On Pattern Avoiding Alternating Permutations

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Abstract

An alternating permutation of length n is a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ such that $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. Let A_n denote the set of alternating permutations of $\{1, 2, \ldots, n\}$, and let $A_n(\sigma)$ be the set of alternating permutations in A_n that avoid a pattern σ. Recently, Lewis used generating trees to enumerate $A_{2n}(1234)$, $A_{2n}(2143)$ and $A_{2n+1}(2143)$, and he posed some conjectures on the Wilf-equivalence of alternating permutations avoiding certain patterns of length four. Some of these conjectures have been proved by Bóna, Xu and Yan. In this paper, we prove two conjectured relations $|A_{2n+1}(1243)| = |A_{2n+1}(2143)|$ and $|A_{2n}(4312)| = |A_{2n}(1234)|$.

Keywords: alternating permutation, pattern avoidance, generating tree

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1 Introduction

The objective of this paper is to prove two conjectures of Lewis on the Wilf-equivalence of alternating permutations avoiding certain patterns of length four.

We begin with some notation and terminology. Let $[n] = \{1, 2, \ldots, n\}$, and let S_n be the set of permutations of $[n]$. A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is said to be an alternating permutation if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. An alternating permutation is also called an up-down permutation. A permutation π is said to be a down-up permutation if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots$. We denote by A_n and A'_n the set of alternating permutations and the set of down-up permutations of $[n]$, respectively. For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, its reverse $\pi^r = \pi^r_1\pi^r_2\cdots\pi^r_n$ is defined by $\pi^r_i = \pi_{n+1-i}$ for $1 \le i \le n$. The complement of π, denoted $\pi^c = \pi^c_1\pi^c_2\cdots\pi^c_n$, is defined by $\pi^c_i = n + 1 - \pi_i$ for $1 \le i \le n$. It is clear that the complement operation gives a one-to-one correspondence between A_n and A'_n.

Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ in S_n and a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_k \in S_k$, where $k \leq n$, we say that π contains a pattern σ if there exists a subsequence $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ ($1 \leq i_1 < i_2 < \cdots < i_k \leq n$) of π that is order isomorphic to σ, in other words, for $1 \leq l < m \leq k$, $\pi_{i_l} < \pi_{i_m}$ if and only if $\sigma_l < \sigma_m$. Otherwise, we say that π avoids a pattern σ, or π is σ-avoiding. For example, 74538126 is 1234-avoiding, while it contains a pattern 3142 corresponding to the subsequence 7486.

Let $A_n(\sigma)$ denote the set of σ-avoiding alternating permutations of $[n]$, and let $A'_n(\sigma)$ denote the set of σ-avoiding down-up permutations of $[n]$. Mansour [8] showed that for $n \geq 0$, $|A_{2n}(132)| = C_n$, where C_n is the Catalan number

$$\frac{1}{n+1}\binom{2n}{n}.$$

Meanwhile, Deutsch and Reifegerste showed that for $n \geq 0$, $|A_{2n}(123)| = C_n$, see Stanley [9]. As pointed out by Lewis [5], $|A_n(\sigma)|$ is a Catalan number for any $n \geq 0$ and any pattern $\sigma \in S_3$. To be more specific, for $n \geq 0$ and $\sigma \in \{123, 132, 213, 231, 312\}$, we have $|A_{2n}(\sigma)| = C_n$. For $n \geq 2$ and $\sigma = 321$, we have $|A_{2n}(\sigma)| = C_{n+1}$. For $n \geq 1$ and $\sigma \in \{123, 213, 312, 321\}$, we have $|A_{2n-1}(\sigma)| = C_n$. For $n \geq 2$ and $\sigma \in \{132, 231\}$, we have $|A_{2n-1}(\sigma)| = C_{n-1}$.

Lewis [5, 6, 7] studied the enumeration of alternating permutations avoiding a pattern of length four. In [7], he constructed generating trees for $A_{2n}(1234)$, $A_{2n}(2143)$ and the set of standard Young tableaux of shape (n, n, n). Then he showed that these generating trees are pairwise isomorphic. From the hook-length formula for the number of standard Young tableaux given by Frame, Robinson and Thrall [3], it follows that

$$|A_{2n}(1234)| = |A_{2n}(2143)| = \frac{2(3n)!}{n!(n+1)!(n+2)!}. \tag{1.1}$$

The above number is called the n-th 3-dimensional Catalan number, and it will be denoted by $C_n^{(3)}$. Notice that $C_n^{(3)}$ also equals the number of walks in 3-dimensions using steps $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ from $(0, 0, 0)$ to (n, n, n) such that after each step we have $z \geq y \geq x$. Lewis also constructed a generating tree for $A_{2n+1}(2143)$ and a generating tree for the set of shifted standard Young tableaux of shape $(n + 2, n + 1, n)$. It turns out that these two generating trees are exactly the same. Using the formula for the number of shifted standard Young tableaux given by Schur [10], we have

$$|A_{2n+1}(2143)| = \frac{2(3n+3)!}{n!(n+1)!(n+2)!(2n+1)(2n+2)(2n+3)}. \tag{1.2}$$

The following conjectures were posed by Lewis [7].

