Rainbow connections for outerplanar graphs with diameter 2 and 3

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Abstract

An edge-colored graph G is rainbow connected if every two vertices are connected by a path whose edges have distinct colors. The rainbow connection number of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. It was proved that computing $rc(G)$ is an NP-Hard problem, as well as that even deciding whether a graph has $rc(G) = 2$ is NP-Complete. Li et al. proved that $rc(G) \leq 5$ if G is a bridgeless graph with diameter 2, while $rc(G) \leq 9$ if G is a bridgeless graph with diameter 3. Furthermore, Uchizawa et al. showed that determining the rainbow connection number of graphs is strongly NP-complete even for outerplanar graphs. In this paper, we give upper bounds of the rainbow connection number of outerplanar graphs with small diameters.

Keywords: rainbow connection; coloring; outerplanar graph; diameter

1. Introduction

All graphs considered here are simple, finite and undirected. We follow the notation and terminology of [23]. An edge-colored graph is *rainbow connected* if every two vertices are connected by a path whose edges have distinct colors (such a path is called a *rainbow path*). Obviously, if G is rainbow connected, then it is also connected. The concept of rainbow connection in

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graphs was introduced by Chartrand et al. in [4]. The *rainbow connection number* of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. Observe that $diam(G) \leq rc(G) \leq n-1$, where $diam(G)$ denotes the diameter of G and n denotes the order of G. It is easy to verify that $rc(G) = 1$ if and only if G is a complete graph, that $rc(G) = n-1$ if and only if G is a tree. Chartrand et al. computed the precise rainbow connection number of several graph classes including complete multipartite graphs [4]. The rainbow connection number has been studied for further graph classes in [1, 12, 16, 17] and for graphs with fixed minimum degree in [1, 13, 20]. There are also some results on the aspect of extremal graph theory, see [21]. In addition, many researches on the rainbow connection of random graphs are stated, such as [7, 8, 9, 10]. For more results on the rainbow connection, we refer to the survey [18] and the monograph [19].

In [2], Chakraborty et al. proved the following result: If G is an n-vertex graph with diameter 2 and minimum degree at least $8\log n$, then $rc(G) \leq 3$. Since a graph with minimum degree $n/2$ is connected and has diameter 2, we have an immediate result: If G is an n-vertex graph with minimum degree at least $n/2$, then $rc(G) \leq 3$. We know that any graph G with $rc(G) = 2$ must have $diam(G) = 2$. So, graphs with $rc(G) = 2$ belong to the graph class with $diam(G) = 2$. Therefore, there is an interesting problem: For any bridgeless graph G with $diam(G) = 2$, determine the smallest constant c such that $rc(G) \leq c$. In [6, 14], Li et al. showed that $c \leq 5$, and moreover, examples are given to show that the bound is best possible.

Theorem 1 ([6, 14]). *If G is a connected bridgeless graph with diameter 2, then $rc(G) \leq 5$. Moreover, the upper bound is sharp.*

In [14], Li et al. also showed that $rc(G) \leq k+2$ if G is connected with diameter 2 and k bridges, where $k \geq 1$. The bound $k+2$ is sharp as there are infinity many graphs with diameter 2 and k bridges whose rainbow connection numbers attain this bound. For diameter 3, Li et al. [15] proved that $rc(G) \leq 9$ if G is a bridgeless graph.

It was proved that the computation of $rc(G)$ is NP-hard [2]. Actually, it is already NP-complete to decide whether $rc(G) = 2$, and in fact it is already NP-complete to decide whether a given edge-colored (with an unbounded number of colors) graph is rainbow connected. In [11], we proved that it is still NP-complete even when the edge-colored graph is a planar bipartite graph.

Uchizawa et al. [22] obtained a stronger result: Determining the rainbow connection number of graphs is strongly NP-complete even for outerplanar graphs. So it is interesting to study the bounds of the rainbow connection number of outerplanar graphs. In [5], uppers bound of rainbow connection number of maximal outerplanar graphs are considered. In this paper, we will show that the rainbow connection number is at most 3 for bridgeless outerplanar graphs with diameter 2, and at most 6 for bridgeless outerplanar graphs with diameter 3.

