Thus $\Gamma(v) = (\Gamma(u))^{g^{-1}} \subseteq B^{g^{-1}} = B$. It follows that Γ has a connected component with vertex set $(\cup_{w \in B}\Gamma(w)) \cup B$. This yields $B = W$. $\Box$

Lemma 2.6 *Let Γ and G be as in Lemma 2.5. Let U and W be the G-orbits on $V\Gamma$. Suppose that G has a normal subgroup N which acts transitively on U. Then*

(1) Γ_N is a $|\Gamma(u)|$-star, where $u \in U$; or

(2) Γ is N-edge-transitive; or

(3) N is regular on both U and W.

Proof. If N is intransitive on W, then part (1) follows from [14, Lemma 5.5]. Thus we assume that N is transitive on W. Take $u \in U$. If N_u is transitive on $\Gamma(u)$ then Γ is N-edge-transitive, and so (2) holds. Suppose that N_u is not transitive on $\Gamma(u)$. Since N_u is normal in G_u and G is locally primitive, N_u fixes $\Gamma(u)$ point-wise. Thus $N_w \geq N_u$ for each $w \in \Gamma(u)$. If N_w is transitive on $\Gamma(w)$ then Γ is N-edge-transitive, and so (2) holds. Thus we may suppose further that $N_w \leq N_{u'}$ for each $u' \in \Gamma(w)$. By the connectedness of Γ, we conclude that $N_u = N_w = 1$. Then (3) follows. $\Box$

Recall that a quasiprimitive group is a permutation group with all minimal normal subgroups transitive. By [14, Theorem 1.1 and Lemma 5.1], the next lemma holds.

Lemma 2.7 *Let Γ and G be as in Lemma 2.5. Suppose that N is a normal subgroup of G which is intransitive on both bipartition subsets of Γ. Then Γ is a normal cover of Γ_N and Γ_N is G/N-locally primitive. If further N is maximal among the normal subgroups of G which are intransitive on both bipartition subsets of Γ, then either Γ_N is a complete bipartite graph, or G/N acts faithfully on both parts and is quasiprimitive on at least one the bipartition subsets of Γ_N.*

For a finite group G, denote by $\mathsf{soc}(G)$ the subgroup generated by all minimal normal subgroups of G, which is called the *socle* of G. The next result describes the basic structural information for quasiprimitive permutation groups, refer to [30].

Lemma 2.8 *Let G be a finite quasiprimitive permutation group on Ω. Then G has at most two minimal normal subgroups, and one of the following statements holds.*

(1) $|\Omega| = p^d$, $\mathsf{soc}(G) \cong \mathbb{Z}_p^d$ *and* $\mathsf{soc}(G)$ *is the unique minimal normal subgroup of* G, *where* $d \geq 1$ *and* p *is a prime; in this case,* G *is primitive on* Ω*;*

(2) $\mathsf{soc}(G) = T^l$ *for* $l \geq 1$ *and a nonabelian simple group* T, *and either* $\mathsf{soc}(G)$ *is the unique minimal normal subgroup of* G, *or* $\mathsf{soc}(G) = M \times N$ *for two minimal normal subgroups* M *and* N *of* G *with* $|M| = |N| = |\Omega|$.

3 The quasiprimitive case

Let Γ be a G-locally primitive regular graph of order $18p$, where $G \leq \mathsf{Aut}\Gamma$ and p is a prime. Assume that G is intransitive on $V\Gamma$. Then Γ is a bipartite graph with two bipartition subsets being G-orbits, say U and W.

Assume next that G acts faithfully on both U and W, and that G is quasiprimitive on one of U and W. If G acts primitively on one of U and W then, by [18], Γ is either arc-transitive or isomorphic to one of the Gray graph and the Tutte 12-cage. Thus we assume in the following that neither G^U nor G^W is a primitive permutation group. Then, by Lemma 2.8, $N := \mathsf{soc}(G)$ is the direct product of some isomorphic non-abelian simple groups. In particular, G is insoluble, and so Γ is not a cycle.

Without loss of generality, we assume that G is quasiprimitive on U. Recall that G^U is not primitive. Take a maximal block B ($\neq U$) of G on U. Then $|B|$ is a proper divisor of $|U| = 9p$ and $|G_B : G_u| = |N_B : N_u| = |B|$ for each $u \in B$. Set $\mathcal{B} = \{B^g \mid g \in G\}$. Then $|\mathcal{B}| = \frac{9p}{|B|}$, and G acts primitively on $\mathcal{B}$. Since G is quasiprimitive on U, we know that G acts faithfully on $\mathcal{B}$. Thus we may view G as a primitive permutation group (on $\mathcal{B}$) of degree $\frac{9p}{|B|}$.

Lemma 3.1 $|B| = 3$ *or* 9.

Proof. It is easy to see that $|B| = 3$, 9 or p. Suppose that $|B| = p$. Then $|\mathcal{B}| = 9$ and, by [7, Appendix B], $N = \mathsf{soc}(G) = \mathrm{A}_9$ or $\mathrm{PSL}(2, 8)$. If $N = \mathrm{A}_9$ then $N_B \cong \mathrm{A}_8$ and $p \leq 7$; however A_8 has no subgroups of index p, a contradiction. Thus $N = \mathrm{PSL}(2, 8)$, $N_B \cong \mathbb{Z}_2^3{:}\mathbb{Z}_7$ and $p = 7$, and so $N_u \cong \mathbb{Z}_2^3$ and $|U| = 63$, where $u \in B$. Since Γ is G-locally primitive, G_u induces a primitive permutation group $G_u^{\Gamma(u)}$. If $G = N$ then $G_u^{\Gamma(u)} \cong \mathbb{Z}_2$, yielding that Γ is a cycle, a contradiction. It follows that $G = \mathrm{P\Sigma L}(2, 8) \cong \mathrm{PSL}(2, 8){:}\mathbb{Z}_3$ and $|G_v| = 24$, where v is an arbitrary vertex of Γ. Checking the subgroups of $\mathrm{PSL}(2, 8)$ in the Atlas [6], we know that N has no proper subgroups of index dividing 21. It implies that N is transitive on W, and so G is also quasiprimitive on W. By the information given for $\mathrm{P\Sigma L}(2, 8)$ in [6], $G_v \cong \mathbb{Z}_2^3{:}\mathbb{Z}_3 \cong \mathbb{Z}_2 \times \mathrm{A}_4$ for each $v \in V\Gamma$. Then either $G_v^{\Gamma(v)} \cong \mathbb{Z}_3$ and Γ is cubic, or $G_v^{\Gamma(v)} \cong \mathrm{A}_4$ and Γ has valency 4. Take $\{u, w\} \in E\Gamma$. Then $G_{uw} \cong \mathbb{Z}_2^3$ or $\mathbb{Z}_6$. It follows that G_u and G_w have the same center, which contradicts Lemma 2.1. $\square$

Therefore, G is a primitive permutation group (on $\mathcal{B}$) of degree p or $3p$. For further argument, we list in Tables 1 and 2 the insoluble primitive groups of degree p and of degree $3p$, respectively. Noting that N_B has a subgroup of index $|B| = 9$ or 3, it is easy to check that $N = \mathrm{A}_6$ or $\mathrm{PSL}(n, q)$. Suppose that $N = \mathrm{A}_6$. Then $|B| = 3$ and $p = 5$. It follows that G_u is a 2-group. Since Γ is G-locally primitive, $G_u^{\Gamma(u)} \cong \mathbb{Z}_2$. Then Γ is a cycle, a contradiction. Thus the next lemma follows.

Degree p	11	11	23	p	$\frac{q^n-1}{q-1}$
Socle	$\mathrm{PSL}(2,11)$	M_{11}	M_{23}	A_p	$\mathrm{PSL}(n,q)$
Stabilizer	A_5	M_{10}	M_{22}	A_{p-1}	
Action					1- or $(n-1)$-subspaces
Remark					prime $n \geq 3$ or $(n,q)=(2,2^{2^s})$

Table 1. Insoluble transitive groups of prime degree (refer to [2, Table 7.4]).

Lemma 3.2 *Either* $|B| = 9$ *and* $N = \mathrm{PSL}(n,q)$ *with* n *prime, or* $|B| = 3$ *and* $N = \mathrm{PSL}(3,q)$ *with* $q \equiv 1 \,(\mathsf{mod}\ 3)$.