Conjecture 1.1 *For $n \geq 1$ and $\sigma \in \{1243, 2134, 1432, 3214, 2341, 4123, 3421, 4312\}$, we have*

$$|A_{2n}(\sigma)| = |A_{2n}(1234)| = |A_{2n}(2143)|. \tag{1.3}$$

Conjecture 1.2 *For $n \geq 0$ and $\sigma \in \{2134, 4312, 3214, 4123\}$, we have*

$$|A_{2n+1}(\sigma)| = |A_{2n+1}(1234)|. \tag{1.4}$$

Conjecture 1.3 *For $n \geq 0$ and $\sigma \in \{1243, 3421, 1432, 2341\}$, we have*

$$|A_{2n+1}(\sigma)| = |A_{2n+1}(2143)|. \tag{1.5}$$

By showing that a classical bijection on pattern avoiding permutations preserves the alternating property, Bóna [2] proved that

$$|A_{2n}(1243)| = |A_{2n}(1234)|, \tag{1.6}$$

$$|A_{2n+1}(2134)| = |A_{2n+1}(1234)|. \tag{1.7}$$

Xu and Yan [11] constructed bijections leading to the following relations

$$|A_{2n}(4123)| = |A_{2n}(1432)| = |A_{2n}(1234)|, \tag{1.8}$$

$$|A_{2n+1}(1432)| = |A_{2n+1}(2143)|, \tag{1.9}$$

$$|A_{2n+1}(4123)| = |A_{2n+1}(1234)|. \tag{1.10}$$

As for the above conjectures, there are essentially two unsolved cases, namely,

$$|A_{2n+1}(1243)| = |A_{2n+1}(2143)|, \tag{1.11}$$

and

$$|A_{2n}(4312)| = |A_{2n}(1234)|, \tag{1.12}$$

because the remaining cases can be deduced by the reverse and complement operations.

In this paper, we prove the relations (1.11) and (1.12) conjectured by Lewis. To prove (1.11), we construct a generating tree for $A_{2n+1}(1243)$ which coincides with the generating tree for $A_{2n+1}(2143)$ as given by Lewis [7]. This construction can be adapted to obtain a generating tree for $A_{2n}(1243)$, which turns out to be isomorphic to the generating tree for $A_{2n}(1234)$ as given by Lewis [7]. This leads to an alternative proof of relation (1.6) conjectured by Lewis and proved by Bóna.

To prove (1.12), we show that the generating tree for $A_{2n+1}(1243)$ constructed in Section 2 is isomorphic to the generating tree for shifted standard Young tableaux of shape $(n+2, n+1, n)$ as given by Lewis [7]. We adopt the notation $SHSYT(\lambda)$ for the set of shifted standard Young tableaux of shape λ. A label (a, b) in the generating tree for $A_{2n+1}(1243)$ corresponds to a label $(a+1, b)$ in the generating tree for $SHSYT(n+2, n+1, n)$. By restricting the correspondence to certain labels of these two generating trees, we obtain that $|A_{2n}(4312)| = |SHSYT(n+2, n, n-2)|$. Since $SHSYT(n+2, n, n-2)$ and $A_{2n}(1234)$ are both enumerated by $C_n^{(3)}$, we arrive at (1.12). Moreover, we obtain a generating tree for $A_{2n}(4312)$, leading to another proof of relation (1.11).

2 Generating trees for $A_{2n+1}(1243)$ and $A_{2n}(1243)$

In this section, we construct a generating tree P_{1243} for $A_{2n+1}(1243)$ which turns out to be the same as the generating tree for $A_{2n+1}(2143)$ given by Lewis. This proves (1.11), that is, $|A_{2n+1}(1243)| = |A_{2n+1}(2143)|$ for $n \geq 0$. We also obtain a generating tree Q_{1243} for $A_{2n}(1243)$ that is easily seen to be isomorphic to the generating tree for $A_{2n}(1234)$ given by Lewis. This provides an alternative proof of relation (1.6), that is, $|A_{2n}(1243)| = |A_{2n}(1234)|$ for $n \geq 1$.