2. Main results

We show that the rainbow connection number is at most 3 for bridgeless outerplanar graphs with diameter 2.

Theorem 2. *If G is a bridgeless outerplanar graph with order n and diameter 2, then $rc(G) \leq 3$, i.e., $rc(G) = 2, 3$.*

Proof. Suppose that G is a bridgeless outerplanar graph with diameter 2. If G has a cut vertex, then this vertex is a dominating vertex of the graph, then $rc(G) \leq 3$. Now we suppose that G is 2-connected and we can embed G in such way that a Hamilton cycle, H, bounds the outer face, and the edges not in H are chords that lie in the interior of H. If G has no chords, then G is a cycle of length at most 5 and thus $rc(G) \leq 3$. In the following we assume G has chords. Let v be a vertex with degree 2 and suppose $N(v) = \{x_1, y_1\}$. Denote by C the induced cycle of G containing vertex v. We will consider the following two cases according to the order of C.

Case 1. $|C| = 4$.

Suppose $C = vx_1zy_1v$. In this case, there is only one vertex outside of C, since each vertex outside of C must be adjacent to both x_1 and z. Observe that $rc(G) = 2$.

Case 2. $|C| = 3$.

For convenience, we assume $H = vx_1x_2 \dots x_{n/2}y_{(n-2)/2} \dots y_2y_1v$ for even n and $H = vx_1x_2 \dots x_{(n-1)/2}y_{(n-1)/2} \dots y_2y_1v$ for odd n. If H has only one chord, then this case is the same as **Case 1**. Otherwise, H has at least two chords, and then $n \geq 5$. There must be one chord e such that one of its end vertices is x_1 or y_1, without loss of generality, say x_1. Then, the other end of e must be y_2 or x_3. Assume $e = x_1y_2$. Then all other vertices in the set $V \setminus \{v, x_1, x_2, y_1, y_2\}$ must be adjacent to x_1, as the diameter of G is 2.

Therefore, in this case, the structure of graph G is a fan, as shown in Figure 1.

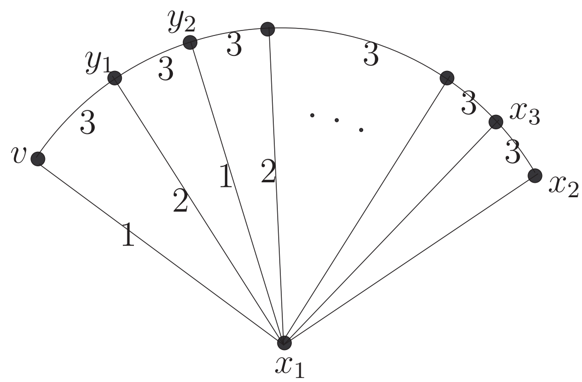

Figure 1: Bridgeless outerplanar graph with diameter two.

For $n = 5$, we can give an edge-coloring c of G such that $rc(G) = 2$, and under this coloring: $c(vx_1) = c(vy_1) = c(x_1y_2) = c(x_2y_2) = 1$ and $c(x_1y_1) = c(y_1y_2) = c(x_1x_2) = 2$. For $n = 6, 7$, the similar discussion can deduce that $rc(G) = 2$. For $n \geq 8$, we observe that $rc(G) = 3$. Notice that 2 colors cannot make G rainbow connected. Now we give an edge-coloring with 3 colors: all edges with x_1 as one of its ends are assigned colors 1 and 2 alternatively in clockwise order; all other edges are assigned color 3, as shown in Figure 1. ∎

A subset D of the vertices in G is called a *dominating set* if every vertex of $G - D$ is adjacent to a vertex of D. Furthermore, if the dominating set D induces a connected subgraph of G, then D is called a *connected dominating set*. Let $X, Y \in V(G)$. We say that X *dominates* Y if every vertex of Y is adjacent to at least one vertex of X. The following lemma will be used in the sequel.