Degree $3p$	Socle	Action	Remark
6	A_5	cosets of D_{10}	
15	A_6	2-subsets or partitions	
21	A_7	2-subsets	
21	$\mathrm{PSL}(3,2)$	$(1,2)$-flags	
57	$\mathrm{PSL}(2,19)$	cosets of A_5	two actions
15	A_7	cosets of $\mathrm{PSL}(2,7)$	two actions
$3p$	A_{3p}		
15	$\mathrm{PSL}(4,2)$	1- or 3-subspaces	
2^f+1	$\mathrm{PSL}(2,2^f)$	1-subspaces	odd prime f
$\frac{q^3-1}{q-1}$	$\mathrm{PSL}(3,q)$	1- or 2-subspaces	$q \equiv 1 \,(\mathsf{mod}\ 3)$

Table 2. Insoluble primitive groups of degree $3p$ (refer to [16]).

Let $\mathbb{F}_q$ be the Galois field of order q, and let $\mathbb{F}_q^n$ be the n-dimensional linear space of row vectors over $\mathbb{F}_q$. Denote by $\mathcal{P}$ and $\mathcal{H}$, respectively, the sets of 1-subspaces and $(n-1)$-subspaces of $\mathbb{F}_q^n$. Then the action of $N = \mathrm{SL}(n,q)/\mathbf{Z}(\mathrm{SL}(n,q))$ on $\mathcal{B}$ is equivalent to one of the actions of N on $\mathcal{P}$ and on $\mathcal{H}$ induced by

$$(x_1, x_2, \cdots, x_n)\mathbf{A} = (\Sigma_{i=1}^n a_{i1}x_i, \Sigma_{i=1}^n a_{i2}x_i, \cdots, \Sigma_{i=1}^n a_{in}x_i),$$

where $\mathbf{A} = (a_{ij})_{n\times n} \in \mathrm{SL}(n,q)$. Let σ be the inverse-transpose automorphism of $\mathrm{SL}(n,q)$, that is,

$$\sigma : \mathrm{SL}(n,q) \to \mathrm{SL}(n,q), \mathbf{a} \mapsto (\mathbf{a}')^{-1}.$$

Then σ gives an automorphism of N of order 2. Define

$$\iota : \mathcal{P} \to \mathcal{H}, \langle (x_1, x_2, \cdots, x_n)\rangle \mapsto \{(y_1, y_2, \cdots, y_n) \mid \Sigma_{i=1}^n x_i y_i = 0\}.$$

Then

$$(\iota(\langle \mathbf{v}\rangle))^{\mathbf{A}} = \iota(\langle \mathbf{v}\mathbf{A}\rangle),\ \forall \mathbf{A} \in \mathrm{SL}(n,q),\ \langle \mathbf{v}\rangle \in \mathcal{P}.$$

For $1 \leq i \leq n$, let $\mathbf{e}_i$ be the vector with the ith entry 1 and other entries 0. Then

$$(\mathrm{SL}(n,q))_{\langle \mathbf{e}_1\rangle} = Q{:}H \text{ and } (\mathrm{SL}(n,q))_{\langle \mathbf{e}_i | 2\leq i\leq n\rangle} = Q^\sigma{:}H,$$

where

$$Q = \left\{ \left(\begin{array}{cc} 1 & \mathbf{0} \\ \mathbf{b}' & \mathbf{I}_{n-1} \end{array} \right) \middle| \ \mathbf{b} \in \mathbb{F}_q^{n-1} \right\},$$

$$H = \left\{ \left(\begin{array}{cc} a & \mathbf{0} \\ \mathbf{0}' & \mathbf{A} \end{array} \right) \middle| \ \mathbf{A} \in \mathrm{GL}(n-1,q), a^{-1} = \det(\mathbf{A}) \right\}.$$

For a subgroup X of $\mathrm{SL}(n,q)$, we denote $\overline{X}$ to be the image of X in N, that is, $\overline{X} = X/\mathbf{Z}(\mathrm{SL}(n,q))$. Then the following lemma holds.

Lemma 3.3 *If $B \in \mathcal{B}$ then N_B is conjugate in N to one of $\overline{Q}{:}\overline{H}$ and $\overline{Q}^{\sigma}{:}\overline{H}$.*

The following simple fact may be shown by simple calculations.

Lemma 3.4 *Set $\mathbb{F}_q \setminus \{0\} = \langle \eta \rangle$ and*

$$L = \left\{ \left(\begin{array}{cc} 1 & \mathbf{0} \\ \mathbf{0}' & \mathbf{A} \end{array} \right) \middle| \ \mathbf{A} \in \mathrm{SL}(n-1,q) \right\}.$$

Then $\overline{Q}{:}\overline{L}$ acts transitively on $\mathcal{P} \setminus \langle \mathbf{e}_1 \rangle$, and has two orbits on $\mathcal{H}$ with length $\frac{q^{n-1}-1}{q-1}$ and q^{n-1}, respectively. Moreover, for each divisor m of $q-1$, $Q{:}H$ has a unique subgroup containing $Q{:}L$ and having index m, which is

$$\left\{ \left(\begin{array}{cc} a & \mathbf{0} \\ \mathbf{b}' & \mathbf{A} \end{array} \right) \middle| \ \mathbf{b} \in \mathbb{F}_q^{n-1}, \mathbf{A} \in \mathrm{GL}(n-1,q), a^{-1} = \det(\mathbf{A}) \in \langle \eta^m \rangle \right\}.$$

Lemma 3.5 *Write $q = r^f$ for a prime r and an integer $f \geq 1$. Assume that $|B| = 9$ for $B \in \mathcal{B}$. Then the following statements hold:*

(1) $(n,q) \neq (2,2)$, $(2,3)$, $(3,2)$, $(3,3)$;

(2) n is an odd prime with $q \not\equiv 1 \pmod{n}$;

(3) n is the smallest prime divisor of nf.

Proof. By Lemma 3.2, $N = \mathrm{PSL}(n,q)$ for a prime n. Since 9 is a divisor of $|N|$ and N is simple, $(n,q) \neq (2,2)$, $(2,3)$, $(3,2)$.

Suppose that $N = \mathrm{PSL}(3,3)$. Then $p = 13$, $G = N$, $|G_B| = 2^4 \cdot 3^3$ and $|G_u| = 48$. Take $w \in \Gamma(u)$. Since Γ is regular, $|G_u| = 48 = |G_w|$. Checking the subgroups of $\mathrm{SL}(3,3)$ (refer to [6]), we have $G_u \cong G_w \cong 2\mathrm{S}_4 \cong \mathrm{GL}(2,3)$. Since Γ is G-locally primitive, $G_u^{\Gamma(u)} \cong \mathrm{S}_4 \cong G_w^{\Gamma(w)}$ and Γ has valency 4. Thus $G_{uw} \cong \mathrm{D}_{12}$. It follows that G_u and G_w have the same center isomorphic to $\mathbb{Z}_2$, which contradicts Lemma 2.1. Thus part (1) follows.

Suppose that $n = 2$. Then, since $p = \frac{r^{nf}-1}{r^f-1}$ is a prime, $r = 2$ and $f = 2^s$ for some integer $s \geq 1$. Thus $N_B \cong \mathbb{Z}_2^{2^s}{:}\mathbb{Z}_{2^{2^s}-1}$, and hence $N_u \cong \mathbb{Z}_2^{2^s}{:}\mathbb{Z}_{\frac{2^{2^s}-1}{9}}$. But $2^{2^s}-1$ is not divisible by 9, a contradiction. This implies that n is an odd prime. If $q \equiv 1 \pmod{n}$ then $p = \Sigma_{i=0}^{n-1} q^i \equiv 0 \pmod{n}$, a contradiction. Then part (2) follows.

If $nf = 6$ and $r = 2$ then $p = \frac{q^n-1}{q-1} = 21$ or 63, a contradiction. Thus, by Zsigmondy's Theorem (refer to [20, p. 508]), there is a prime which divides $r^{nf}-1$ but not divides r^i-1 for all $1 \leq i \leq nf-1$. Clearly, such a prime is p. Suppose that f has a prime divisor s such that $s < n$. Then q^n-1 has a divisor $r^{\frac{nf}{s}}-1$. By Zsigmondy's Theorem, either $(r, \frac{nf}{s}) = (2,6)$, or $r^{\frac{nf}{s}}-1$ has a prime divisor which does not divide r^f-1. The latter case yields that $\frac{q^n-1}{q-1}$ has two (distinct) prime divisors, a contradiction. Thus $(r, \frac{nf}{s}) = (2, 6)$, yielding that $n = 3$ and $f = 4$. Then $p = \frac{q^n-1}{q-1} = \frac{2^{12}-1}{2^4-1} = 273$, a contradiction. Then part (3) follows. $\Box$

Lemma 3.6 *Let $B \in \mathcal{B}$. If $(n, q) = (3, 8)$ then $|B| = 9$ and Γ is arc-transitive and of valency* 8 *or* 64.