Theorem 2.1 *The generating tree P_{1243} for $\{A_{2n+1}(1243)\}_{n\geq 0}$ is given by*

$$\begin{cases} \textit{root}: & (0,2), \\ \textit{rule}: & (a,b) \mapsto \{(x,y)\,|\,1 \leq x \leq a+1 \textit{ and } x+2 \leq y \leq b+2\}. \end{cases} \tag{2.1}$$

Theorem 2.2 *The generating tree Q_{1243} for $\{A_{2n}(1243)\}_{n\geq 1}$ is given by*

$$\begin{cases} \textit{root}: & (1,3), \\ \textit{rule}: & (a,b) \mapsto \{(x,y)\,|\,1 \leq x \leq a+1 \textit{ and } x+2 \leq y \leq b+2\}. \end{cases} \tag{2.2}$$

It is clear that the above generating tree Q_{1243} is isomorphic to the following generating tree for $\{A_{2n}(1234)\}_{n\geq 1}$ due to Lewis

$$\begin{cases} \textit{root}: & (2,3), \\ \textit{rule}: & (a,b) \mapsto \{(x,y)\,|\,2 \leq x \leq a+1 \text{ and } x+1 \leq y \leq b+2\}. \end{cases} \tag{2.3}$$

To present the proofs of the above theorems, let us give an overview of the terminology on generating trees. Given a sequence $\{\Sigma_n\}_{n\geq 1}$ of finite sets with $|\Sigma_1| = 1$, a generating tree is a rooted, labeled tree such that the vertices at level n are the elements of Σ_n and the label of each vertex determines the labels of its children. A generating tree may be described by a recursive definition consisting of

(1) the label of the root,

(2) a set of succession rules explaining how to derive, given the label of a parent, the labels of its children.

A generating tree for a sequence $\{\Sigma_n\}_{n\geq 1}$ is also called a generating tree for the set Σ_n. To illustrate the idea of generating trees, we consider the construction of a generating tree for S_n. We need to determine the children of each permutation in S_n. Given $\pi \in S_n$, we can generate $n+1$ permutations in S_{n+1}. For $1 \leq i \leq n+1$, let $i \mapsto \pi$ denote the permutation $\tau = \tau_1\tau_2\cdots\tau_{n+1}$ in S_{n+1} generated by π, where $\tau_1 = i$ and $\tau_2\tau_3\cdots\tau_{n+1}$ is order isomorphic to π. Apparently, $i \mapsto \pi$ can be obtained from π by adding i to the beginning of π and increasing each element in $\{i, i+1, \ldots, n\}$ by 1. For example, $3 \mapsto 3142 = 34152$ is a child of 3142 in the generating tree.

Notice that Lewis [7] used the notation $\pi \leftarrow i$ to denote the permutation $\tau = \tau_1\tau_2\cdots\tau_{n+1}$ in S_{n+1} such that $\tau_{n+1} = i$ and $\tau_1\tau_2\cdots\tau_n$ is order isomorphic to π. The idea of a generating tree is to give succession rules for the structure of the generating tree by assigning labels to the vertices. For the case of permutations, given $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, we associate it with a label (π_1, n). Then we obtain the following generating tree for S_n

$$\begin{cases} \mathit{root}: & (1,1), \\ \mathit{rule}: & (i,n) \rightarrow \{(j, n+1) \,|\, 1 \leq j \leq n+1\}. \end{cases}$$

By the recursive construction of alternating permutations, Lewis [7] obtained generating schemes for A_{2n} and $A_{2n}(\sigma)$, where σ is a given permutation pattern in S_k. Here we describe the recursive constructions of A_{2n} and $A_{2n}(\sigma)$ by adding elements at the beginning.

For $n \geq 1$, let $u = u_1u_2\cdots u_{2n}$ be an alternating permutation in A_{2n}. The generating tree is constructed based on the following procedure. Consider alternating permutations $w = w_1w_2w_3\cdots w_{2n+2}$ in A_{2n+2} such that $w_3w_4\cdots w_{2n+2}$ is order isomorphic to u. Such permutations are set to be the children of u in the generating tree. One can also use this recursive procedure to generate pattern avoiding alternating permutations. Given $u \in A_{2n}(\sigma)$, where σ is a permutation pattern in S_k, the set of children of u is given by

$$\{w \mid w = v_1 \mapsto (v_2 \mapsto u),\ w \in A_{2n+2}(\sigma)\}.$$

Analogously, one can describe the procedure to generate pattern avoiding alternating permutations of odd length.