Lemma 1 ([3]). *Let G be a connected graph with minimum degree at least 2, D a connected dominating set of G. Then $rc(G) \leq rc(G[D]) + 3$.*

Theorem 3. *If G is a bridgeless outerplanar graph with order n and diameter 3, then $3 \leq rc(G) \leq 6$.*

Proof. Suppose that $G = (V, E)$ is a bridgeless outerplanar graph with diameter 3. Since the rainbow connection number is at least the diameter, then we have $rc(G) \geq 3$. Suppose that G is not 2-connected and let v be a cut vertex of G. There is a partition of $V - \{v\}$ into two sets A and B such that the vertex v dominates either A or B. Without loss of generality, we may assume that v dominates A. Denote by B_1 the vertices of B that are adjacent to v and $B_2 = B - B_1$. Choose a minimum cardinality subset S of B_1 such that S dominates B_2. Then $S \cup \{v\}$ is a connected dominating set. We claim that $|S| \leq 2$. Suppose that $|S| \geq 3$ and let $S = \{s_1, s_2, s_3\}$. By the minimality of S, there exist three vertices $x_1, x_2, x_3 \in B_2$ satisfying that among three vertices s_1, s_2, s_3, x_i is only adjacent to vertex s_i for $1 \leq i \leq 3$. Take $a \in A$. Without loss of generality, we may assume that an embedding of G as an outerplanar graph has vertices a, s_1, s_2, s_3 in clockwise order, adjacent to v. Since all the vertices of G lie on a common face, there is no way to obtain a path of length at most 3 between x_1 and x_3. Thus, $|S| \leq 2$, which yields that $rc(G) \leq rc(G[S \cup \{v\}]) + 3 = 5$.

Now suppose that G is 2-connected. It follows that G can be embedded in such a way that a Hamilton cycle H bounds the outer face, and the edges not in H are chords that lie in the interior of H. If H has no chords, then G is a cycle of length at most 7, and thus $rc(G) = 3$ or $rc(G) = 4$. Thus, in the following we assume that H has at least one chord.

Suppose that xy is a chord of H. The cycle H is divided into two xy-paths. We denote the path goes in clockwise direction from x to y by the xy-segment of H, and denote the other path by the yx-segment of H.

Now suppose that H has precisely one chord xy. In this case, $\{x, y\}$ is a vertex cut of G. Since G has diameter 3, then $\{x, y\}$ dominates either the xy-segment of H or the yx-segment of H. Without loss of generality, we suppose that xy-segment is dominated. Since there are no other chords, the xy-segment of H is a path of length 2 or 3. If it is 2, then the yx-segment of H is a path of length 4 or 5 and thus we can check that $rc(G) = 3$. Otherwise, the yx-segment of H is a path of length 3 or 4 and thus $rc(G) = 3$ or 4.

Suppose that H has at least two chords. Among all vertex cuts with two vertices, we choose $\{a, b\}$ as a vertex cut such that it dominates a maximum number of vertices. Note that a and b may not correspond to the ends of a chord of H. Since G has diameter 3, $\{a, b\}$ dominates one segment of H. Without loss of generality, we assume that the ba-segment of H is dominated by $\{a, b\}$. Consider the ab-segment of H.

Case 1. There are no chords with both ends on the ab-segment of H.

In this case, there are at least two chords in the ba-segment. It follows that there are at most three internal vertices in the ab-segment of H. Now we suppose that there are three internal vertices in the ab-segment of H, since it is easy to check that $rc(G) \leq 6$ for the other two cases. If $ab \in E(G)$, then there exists a connected dominating set with three vertices, and then $rc(G) \leq 5$. Otherwise, we claim that there exists a vertex v in the ba-segment such that $va,\ vb \in E(G)$, since G has diameter 3 and at least two chords. It implies that G has a connected dominating set with four vertices, and then $rc(G) \leq 6$.

Case 2. There are some chords with both ends on the ab-segment of H.

Choose a vertex cut of size 2, $\{c, d\}$, such that any other vertex cut of size 2 with both vertices in the ab-segment of H has at least one vertex in the cd-segment of H, where the cd-segment is a part of the ab-segment.

Subcase 2.1. a, b, c, d are not all distinct vertices.

Without loss of generality, we suppose $b = d$. By our choice, any vertex on the ac-segment of H does not form of a vertex cut with b, and hence ac must be an edge of G (ac may be an edge of H or a chord of H).