Proof. Assume that $(n, q) = (3, 8)$. Then $N \cong \mathrm{SL}(3, 8)$, $p = 73$ and $|G : N| = 1$ or 3. By Lemma 3.2, $|B| = 9$. Without loss of generality, we assume that $N = \mathrm{SL}(3, 8)$ and choose B such that $N_B = P{:}H$, where $P \cong \mathbb{Z}_2^6$ and H is defined as above Lemma 3.3.

Since N_B is transitive on B, it is easily shown that P acts trivially on B, and so H acts transitively on B. Then $|H : H_u| = 9$. Note that $H \cong \mathrm{GL}(2, 8) \cong \mathbb{Z}_7{\times}\mathrm{PSL}(2, 8)$. Checking the subgroups of $\mathrm{PSL}(2, 8)$, we conclude that the action of H on B is equivalent to the action of H on the 1-subspaces of $\mathbb{F}_8^2$. Then, without loss of generality, we may assume that H_u is conjugate to

$$\left\{ \left. \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & b & a_3 \end{pmatrix} \right| a_1, a_2, a_3, b \in \mathbb{F}_8, a_1a_2a_3 = 1 \right\}.$$

Recall that a $(1, 2)$-flag of $\mathbb{F}_8^3$ is a pair $\{V_1, V_2\}$ of a 1-subspace and a 2-subspace with the 1-subspace contained in the 2-subspace. Since $P \le N_u$, we have $N_u = N_u \cap (PH) = PH_u \cong \mathbb{Z}_2^6{:}(\mathbb{Z}_2^3{:}\mathbb{Z}_7^2)$. It is easily shown that N_u is the stabilizer of some $(1, 2)$-flag $\{V_1, V_2\}$ in N. It follows that the action of N on U is equivalent to the action of N on the set $\mathcal{F}$ of $(1, 2)$-flags of $\mathbb{F}_8^3$.

Now we show that the actions of N on U and W are equivalent. Note that $|G : N| = 1$ or 3. Thus, since W is a G-orbit, either N is transitive on W or N has 3 orbits on W. Checking the subgroups of $\mathrm{SL}(3, 8)$, we know that N has no subgroups of index 219. It follows that N is transitive on W. Note that $N = \mathrm{SL}(3, 8)$ has no subgroups of indices 3, 9 and 219. It follows that a maximal block of N on W has size 9. Then a similar argument as above implies the action of N on W is also equivalent to that on $\mathcal{F}$. Moreover, Γ is N-edge-transitive by Lemma 2.6.

Identifying U with $\mathcal{F}$, by Lemma 2.3, $\Gamma \cong B(N, \mathcal{F}, \Delta)$, where Δ is an N-orbital on $\mathcal{F}$. Without loss of generality, choose u to be the flag $\{\langle \mathbf{e}_3 \rangle, \langle \mathbf{e}_2, \mathbf{e}_3 \rangle\}$. Calculation shows that $\Delta(u)$ is one of the following 5 suborbits:

(i) $\{\{\langle \mathbf{e}_2 + a\mathbf{e}_3 \rangle, \langle \mathbf{e}_2, \mathbf{e}_3 \rangle\} \mid a \in \mathbb{F}_8\}$ and $\{\{\langle \mathbf{e}_3 \rangle, \langle \mathbf{e}_3, \mathbf{e}_1 + a\mathbf{e}_2 \rangle\} \mid a \in \mathbb{F}_8\}$, which are self-paired and of length 2^3;

(ii) $\{\{\langle \mathbf{e}_2 + a\mathbf{e}_3 \rangle, \langle \mathbf{e}_1 + b\mathbf{e}_2, \mathbf{e}_2 + a\mathbf{e}_3 \rangle\} \mid a, b \in \mathbb{F}_8\}$ and $\{\{\langle \mathbf{e}_1 + a\mathbf{e}_2 + b\mathbf{e}_3 \rangle, \langle \mathbf{e}_1 + a\mathbf{e}_2, \mathbf{e}_3 \rangle\} \mid a, b \in \mathbb{F}_8\}$, which are paired to each other and of length 2^6;

(iii) $\{\{\langle \mathbf{e}_1 + a\mathbf{e}_2 + b\mathbf{e}_3 \rangle, \langle \mathbf{e}_1 + a\mathbf{e}_2 + b\mathbf{e}_3, \mathbf{e}_2 + c\mathbf{e}_3 \rangle\} \mid a, b, c \in \mathbb{F}_8\}$, which is self-paired and of length 2^9.

Suppose that $\Delta(u)$ is the suborbit in (iii). Then Γ has valency 2^9. Recall that $|G : N| = 1$ or 3, and $N_u = PH_u \cong \mathbb{Z}_2^6{:}(\mathbb{Z}_2^3{:}\mathbb{Z}_7^2)$. It follows that G_u/N_u is cyclic, and hence G_u is soluble. Since Γ is G-locally primitive, $G_u^{\Gamma(u)}$ is a soluble primitive permutation group of degree 2^9. In particular, $\mathsf{soc}(G_u^{\Gamma(u)}) \cong \mathbb{Z}_2^9$ and $\mathsf{soc}(G_u^{\Gamma(u)})$ is the unique minimal normal subgroup of $G_u^{\Gamma(u)}$. Thus $N_u^{\Gamma(u)} \ge \mathsf{soc}(G_u^{\Gamma(u)})$ as N_u induces a normal transitive subgroup of $G^{\Gamma(u)}$. However, the unique Sylow 2-subgroup of N_u is non-abelian and has order 2^9, a contradiction.

If $\Delta(u)$ is described in (i) then Γ has valency 8 and, by Lemma 2.3, Γ is arc-transitive.

Assume that $\Delta(u)$ is one of the suborbits in (ii). Then Γ has valency 64. Let σ is the inverse-transpose automorphism of $N = \mathrm{SL}(3,8)$. Then $\mathcal{F}$ is σ-invariant. Consider that action $N{:}\langle\sigma\rangle$ on $\mathcal{F}$ and take $\mathbf{a} \in \mathrm{SL}(3,8)$ with

$$\mathbf{a} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $(N\langle\sigma\rangle)_u = N_u{:}\langle\sigma\mathbf{a}\rangle$, which interchanges the two suborbits in (ii). It follows from Lemma 2.2 that Γ is arc-transitive. $\square$

Lemma 3.7 *Assume that* $(n,q) \neq (3,8)$. *Then there is* $u \in U$ *with* $N_u \geq \overline{Q}{:}\overline{L}$ *or* $\overline{Q}^{\sigma}{:}\overline{L}$, *where* σ *is the inverse-transpose automorphism of* $\mathrm{SL}(n,q)$, $\overline{Q}$ *and* $\overline{L}$ *are described as in Lemmas* 3.3 *and* 3.4. *In particular,* $q \equiv 1(\mathsf{mod}\ |B|)$.

Proof. Recall that the action of $N = \mathrm{SL}(n,q)/\mathbf{Z}(\mathrm{SL}(n,q))$ on $\mathcal{B}$ is equivalent to one of the actions of N on $\mathcal{P}$ and on $\mathcal{H}$. Without loss of generality, we may choose $B \in \mathcal{B}$ such that $N_B = R{:}\overline{H}$, where $R = \overline{Q}$ or $\overline{Q}^{\sigma}$, and $\overline{H}$ is described as in Lemma 3.3. Set $q = r^f$ for some prime r and integer $f \geq 1$. Then R is a nontrivial r-group.

Take $u \in B$. Then $|N_B : N_u| = |B| = 3$ or 9. Suppose that $R \not\leq N_u$. Noting that RN_u is a subgroup of N_B as R is normal in N_B, it follows that $|R : (R \cap N_u)| = |(RN_u) : N_u| = 3$ or 9. In particular, R is a 3-group, and hence $|B| = 9$ by Lemma 3.2. Then, by Lemma 3.5, n and $q-1$ are coprime, and so $\mathbf{Z}(\mathrm{SL}(n,q)) = 1$. Thus $N \cong \mathrm{SL}(n,q)$ and $R \cong Q \cong \mathbb{Z}_3^{(n-1)f}$. Assume that $|(RN_u) : N_u| = 9$. Then $N_B = RN_u$. It implies that $R \cap N_u$ is normal in N_B. Then $N_u > R \cap N_u = \langle (R \cap N_u)^x \mid x \in N_B \rangle = R$, yielding $R \cap N_u = 1$. It follows that $R \cong \mathbb{Z}_3^2$. By Lemma 3.5, we conclude that $n = 3$ and $f = 1$, that is, $(n,q) = (3,3)$, a contradiction. Thus $|N_B : (RN_u)| = 3$. Noting that $\mathrm{GL}(n-1,3^f) \cong H \cong \overline{H} \cong N_B/R$, it follows that $\mathrm{GL}(n-1,3^f)$ has a subgroup of index 3. Note that $\mathrm{GL}(n-1,3^f) = \mathbb{Z}_{3^f-1}.(\mathrm{PSL}(n-1,3^f).\mathbb{Z}_d$, where d is the largest common divisor of $n-1$ and 3^f-1. It implies that $\mathrm{PSL}(n-1,3^f)$ has a subgroup of index 3. Then $n = 3$ and $f = 1$, a contradiction. Therefore, R is contained in N_u.