We now proceed to construct the generating trees P_{1243} and Q_{1243} for $A_{2n+1}(1243)$ and $A_{2n}(1243)$, respectively. It turns out that these two generating trees have the same succession rules, but with different roots. Here we shall only present the derivation of the succession rules of P_{1243}. First, we introduce two statistics on sequences of positive integers. For $n \geq 0$, let $s = s_1s_2\cdots s_n$ be a sequence of positive integers. We define

$$f(s) = \max\{0,\ s_j \,|\, \text{there exists } i \text{ such that } i < j \text{ and } s_i > s_j\},$$

$$e(s) = \max\{0,\ s_i \,|\, \text{there exist } j \text{ and } k \text{ such that } i < j < k \text{ and } s_i < s_k < s_j\}.$$

For example, let $s = 48152967$, which is a permutation of $[9]\backslash\{3\}$. Then we have $f(s) = 7$ and $e(s) = 5$. In fact, for any permutation $u \in S_n$, $f(u) = 0$ if u contains no 21-pattern; otherwise, $f(u)$ is the largest entry among the smaller elements of 21-patterns of u. Similarly, $e(u) = 0$ if u contains no 132-pattern; otherwise, $e(u)$ is the largest entry among the smallest elements of 132-patterns of u. Since each 132-pattern contains a 21-pattern, we have $f(u) \geq e(u)$. Moreover, $f(u) = e(u)$ if and only if $u = 12\cdots n$.

To derive the succession rules of P_{1243}, we need to characterize the set of 1243-avoiding alternating permutations in A_{2n+3} that are generated by an alternating permutation u in $A_{2n+1}(1243)$. Recall that for an alternating permutation u in $A_{2n+1}(1243)$, a permutation

$w \in A_{2n+3}(1243)$ is said to be a child of u if w is of the form $w = v_1 \mapsto (v_2 \mapsto u)$. The following theorem shows exactly how to generate the set of children of an alternating permutation in $A_{2n+1}(1243)$.

Theorem 2.3 *For $n \geq 0$, given a permutation $u = u_1u_2 \cdots u_{2n+1} \in A_{2n+1}(1243)$, the set of children of u consists of sequences of the form $w = v_1 \mapsto (v_2 \mapsto u)$, where*

$$e(u) < v_1 \leq v_2, \tag{2.4}$$

$$max\{u_1 + 1, f(u) + 1\} \leq v_2 \leq 2n + 2. \tag{2.5}$$

Proof. Suppose that $w = w_1w_2 \cdots w_{2n+3}$ is a child of u, that is, w is an alternating permutation in $A_{2n+3}(1243)$ and it is of the form $w = v_1 \mapsto (v_2 \mapsto u)$. We proceed to prove relations (2.4) and (2.5). Since w is alternating on $[2n+3]$, we have $v_1 \leq v_2 \leq 2n+2$ and $v_2 \geq u_1 + 1$. By the order of the insertions of v_1 and v_2, we see that $w_1 = v_1$ and $w_2 = v_2 + 1$. Since w is 1243-avoiding, we claim that $v_2 \geq f(u) + 1$. Assume to the contrary that $v_2 \leq f(u)$. Then there exist $i < j$ such that $u_i > u_j$ and $v_2 \leq u_j$. This implies that $w_1w_2w_{i+2}w_{j+2}$ forms a 1243-pattern, a contradiction. So the claim is proved. We continue to show that $v_1 > e(u)$. Assume to the contrary that $v_1 \leq e(u)$. Then there exist $i < j < k$ such that $u_i < u_k < u_j$ and $v_1 \leq u_i$. Using the fact that $w_{i+2} > u_i \geq v_1$, we deduce that $w_1w_{i+2}w_{j+2}w_{k+2}$ is of pattern 1243, a contradiction. This proves that $v_1 > e(u)$. Hence (2.4) and (2.5) are verified.

Conversely, we assume that $w = v_1 \mapsto (v_2 \mapsto u)$, where v_1 and v_2 satisfy conditions (2.4) and (2.5). We proceed to show that w is an alternating permutation in $A_{2n+3}(1243)$. Since u is alternating, it is easy to see that w is alternating. It remains to verify that w is 1243-avoiding. Assume to the contrary that w contains a 1243-pattern, that is, there exist $t < i < j < k$ such that $w_tw_iw_jw_k$ is of pattern 1243. Since u is 1243-avoiding, we deduce that $t = 1$ or 2. If $w_2w_iw_jw_k$ forms a 1243-pattern, then $w_1w_iw_jw_k$ is also a 1243-pattern. Hence we can always choose $t = 1$. To prove w is 1243-avoiding, we show that it is impossible for $w_1w_iw_jw_k$ to form a 1243-pattern.