Suppose that $\{c, b\}$ can not dominate the cb-segment. Let v be a vertex on the cb-segment such that $d(v, c) \geq 2$ and $d(v, b) \geq 2$. Then all vertices in the ba-segment must be adjacent to vertex b. Therefore, all vertices in the ac-segment must be adjacent to vertex c, since otherwise, if there exists a vertex w such that $wa \in E(G)$ and $wc \notin E(G)$, then $d(w, v) \geq 4$. Thus, $\{b, c\}$ is a vertex cut with two vertices, which dominates more vertices than $\{a, b\}$, a contradiction to the choice of $\{a, b\}$.

Now suppose that $\{c, b\}$ dominates the cb-segment. Thus, $\{a, b, c\}$ must be a dominating set of G. If one of ab and bc is an edge of G, then $\{a, b, c\}$ is a connected dominating set of G and thus $rc(G) \leq 2 + 3 = 5$. Now suppose neither ab nor bc is an edge of G.

Subsubcase 2.1.1. There is vertex v in the ba-segment (or cb-segment) such that v is adjacent to both a and b (or c and b).

In this situation, $\{a, b, c, v\}$ is a connected dominating set of G and thus $rc(G) \leq 3 + 3 = 6$.

Subsubcase 2.1.2. Otherwise, there does not exist such a vertex.

Each vertex in the ba-segment is only adjacent to one of a and b, and each vertex in the cb-segment is only adjacent to one of c and b. Now in this case, each of the ba-segment and cb-segment of H has at least two internal vertices. We claim that each of the ba-segment and cb-segment of H has exactly two internal vertices, since otherwise, we can always find two vertices with distance at least 4. Since G has at least two chords, then we can assume that the ac-segment has at least two internal vertices, which also implies that one pair of vertices has distance at least 4.

Subcase 2.2. a, b, c, d are distinct vertices.

The choice of $\{c, d\}$ implies that neither ad nor bc is an edge of G. From the way that $\{a, b\}$ and $\{c, d\}$ was chosen, we know that $\{a, b\}$ dominates the ba-segment and $\{c, d\}$ dominates the cd-segment. Moreover, ac and bd must be edges of G. If there is one vertex p in the ba-segment such that it is adjacent to a but not adjacent to b, and also one vertex q in the cd-segment such that it is adjacent to d but not adjacent to c, then $d(p, q) \geq 4$, a contradiction. Therefore, either $\{a, b, c\}$ or $\{b, c, d\}$ is a dominating set of G. We assume that $\{a, b, c\}$ is a dominating set of G, since the other case is similar. If ab is an edge of G, then $\{a, b, c\}$ is a connected dominating set of G and thus $rc(G) \leq 2 + 3 = 5$. Now suppose that ab is not an edge of G.

Subsubcase 2.2.1. There is vertex v in the ba-segment such that v is adjacent to both a and b.

In this situation, $\{a, b, c, v\}$ is a connected dominating set of G and thus $rc(G) \leq 3 + 3 = 6$.

Subsubcase 2.2.2. Otherwise, there does not exist such a vertex.

In this situation, each vertex in the cd-segment must be adjacent to both c and d, which implies that the cd-segment contains exactly one internal vertex. Similarly, there exist exactly two internal vertices in the ba-segment. Since G has two chords, then there exist some internal vertices in the ac-segment and bd-segment. In each case, we can find two vertices with distance at least 4.

The proof is thus completed. ∎

3. Conclusion

In [11], the authors proved that it is still NP-complete even when the edge-colored graph is a planar bipartite graph. Uchizawa et al. [22] obtained a stronger result: Determining the rainbow connection number of graphs is strongly NP-complete even for outerplanar graphs. So it is interesting to study the bounds of the rainbow connection number of outerplanar graphs. In this paper, we show that the rainbow connection number is at most 3 for bridgeless outerplanar graphs with diameter 2, and at most 6 for bridgeless outerplanar graphs with diameter 3.

In the future, we would like to generalize our results to the bridgeless outerplanar graphs with diameter $D \geq 4$. In addition, it is interesting to study the bounds for planar graphs with a given diameter.

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