Since $R{:}\overline{L}$ is normal in N_B, we know that $\overline{L}N_u = (R{:}\overline{L})N_u$ is a subgroup of N_B. Suppose that $R{:}\overline{L} \not\leq N_u$. Then $|\overline{L} : (\overline{L} \cap N_u)| = |(\overline{L}N_u) : N_u| = 3$ or 9. Let Z be the center of $\overline{L}$. Then $\overline{L}/Z \cong \mathrm{PSL}(n-1,q)$ and $|\overline{L}/Z : (\overline{L} \cap N_u)Z/Z|$ divides 9. By Lemma 3.2 and 3.5, $n \geq 3$ and $(n,q) \neq (3,2), (3,3)$. Thus $\overline{L}/Z$ is simple, and hence it has no subgroups of order 3. Suppose that $|\overline{L}/Z : (\overline{L} \cap N_u)Z/Z| = 9$. Then $\overline{L}/Z$ has a primitive permutation representation of degree 9. By [7, Appendix B], we conclude that $\overline{L}/Z \cong \mathrm{PSL}(2,8)$. Then $(n,q) = (3,8)$, a contradiction. It follows that $|\overline{L}/Z : (\overline{L} \cap N_u)Z/Z| = 1$, that is, $\overline{L} = (\overline{L} \cap N_u)Z$. Consider the commutator subgroups of L and $\overline{L}$. By [19, Chapter II, 6.10], $L' = L$, hence $\overline{L} = \overline{L}' = (\overline{L} \cap N_u)' \leq \overline{L} \cap N_u \neq N_u$, a contradiction. Therefore, the first part of this lemma follows.

Let X and Y be the pre-images of N_B and N_u in $\mathrm{SL}(n,q)$. Then $|X : Y| = |N_B : N_u| = |B|$. Moreover $X = Q{:}H$ or $Q^{\sigma}{:}H$ and $Y \geq Q{:}L$ or $Q^{\sigma}{:}L$, respectively. It follows that $|B|$ is divisor of $|H : L| = q-1$. Then $q \equiv 1\,(\mathsf{mod}\ |B|)$. $\square$

Theorem 3.1 *Γ is an arc-transitive graph, and one of the following holds.*

(1) $N = \mathrm{PSL}(3,8)$, $p = 73$ *and* Γ *has valency* 8 *or* 64*;*

(2) $N = \mathrm{PSL}(n,q)$, $p = \frac{q^n-1}{q-1}$ *and* Γ *has valency* q^{n-1}, *where* $q \equiv 1 \,(\mathsf{mod}\ 9)$, $n \geq 5$ *and* (n,q) *satisfies Lemma* 3.5*;*

(3) $N = \mathrm{PSL}(3,q)$, $3p = q^2+q+1$ *and* Γ *has valency* q^2, *where* $q \equiv 1 \,(\mathsf{mod}\ 3)$.

Proof. By Lemmas 3.2 and 3.5, $N = \mathsf{soc}(G) = \mathrm{PSL}(n,q)$ for some odd prime n. If $(n,q) = (3,8)$ then part (1) of the theorem follows from Lemma 3.6. Thus we assume that $(n,q) \neq (3,8)$ in the following. Write $q = r^f$ for a prime r and an integer $f \geq 1$.

Case 1. Assume that $|B| = 9$. Then $|\mathcal{B}| = p = \frac{q^n-1}{q-1}$ is a prime. By Lemma 3.7, $q \equiv 1 \,(\mathsf{mod}\ 3)$, and so $n < p = \Sigma_{i=0}^{n-1} q^i \equiv n \,(\mathsf{mod}\ 3)$. It follows that $n \neq 3$. By Lemmas 3.5, nf has no prime divisors less that 5. Note that $|G : N|$ divides nf and G is transitive on W. It follows that the number of N-orbits on W is a divisor of nf. It implies that N is transitive on W, and hence G is quasiprimitive on W.

Recall that G is faithful and imprimitive on W. Take a maximal block C of G on W, and set $\mathcal{C} = \{C^g \mid g \in G\}$. Then G acts primitively on $\mathcal{C}$.

Since $n \geq 5$, checking Table 2, we conclude that G has no primitive permutation representation of degree $3p$. Then $|C| \neq 3$. In addition, G has no subgroups of index 9, and so $|C| \neq p$. It follows that $|C| = 9$ and $|\mathcal{C}| = p$. Then the argument for the actions of N on $\mathcal{B}$ and on U is available for the actions on $\mathcal{C}$ and on W. This allows us to view $\mathcal{B}$ as a copy of $\mathcal{P}$ and $\mathcal{C}$ a copy of $\mathcal{P}$ or $\mathcal{H}$.

Choose $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $N_B = \overline{Q}{:}\overline{H}$ and $N_C = N_B$ or N_B^σ. Then, by Lemmas 3.4 and 3.7, $\overline{Q}{:}\overline{L} \leq N_u = X/\mathbf{Z}(\mathrm{SL}(n,q))$ and $N_w = N_u$ or N_u^σ, where $u \in B$, $w \in C$ and X is a subgroup of $\mathrm{SL}(n,q)$ consists matrices of the following form:

$$\left(\begin{array}{cc} a & \mathbf{0} \\ \mathbf{b}', & \mathbf{A} \end{array}\right),\ \mathbf{b} \in \mathbb{F}_q^{n-1}, \mathbf{A} \in \mathrm{GL}(n-1,q), a^{-1} = \det(\mathbf{A}) \in \langle \eta^9 \rangle.$$

Note Γ is G-locally primitive and N is not regular on both U and W. By Lemma 2.6, Γ is N-edge-transitive. Then $\Gamma(u)$ is an N_u-orbit on W. Thus, for an N_u-orbit $\mathcal{C}'$ on $\mathcal{C}$, either $\Gamma(u) = \cup_{C' \in \mathcal{C}'}(\Gamma(u) \cap C')$, or $\Gamma(u) \cap C' = \emptyset$ for each $C' \in \mathcal{C}'$.

Suppose that $N_C = N_B$. Then both B and C are corresponding to $\langle \mathbf{e}_1 \rangle$. By Lemma 3.4, for each $u \in B$, the stabilizer N_u is transitive on $\mathcal{C} \setminus \{C\}$. Thus either $\Gamma(u) \subseteq C$ or $\Gamma(u) = \cup_{C' \in \mathcal{C} \setminus \{C\}} \Gamma(u) \cap C'$. Note that N_u fixes C point-wise as $N_u = N_w$ is normal in $N_B = N_C$, where $w \in C$. Then $\Gamma(u) = \cup_{C' \in \mathcal{C} \setminus \{C\}} \Gamma(u) \cap C'$. Choose $C' \in \mathcal{C}$ corresponding to $\langle \mathbf{e}_2 \rangle$, and take $w' \in C'$. Let Y_1 and Y_2 be the pre-images of $N_u \cap N_{C'}$ and $N_u \cap N_{w'}$, respectively. Then

$$Y_1 = \left\{ \left(\begin{array}{ccc} a & 0 & \mathbf{0} \\ 0 & b & \mathbf{0} \\ \mathbf{b}_1' & \mathbf{b}_2' & \mathbf{a}_1 \end{array} \right) \middle| \ \mathbf{a}_1 \in \mathrm{GL}(n-2,q), a^{-1} = b\det(\mathbf{a}_1) \in \langle \eta^9 \rangle \right\},$$

$$Y_2 = \left\{ \left(\begin{array}{ccc} a & 0 & \mathbf{0} \\ 0 & b & \mathbf{0} \\ \mathbf{b}_1' & \mathbf{b}_2' & \mathbf{a}_1 \end{array} \right) \middle| \ \mathbf{a}_1 \in \mathrm{GL}(n-2,q), ab\det(\mathbf{a}_1) = 1, a, b \in \langle \eta^9 \rangle \right\}.$$

It follows that $|(N_u \cap N_{C'}) : (N_u \cap N_{w'})| = |Y_1 : Y_2| = |9| = |C'|$, and so $N_u \cap N_{C'}$ is transitive on C'. Then $C' \subset \Gamma(u)$, which contradicts Lemma 2.5.

Now let $N_C = N_B^\sigma$. Then B and C are corresponding to $\langle \mathbf{e}_1 \rangle$ and $\langle \mathbf{e}_i \mid 2 \leq i \leq n \rangle$, respectively. By Lemma 3.4, N_u has two orbits $\mathcal{C}_1$ and $\mathcal{C}_2$ on $\mathcal{C}$, where $\mathcal{C}_1$ has length $\frac{q^{n-1}-1}{q-1}$

and contains C_1 corresponding to $\langle \mathbf{e}_i \mid 1 \leq i \leq n-1 \rangle\rangle$, and $\mathcal{C}_2$ has length q^{n-1} and contains C_2 corresponding to $\langle \mathbf{e}_i \mid 2 \leq i \leq n \rangle\rangle$. Calculation shows that $|(N_u \cap N_{C_1}) : (N_u \cap N_{w_1})| = 9$ and $N_u \cap N_{C_2} = N_u \cap N_{w_2}$, where $w_1 \in C_1$ and $w_2 \in C_2$. If $\Gamma(u) \subseteq \cup_{C' \in \mathcal{C}_1} C'$ then we get a similar contradiction as above. Thus $\Gamma(u) \subseteq \cup_{C' \in \mathcal{C}_2} C'$, and $\Gamma(u)$ is one of the 9-orbits of N_u on $\cup_{C' \in \mathcal{C}_2} C'$. Note that $N_u^\sigma = N_w$ for $w \in C_2$. Then Γ is arc-transitive by Lemma 2.1, and part (2) follows.

Case 2. Assume that $|B| = 3$. Then $N = \mathrm{PSL}(3, q)$ with $q = r^f \equiv 1 \pmod 3$. In particular, $|N|$ has at least 4 distinct prime divisors, refer to [15, pp. 12].

Let W_1 be an arbitrary N-orbit on W. Take $w \in W_1$. Then $|W_1| = |N : N_w| = 3$, 9, p, $3p$ or $9p$. Since N is simple, N has no subgroups of index 3. By [7, Appendix B], N has no subgroups of index 9. By Table 1, N has no primitive permutation representations of prime degree, hence N has no subgroups of index p. Thus $|W_1| = 3p$ or $9p$. Suppose that $|W_1| = 3p$. Then N has exactly three orbits on W. Since N is normal in G, each N-orbit on W is a block of G. By Lemma 2.5, $|\Gamma(u) \cap W_1| \leq 1$ for $u \in U$, yielding $|\Gamma(u)| \leq 3$. By Lemma 2.1, $|G_u| = 2^s \cdot 3^t$ for some integer $s, t \geq 0$. Then $|G| = 2^s \cdot 3^{t+2} \cdot p$. Thus $|N|$ has at most 3 distinct prime divisors, a contradiction. Then $|W_1| = 9p$, that is, N is transitive on W.

Take a maximal block C of G on W, and set $\mathcal{C} = \{C^g \mid g \in G\}$. Then G acts primitively on $\mathcal{C}$. Recall that N has no subgroups of indices 3, 9 and p. It implies that $|\mathcal{C}| = 3p$. Then part (3) of this theorem follows from an analogous argument given in Case 1. □

4 The proof of Theorem 1.1

Let Γ be a G-locally primitive regular graph of order $18p$, where $G \leq \mathsf{Aut}\Gamma$ and p is a prime. Assume that G is intransitive on $V\Gamma$. Let U and W be the G-orbits on $V\Gamma$. If G acts unfaithfully on one of U and W, then Γ is the complete bipartite graph $\mathsf{K}_{9p,9p}$, and hence Γ is arc-transitive. Thus we assume that G is faithful on both U and W. By the argument in Section 3, we assume further that G has non-trivial normal subgroups which are intransitive on both U and W. Let M be maximal one of such normal subgroups of G. Denote by $\widetilde{U}$ and $\widetilde{W}$ be the sets of M-orbits on U and W, respectively. For each $v \in V\Gamma$, denote by $\tilde{v}$ the M-orbit containing v.

By Lemma 2.7, Γ is a normal cover of Γ_M. Then M is semiregular on both U and W; in particular $|M| = 3$, 9, p or $3p$ and $|\widetilde{U}| = |\widetilde{W}| = \frac{9p}{|M|} = 3p$, p, 9 or 3, respectively. Note that M is the kernel of G acting on $V\Gamma_M = \widetilde{U} \cup \widetilde{W}$. Then we identify $X := G/M$ with a subgroup group of $\mathsf{Aut}\Gamma_M$. Then Γ_M is X-locally primitive.

Next we finish the proof of Theorem 1.1 in two subsections depending on whether or not Γ_M is a bipartite complete graph.

4.1

In this subsection we assume that Γ_M is a complete bipartite graph, that is, $\Gamma_M \cong \mathsf{K}_{\frac{9p}{|M|}, \frac{9p}{|M|}}$. Let $u \in U$ and $w \in W$. Then $X_{\tilde{u}}$ and $X_{\tilde{w}}$ acts primitively on $\widetilde{W}$ and $\widetilde{U}$, respectively. Thus X acts primitively on both $\widetilde{U}$ and $\widetilde{W}$. Moreover, $|X_{\tilde{u}} : X_{\tilde{u}\tilde{w}}| = \frac{9p}{|M|} = |X : X_{\tilde{u}}|$, and so $\frac{81p^2}{|M|^2}$ is a divisor of $|X|$.

Lemma 4.1 *Assume that X is faithful on one of $\widetilde{U}$ and $\widetilde{W}$. Then Γ is an arc-transitive graph of order* 36 *and valency* 6.

Proof. Without loss of generality, we may assume that X is faithful on $\widetilde{W}$. Then both X and $X_{\tilde{u}}$ are primitive permutation groups on $\widetilde{W}$. If $|M| = 3p$ then $X \cong \mathbb{Z}_3$ or S_3, and hence X is intransitive on the edges of Γ_M, a contradiction. If $|M| = 9$ then p^2 is a divisor of X; however each permutation group of degree prime p has order indivisible by p^2, a contradiction. If $|M| = p$ then $\mathsf{soc}(X)$ and $\mathsf{soc}(X_{\tilde{u}})$ are one of A_9, $\mathrm{PSL}(2,8)$ or $\mathbb{Z}_3^2$, yielding $9 = |\widetilde{U}| = |X : X_{\tilde{u}}| \neq 9$, a contradiction.

Now let $|M| = 3$. Then $M \cong \mathbb{Z}_3$ and $|\widetilde{W}| = 3p$. Since $9p^2$ is a divisor of $|X|$, checking Table 2 implies that $\mathsf{soc}(X) = \mathrm{A}_{3p}$ or A_5. Note $|X : X_{\tilde{u}}| = 3p$ and $|X_{\tilde{u}} : X_{\tilde{u}\tilde{w}}| = 3p$. It follows that $p = 2$, $\Gamma_M \cong \mathsf{K}_{6,6}$, $\mathsf{soc}(X) \cong \mathrm{A}_6$ and $\mathsf{soc}(X_{\tilde{u}}) \cong \mathrm{A}_5$.

By $\mathsf{soc}(X) \cong \mathrm{A}_6$, we know that X is isomorphic to a subgroup of $\mathsf{Aut}(\mathrm{A}_6) = \mathrm{A}_6.\mathbb{Z}_2^2$. In particular, $|X : \mathsf{soc}(X)|$ is a divisor of 4. Since $\mathsf{soc}(X)$ is normal in X, all $\mathsf{soc}(X)$-orbits on $\widetilde{U}$ have that same length dividing $3p$. Thus the number of $\mathsf{soc}(X)$-orbits on $\widetilde{U}$ is a common divisor of 4 and $3p$. It follows that $\mathsf{soc}(X)$ acts transitively on $\widetilde{U}$. In addition $\mathsf{soc}(X)$ is transitive $\widetilde{W}$ as X is faithful and primitive on $\widetilde{W}$. Then Γ_M is $\mathsf{soc}(X)$-edge-transitive by Lemma 2.6. In particular, $\mathsf{soc}(X)_{\tilde{u}}$ and $\mathsf{soc}(X)_{\tilde{w}}$ acts transitively on $\widetilde{W}$ and $\widetilde{U}$, respectively. Checking the subgroups of A_6, we conclude that $\mathsf{soc}(X)_{\tilde{u}} \cong \mathsf{soc}(X)_{\tilde{w}} \cong \mathrm{A}_5$, and $\mathsf{soc}(X)_{\tilde{u}}$ and $\mathsf{soc}(X)_{\tilde{w}}$ are not conjugate in $\mathsf{soc}(X)$. It is easy to see that Γ is $\mathsf{soc}(X)$-locally primitive.

Let H be the pre-image of $\mathsf{soc}(X)$ in G. Then $H = M.\mathsf{soc}(X)$, $M = \mathbf{Z}(H)$ and Γ is H-locally primitive. Let H' be the commutator subgroup of H. Suppose that $H' \neq H$. Then $H = M \times H'$ and $H' \cong \mathrm{A}_6$. Thus H' is normal in H and intransitive on both U and W. By Lemma 2.7, H' is semiregular on $V\Gamma$, which is impossible. Therefore, $H = H'$. By the information given in [6], we know that H has an automorphism σ of order 2 with $H_{\tilde{u}}^{\sigma} = H_{\tilde{w}}$ for suitable $\tilde{u} \in \widetilde{U}$ and $\tilde{w} \in \widetilde{W}$. Noting that $H_{\tilde{u}} = M \times H_{u'}$ and $H_{\tilde{w}} = M \times H_{w'}$ for arbitrary $u' \in \tilde{u}$ and $w' \in \tilde{w}$, it follows that $H_{u'}^{\sigma} = H_{w'}$. Then, by Lemma 2.1, Γ is an arc-transitive graph. □

Lemma 4.2 *Assume that X acts unfaithfully on both $\widetilde{U}$ and $\widetilde{W}$. Then Γ has valency* 2, 3 *or p, and Γ is either arc-transitive or isomorphic to the Gray graph.*

Proof. Let Y_1 and Y_2 be the corresponding kernels. Then $Y_1 \cap Y_2 = 1$ and $Y_1Y_2 = Y_1 \times Y_2$. Since X acts primitively on both $\widetilde{U}$ and $\widetilde{W}$, we conclude that Y_1 and Y_2 act transitively on $\widetilde{W}$ and $\widetilde{U}$, respectively. It follows that $\mathsf{soc}(X/Y_i) \leq Y_1Y_2/Y_{3-i}$, where $i = 1, 2$. Checking primitive permutation groups of degree $\frac{9p}{|M|}$, we conclude that $Y_1 \times Y_2$ contains a normal subgroup $Y = T_1 \times T_2$ which is transitive on $E\Gamma_M$ such that $Y_i \geq T_i \cong \mathsf{soc}(X/Y_i)$ and one of the following conditions holds:

(i) $p = 2$ and Γ_M is a 4-cycle;

(ii) $|M| = 9$, $p \geq 5$, $\Gamma_M \cong \mathsf{K}_{p,p}$, $T_1 = \mathsf{soc}(Y_1) \cong T_2 = \mathsf{soc}(Y_2)$ and T_1 is simple;

(iii) $\Gamma_M \cong \mathsf{K}_{3,3}$, $T_1 = \mathsf{soc}(Y_1) \cong T_2 = \mathsf{soc}(Y_2) \cong \mathbb{Z}_3$;

(iv) $\Gamma_M \cong \mathsf{K}_{9,9}$, $T_1 \cong T_2 \cong \mathbb{Z}_3^2$;

(v) $|M| = 3$ or p, $T_1 = \mathsf{soc}(Y_1) \cong T_2 = \mathsf{soc}(Y_2)$ and T_1 is non-abelian simple.

Let N be the pre-image of Y in G. Then Γ is N-edge-transitive. In particular, N is not regular on U and W. Noting that N is faithful on both U and W, it follows that N is not abelian.

If (i) occurs then Γ_M is a cycle, and so Γ is arc-transitive.

Assume that (ii) occurs. Then Y has a subgroup which has order p and acts regularly on both $\widetilde{U}$ and $\widetilde{W}$. Thus N has a subgroup $M.\mathbb{Z}_p$ acting regularly on both U and W. By the Sylow Theorem, it is easily shown that $N.\mathbb{Z}_p \cong \mathbb{Z}_3^2\times\mathbb{Z}_p$ or $\mathbb{Z}_9\times\mathbb{Z}_p$. It follows from Lemma 2.1 that Γ is vertex-transitive, hence Γ is arc-transitive.

Assume that (iii) occurs. Then $|M| = 3p$ and $N = M.\mathbb{Z}_3^2$. If $p = 3$ then either Γ is arc-transitive or, by [26] or [27], Γ is isomorphic to the Gray graph. Assume that $p = 2$. Then M has a characteristic subgroup $K \cong \mathbb{Z}_3$, and hence K is normal in N. It is easily shown that Γ is a normal cover of Γ_K with respect to N and K. Thus Γ_K is a cubic edge-transitive graph of order 12. However, by [3, 5], there are no such graphs, a contradiction. Thus assume that $p \geq 5$. Then M has a unique Sylow p-subgroup. Let P be the unique Sylow p-subgroup of M. Then $P \cong \mathbb{Z}_p$ and P is normal in N. Since Γ is cubic, Γ is N-locally primitive. Thus Γ is a normal cover of Γ_P, and hence Γ_P is an N/P-edge-transitive cubic graph of order 18. Write $N = P{:}Q$, where Q is a Sylow 3-subgroup of N. Then $Q \cong N/P$ is non-abelian.

Let S be the Sylow 3-subgroup of $\mathrm{C}_N(P)$. Then S is normal in N. It is easily shown that S fixes both U and W set-wise, and so S is intransitive on both U and W as $|U| = |W| = 9p$ and $p \neq 3$. Then S is semiregular on both U and W, and so $|S| = 1$, 3 or 9; in particular, S is ablelian. It implies that $PS = P\times S$ is abelian and semiregular on both U and W. Assume $|S| = 3$. Since S is normal in Q, it implies that S lies in the center of Q. Note that $Q/S = Q/Q\cap\mathrm{C}_N(P) \cong Q\mathrm{C}_N(P)/\mathrm{C}_N(P) \leq N/\mathrm{C}_N(P) \lesssim \mathsf{Aut}(P) \cong \mathbb{Z}_{p-1}$. Then Q/S is cyclic. It follows that Q is abelian, a contradiction. Therefore $|S| = 9$, and hence PS is regular on both U and W. Thus Γ is arc-transitive by Lemma 2.1.

Next we finish the proof by excluding (iv) and (v).

Suppose that (iv) occurs. Write $N = P{:}Q$, where Q is a Sylow 3-subgroup of N. Then $Q \cong \mathbb{Z}_3^4$. Let S be the Sylow 3-subgroup of $\mathrm{C}_N(P)$. Then S is normal in N. Since N is non-abelian, $Q \neq S$. Consider the quotient $N/\mathrm{C}_N(P)$. We conclude that $S \cong \mathbb{Z}_3^3$. Since Γ is bipartite, it is easily shown that S fixes the bipartition of Γ. If $p \neq 3$ then S is neither transitive nor semiregular on both U and W, which contradicts Lemma 2.7. Thus $p = 3$, and so $|V\Gamma| = 54$ and $|\mathsf{Aut}\Gamma|$ is divisible by 3^5. By [3, 5], there exists no such a cubic edge-transitive graph, a contradiction.

Suppose that (v) occurs. Note that $(N/M)/(\mathrm{C}_N(M)/M) \cong N/\mathrm{C}_N(M) \lesssim \mathsf{Aut}(M) \cong \mathbb{Z}_{p-1}$ or $\mathbb{Z}_2$. Since $Y = N/M$ is the direct product of two isomorphic non-abelian simple groups, it follows that $N/M = \mathrm{C}_N(M)/M$, and so $N = \mathrm{C}_N(M)$. Then M is the center of N. Take $u \in U$. Then $N_{\tilde{u}} = M\times N_u$, and so $N_u \cong N_{\tilde{u}}/M = Y_{\tilde{u}} = (T_2)_{\tilde{u}}\times T_1$. Then N_u acts transitively on $\widetilde{W}$, and hence $N_{\tilde{u}}$ acts transitively on W. Note that N_u has a normal subgroup $K \cong (T_2)_{\tilde{u}}$ which acts trivially on $\widetilde{W}$. Then K fixes set-wise each M-orbit on W. It is easily shown that K is normal in $N_{\tilde{u}}$. It follows that all K-orbits on W have the same length. Thus either K acts trivially on W, or K acts transitively on each M-orbit on W. The latter case implies that $\Gamma \cong \mathsf{K}_{9p,9p}$, a contradiction. Thus $K = 1$ as G is faithful on both U and W, and so $(T_2)_{\tilde{u}} = 1$. Noting that T_2 is transitive on $\widetilde{U}$, it follows that $|T_2| = |T_2 : (T_2)_{\tilde{u}}| = |\widetilde{U}| = 9$ or $3p$, which contradicts that T_2 is simple. $\square$

4.2

Now we assume that Γ_M is not a complete bipartite. Then X acts faithfully on both $\widetilde{U}$ and $\widetilde{W}$. By Lemma 2.7, X is quasiprimitive on one of $\widetilde{U}$ and $\widetilde{W}$. Recall that $|\widetilde{U}| = |\widetilde{W}| = \frac{9p}{|M|} = 3p$, p, 9 or 3.

Lemma 4.3 $|\widetilde{U}| = |\widetilde{W}| \neq 9$.

Proof. Suppose that $|\widetilde{U}| = |\widetilde{W}| = 9$. Without loss of generality, we assume that X is quasiprimitive on $\widetilde{U}$. Then it is easily shown that X is primitive on $\widetilde{U}$. Thus $\mathsf{soc}(X)$ is isomorphic to one of A_9, $\mathrm{PSL}(2,8)$ or $\mathbb{Z}_3^2$. Let $N \leq G$ with $N/M = \mathsf{soc}(X)$.

Assume that $\mathsf{soc}(X) \cong \mathrm{PSL}(2,8)$. Then X is 3-transitive on both $\widetilde{U}$ and $\widetilde{W}$. It follows that $\Gamma_M \cong \mathsf{K}_{9,9} - 9\mathsf{K}_2$, and that Γ is N-locally primitive. Moreover, it is easily shown that M is the center of N. By [6], $\mathrm{PSL}(2,8)$ has Schur Multiplier 1. This implies that $N = M{\times}K$ with $\mathrm{PSL}(2,8) \cong K < N$. Thus N has a normal subgroup K acting neither transitively nor semiregularly on each of U and W, which contradicts Lemma 2.7.

Assume that $\mathsf{soc}(X) \cong \mathrm{A}_9$. A similar argument as above implies that $\Gamma_M \cong \mathsf{K}_{9,9} - 9\mathsf{K}_2$ and Γ is N-locally primitive. Moreover, N is a central extension of M by A_9. If $p \neq 2$ then, noting that A_9 has Schur Multiplier $\mathbb{Z}_2$, we have $N = M{\times}K$ for $K < N$ with $K \cong \mathrm{A}_9$, which yields a similar contradiction as above. Suppose that $p = 2$. Take $u \in \widetilde{U}$. Then $N_{\tilde{u}} = M{\times}N_u$, and so $N_u \cong N_{\tilde{u}}/M \cong \mathrm{A}_8$. Noting that $M \cong \mathbb{Z}_2$ and $N_{\tilde{u}}$ contains a Sylow 2-subgroup of N, it follows from Gaschtz' Theorem (see [1, 10.4]) that the extension $N = M.\mathsf{soc}(X)$ splits over M, that is, $N = M{\times}K$ for $K < N$ with $K \cong \mathrm{A}_9$, again a contradiction.

Assume that $\mathsf{soc}(X) \cong \mathbb{Z}_3^2$. Then $X \lesssim \mathrm{AGL}(2,3)$ and, for some $\tilde{u} \in \widetilde{W}$, the stabilizer $X_{\tilde{u}}$ is isomorphic to an irreducible subgroup of $\mathrm{GL}(2,3)$. By [13, Theorem 2], there are no semisymmetric graphs of order 18. It follows from [17, Lemma 2.5] that $\mathsf{soc}(X)$ acts transitively on $\widetilde{W}$. Thus $\mathsf{soc}(X)$ is regular on both $\widetilde{U}$ and $\widetilde{W}$. By [25], $X_{\tilde{u}}$ acts faithfully on the neighbors of $\tilde{u}$. In addition, since Γ_M is X-locally primitive, $X_{\tilde{u}}$ is a primitive permutation group on $\Gamma_M(\tilde{u})$. However, it is easy to check that $\mathrm{GL}(2,3)$ has no irreducible subgroups satisfying the conditions for $X_{\tilde{u}}$, a contradiction. $\square$

Lemma 4.4 *If* $|\widetilde{U}| = |\widetilde{W}| = 3$ *or* p*, then* Γ *is arc-transitive.*

Proof. If $|\widetilde{U}| = 2$ then $X \cong \mathbb{Z}_2$ and Γ_M is 4-cycle, which is impossible. If $|\widetilde{U}| = 3$ then $X \cong \mathrm{S}_3$ and Γ_M is 6-cycle, and hence Γ is a cycle. Thus we assume that $|\widetilde{U}| = p \geq 5$. Then $|M| = 9$, and either $X = G/M \leq \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$ or X is a permutation group with $\mathsf{soc}(X)$ listed in Table 1. In particular, G has a subgroup $R = M.\mathbb{Z}_p$ which acts regularly on both U and W. By the Sylow Theorem, it is easily shown that $R \cong M{\times}P$, where P is a Sylow p-subgroup of R. Then R is abelian, and hence Γ is arc-transitive by Lemma 2.1. $\square$

Finally, we deal with the case where $|\widetilde{U}| = 3p \neq 9$, that is, $p \neq 3$ and $M \cong \mathbb{Z}_3$.

Lemma 4.5 *Assume that* $|\widetilde{U}| = 3p \neq 9$*. Then* Γ *is arc-transitive.*

Proof. Without loss of generality, we assume that $X = G/M$ is a quasiprimitive group on $\widetilde{U}$. Since $|\widetilde{U}| = 3p \neq 9$, by Lemma 2.8, $\mathsf{soc}(X)$ is insoluble.

Case 1. Assume that $X = G/M$ is primitive on $\widetilde{U}$. Then X is known as in Table 2. Since $\mathsf{soc}(X)$ is non-abelian simple, it has no proper subgroups of index less than 5. Suppose that $\mathsf{soc}(X)$ is not primitive on $\widetilde{W}$. Then either each $\mathsf{soc}(X)$-orbit on $\widetilde{W}$ has length p, or $\mathsf{soc}(X)$ is transitive on $\widetilde{W}$ with a block of size 3; moreover, $p > 3$ in both cases. Thus, for these two cases, $\mathsf{soc}(X)$ can be viewed as a transitive permutation group of prime degree. Checking Table 1 and 2, we conclude that $\mathsf{soc}(X) \cong \mathrm{A}_7$ and $\mathsf{soc}(X)_\alpha \cong \mathrm{A}_6$, where α is either an M-orbit on $\widetilde{W}$ or a block of $\mathsf{soc}(X)$ with size 3 on $\widetilde{W}$. For the former case, $3p = |\widetilde{W}| = |X : X_\alpha| \leq |X : \mathsf{soc}(X)_\alpha| \leq |\mathrm{S}_7{:}\mathrm{A}_6| = 14$, a contradiction; for the latter case, A_6 has a subgroup of index 3, which is impossible. It follows that $\mathsf{soc}(X)$ is primitive on both $\widetilde{U}$ and $\widetilde{W}$; in particular, Γ_M is $\mathsf{soc}(X)$-edge-transitive.

Let $N \leq G$ with $N/M = \mathsf{soc}(X)$. Clearly, N is normal in G and Γ is N-edge-transitive. Moreover, it is easily shown that M is the center of N.

Subcase 1.1. Assume that the extension $N = M.\mathsf{soc}(X)$ splits over M, that is, $N = M \times K$ for $\mathsf{soc}(X) \cong K < N$. Then K is a normal subgroup of G, and K acts primitively on both $\widetilde{U}$ and $\widetilde{W}$. Since K is a non-abelian simple group, its order has at least three distinct prime divisor. It follows that K is not semiregular on both U and W. Then K is transitive on one of U and W. This implies that $9p$ is a divisor of $|K|$, and so K is not isomorphic to one of A_5, $\mathrm{PSL}(3, 2)$ and $\mathrm{PSL}(2, 2^f)$.

Without loss of generality, assume that K is transitive on U. Then, for $u \in U$, the stabilizer $K_{\tilde{u}}$ is transitive on the M-orbit $\tilde{u}$. Thus $3 = |M| = |\tilde{u}| = |K_{\tilde{u}} : K_u|$, and so K has a subgroup of index 3. Noting that $N_{\tilde{u}} = MK_{\tilde{u}}$, it implies that $K_{\tilde{u}} \cong N_{\tilde{u}}/M = \mathsf{soc}(X)_{\tilde{u}}$. Checking the subgroups of $\mathsf{soc}(X)_{\tilde{u}}$, we know that either $K \cong \mathsf{soc}(X) = \mathrm{A}_6$ and $p = 5$, or $K \cong \mathsf{soc}(X) = \mathrm{PSL}(3, q)$ and $3p = q^2 + q + 1$, where q is a power of a prime with $q \equiv 1 \pmod 3$.

Assume that $\mathsf{soc}(X) = \mathrm{A}_6$. Then Γ has order 90. Suppose that K is intransitive on W. Then K has three orbits on W, and so Γ is cubic by Lemma 2.6. Thus Γ is a semisymmetric cubic graph by [5, Theorem 5.2]. Again by [5], there is no semisymmetric cubic graphs of order 90, a contradiction. Then K is also transitive on W. By Lemma 2.6, Γ is K-edge-transitive. Checking the subgroups of A_6, we know that $K_u \cong \mathrm{D}_8$ for $u \in U$. It follows that Γ has valency 4 or 8. Since Γ is G-locally primitive, $G_u^{\Gamma(u)}$ is a primitive group of degree 4 or 8. Since $K_u^{\Gamma(u)}$ is a transitive normal subgroup of $G_u^{\Gamma(u)}$, it follows that Γ has valency 4. Then Γ_M has valency 4. Consider the actions of $\mathsf{soc}(X)$ on $\widetilde{U}$ and $\widetilde{W}$. If these two actions are equivalent then Γ_M has valency 6 or 8; otherwise, Γ_M has valency 3 or 12. This is a contradiction.

Assume that $\mathsf{soc}(X) = \mathrm{PSL}(3, q)$. Then Γ_M has valency q^2, $q+1$ or q^2+q. If K is intransitive on W then K has three orbits on W, and hence Γ is cubic by Lemma 2.6, a contradiction. Thus K is also transitive on W, and so Γ is K-edge-transitive. Arguing similarly as in the proof of Theorem 3.1, we conclude that Γ is arc-transitive and has valency q^2.

Subcase 1.2. Assume that the extension $N = M.\mathsf{soc}(X)$ does not split over M. Then, checking the Schur multipliers of the simple groups in Table 2, we conclude that $N = 3.\mathrm{A}_6$ with $p = 5$ or 2, or $N = 3.\mathrm{A}_7$ with $p = 5$ or 7, or $N = \mathrm{SL}(3, q)$ with $3 \mid q - 1$.

Let $N = \mathrm{SL}(3, q)$ with $3 \mid q - 1$. Using the notation defined above Lemma 3.3, we identify $\widetilde{U}$ with $\mathcal{P}$ and $\widetilde{W}$ with $\mathcal{P}$ or $\mathcal{H}$. Then there are $\tilde{u} \in \widetilde{U}$ and $\tilde{w} \in \widetilde{W}$ such that

$$N_{\tilde{u}} = \left\{ \left(\begin{array}{cc} a & \mathbf{0} \\ \mathbf{b}' & \mathbf{A} \end{array} \right) \middle| \, \mathbf{b} \in \mathbb{F}_q^2, \mathbf{A} \in \mathrm{GL}(2, q), a^{-1} = \det(\mathbf{A}) \right\}$$

and $N_{\tilde{w}} = N_{\tilde{u}}$ or $N_{\tilde{u}}^{\sigma}$. By Lemma 3.4 and a similar argument in the proof of Theorem 3.1, it is easily shown that Γ is an arc-transitive graph of valency q^2.

Let $N = 3.\mathrm{A}_6$. If $p = 2$ then $\Gamma_M \cong \mathsf{K}_{6,6} - 6\mathsf{K}_2$, and hence Γ is arc-transitive by Lemma 2.4. Now let $p = 5$. Then Γ_M has valency 6, 8, 3 or 12. Take $u \in U$. Then $N_{\tilde{u}} = M \times N_u$, and so $N_u \cong N_{\tilde{u}}/M = \mathsf{soc}(X)_{\tilde{u}} \cong \mathrm{S}_4$. Since Γ is G-locally primitive, $G_u^{\Gamma(u)}$ is a primitive group. Noting that $N_u^{\Gamma(u)}$ is a transitive normal subgroup of $G_u^{\Gamma(u)}$, it follows that Γ has valency 4 or 3. Since Γ is a normal cover of Γ_M, we conclude that Γ has valency 3. By [5], there is no semisymmetric cubic graphs of order 90. Thus Γ is arc-transitive.

Let $N = 3.\mathrm{A}_7$ with $p = 5$ or 7. Assume first that $\mathsf{soc}(X)$ acts equivalently on $\widetilde{U}$ and $\widetilde{W}$. Then, by Lemma 2.3, Γ_M is isomorphic to an orbital bipartite graph of $\mathsf{soc}(X)$ on $\widetilde{U}$. Calculation shows that the suborbits of $\mathsf{soc}(X)$ on $\widetilde{U}$ are all self-paired. Then Γ is arc-transitive by Colloray 2.1. If $p = 5$ then $X = \mathsf{soc}(X) \cong \mathrm{A}_7$, $X_{\tilde{u}} \cong \mathrm{PSL}(2,7)$ and Γ_M has valency 14; however $\mathrm{PSL}(2,7)$ has no primitive permutation representations of degree 14, a contradiction. Then $p = 7$. It is easily shown that Γ has valency 10.

Assume that the actions of $\mathsf{soc}(X)$ on $\widetilde{U}$ and $\widetilde{W}$ are not equivalent. Then $X = \mathsf{soc}(X) = \mathrm{A}_7$ and $X_{\tilde{u}} \cong \mathrm{PSL}(2,7)$, and so $G = N = 3.\mathrm{A}_7$. In particular, $p = 5$ and Γ_M has order 30. Take $\tilde{w} \in \Gamma_M(\tilde{u})$. Checking the subgroups of A_7, we conclude that $|X_{\tilde{u}} : (X_{\tilde{u}} \cap X_{\tilde{w}})| = 7$ or 8. Then Γ_M has valency 7 or 8, and so does Γ. Verified by GAP, there are two involutions σ_1, $\sigma_2 \in \mathrm{S}_7$ such that $|X_{\tilde{u}} : (X_{\tilde{u}} \cap X_{\tilde{u}}^{\sigma_1})| = 7$ and $|X_{\tilde{u}} : (X_{\tilde{u}} \cap X_{\tilde{u}}^{\sigma_2})| = 8$. Note that $G_{\tilde{v}} = N \times G_v$ and $X_{\tilde{v}} \cong G_v$ for $v \in V\Gamma$. Thus we may choose a suitable $w \in \Gamma(u)$ such that $G_u^{\sigma} = G_w$ for an automorphism of G of order 2. Then Γ is arc-transitive by Lemma 2.1.

Case 2. Assume that $X = G/M$ is quasiprimitive but not primitive on $\widetilde{U}$. Let B be a maximal block of X on $\widetilde{U}$. Then $|B| = 3$. Set $\mathcal{B} = \{B^x \mid x \in X\}$. Then $|\mathcal{B}| = p$ and X acts faithfully on $\mathcal{B}$. Thus X is known as in Table 1. Let $\tilde{u} \in B$. Then $|X_B : X_u| = |B| = 3$. Checking one by one the groups listed in Table 1, we conclude that $\mathsf{soc}(X) = \mathrm{PSL}(n,q)$ with $p = \frac{q^n-1}{q-1}$.

Suppose that $n = 2$. Then $q = 2^{2^s}$ for some integer $s \geq 1$, and $N = M.\mathsf{soc}(X) \cong \mathbb{Z}_3 \times \mathrm{PSL}(2, 2^{2^s})$. It follows that G has a normal subgroup K isomorphic to $\mathrm{PSL}(2, 2^{2^s})$. Note that 9 is not a divisor of $|K|$. It follows that K is intransitive on both U and W. By Lemma 2.7, K is semiregular on U, which is impossible. Then $n \geq 3$.

A similar argument as above implies that $(n,q) \neq (3,2)$. Then, by [15, pp. 12], $|\mathsf{soc}(X)|$ has at least four distinct prime divisors. Noting $|X| = 3p|X_{\tilde{u}}|$, it follows that $|X_{\tilde{u}}|$ has an odd prime divisor other than 3. This implies that the valency of Γ_M is no less than 5. If $\mathsf{soc}(X)$ is intransitive on $\widetilde{W}$, then $\mathsf{soc}(X)$ has exactly three orbits on $\widetilde{W}$, and so Γ_M has valency 3 by Lemma 2.6, a contradiction. Therefore, $\mathsf{soc}(X)$ is transitive on $\widetilde{W}$, and hence Γ_M is $\mathsf{soc}(X)$-edge-transitive. Let $N \leq G$ with $N/M = \mathsf{soc}(X)$. Then N is normal in G and Γ is N-edge-transitive.

It is easily shown that n is an odd prime with $q \not\equiv 1 \pmod{n}$, see the proof of Lemma 3.5. Then the Schur Multiplier of $\mathrm{PSL}(n,q)$ is 1. Recalling $M \cong \mathbb{Z}_3$, it yields that $N = M \times K$, where $K \cong \mathrm{PSL}(n,q)$. Clearly, K is a normal subgroup of G. Recalling $\mathsf{soc}(X)$ is transitive on both $\widetilde{U}$ and $\widetilde{W}$, we conclude that each K-orbit on $V\Gamma$ has length at least $3p$. Since K is not semiregular and Γ has valency no less than 5, by Lemma 2.6, we know that Γ is K-edge-transitive. Then the argument in Section 3 implies that Γ is an arc-transitive graph. $\square$

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