We now assume that $w_1w_iw_jw_k$ forms a 1243-pattern. If $i = 2$, we have $w_2 < w_k$. Since $w_2 = v_2+1$ and $w_k \leq u_{k-2}+2$, we get $v_2 \leq u_{k-2}$. Using the fact that $u_{j-2}u_{k-2}$ forms a 21-pattern, we find that $u_{k-2} \leq f(u)$. It follows that $v_2 \leq f(u)$, which contradicts the condition that $v_2 \geq f(u) + 1$. Hence we get $i > 2$.

We claim that $w_1 \leq u_{i-2}$. Assume to the contrary that $w_1 > u_{i-2}$. Since $w_1 = v_1$ and $v_1 \leq v_2$, we see that $u_{i-2} < v_1 \leq v_2$. Moreover, we have $w_i = u_{i-2}$ since $w = v_1 \mapsto (v_2 \mapsto u)$. This yields $w_i < w_1$, which contradicts the assumption that $w_1w_iw_jw_k$ forms a 1243-pattern. So the claim is proved. Clearly, $u_{i-2}u_{j-2}u_{k-2}$ is a 132-pattern. It follows that $u_{i-2} \leq e(u)$. Thus $v_1 = w_1 \leq u_{i-2} \leq e(u)$, which contradicts the condition $v_1 > e(u)$. So the assumption that $w_1w_iw_jw_k$ is a 1243-pattern is false. Hence w is 1243-avoiding and the proof is complete. ∎

In light of the above characterization, we are led to a labeling scheme for alternating permutations in $A_{2n+1}(1243)$. For $u \in A_{2n+1}(1243)$, we associate a label (a, b) to u, where

$$a = 2n + 2 - \max\{u_1 + 1, f(u) + 1\}, \tag{2.6}$$

$$b = 2n + 2 - e(u). \tag{2.7}$$

For example, the permutation $1 \in A_1(1243)$ has label $(0, 2)$, and the permutation $2546173 \in A_7(1243)$ has label $(3, 6)$.

The above labeling scheme enables us to derive succession rules of the generating tree P_{1243} for $A_{2n+1}(1243)$.

Theorem 2.4 *For $n \geq 0$, given $u = u_1u_2 \cdots u_{2n+1} \in A_{2n+1}(1243)$ with label (a, b) , the set of labels of children of u is given by*

$$\{(x, y) \,|\, 1 \leq x \leq a + 1, x + 2 \leq y \leq b + 2\}.$$

Proof. Assume that $w = v_1 \mapsto (v_2 \mapsto u)$ is a child of u. Write $w = w_1w_2 \cdots w_{2n+3}$. According to Theorem 2.3, we have $e(u) < v_1 \leq v_2$ and $\max\{u_1 + 1, f(u) + 1\} \leq v_2 \leq 2n+2$. Let (x, y) be the label of w. Since $w \in A_{2n+3}(1243)$, from the labeling rules (2.6) and (2.7) it follows that

$$x = 2n + 4 - \max\{w_1 + 1, f(w) + 1\}, \tag{2.8}$$

$$y = 2n + 4 - e(w). \tag{2.9}$$

To determine the range of the label (x, y), we proceed to compute $f(w)$ and $e(w)$. Notice that the insertions of v_1 and v_2 to u may cause new 21-patterns and new 132-patterns. Let $s = w_3w_4 \cdots w_{2n+3}$. To determine $f(w)$, it suffices to compare $f(s)$ with the smaller element in each new 21-pattern. Similarly, $e(w)$ can be obtained by comparing $e(s)$ with the smallest element in each new 132-pattern. Here are two cases.

Case 1: $e(u) < v_1 < v_2$. It is easily seen that $e(s) = e(u)$. To compute $e(w)$, we consider new 132-patterns caused by the insertions of v_1 and v_2 into u. Since $w_2 = v_2 + 1$ and $v_2 > f(u)$, we find that w_2 does not appear as the smallest entry of any 132-pattern of w. Since $v_1(v_2 + 1)v_2$ is a 132-pattern and $v_1 > e(u) = e(s)$, we see that $e(w) = \max\{v_1, e(s)\} = v_1$.

To compute $f(w)$, we first determine $f(s)$. There are two cases. If $e(u) < v_1 \leq f(u)$, then $f(u) \neq 0$. Noting that $v_2 > f(u) \geq v_1$, we get $f(s) = f(u) + 1$. If $f(u) < v_1 < v_2$, it is obvious that $f(s) = f(u)$. Therefore, in either case we have $f(s) \leq f(u) + 1$.

We now consider new 21-patterns caused by the insertions of v_1 and v_2 into u. Since $v_1 < v_2$ and $w_2 = v_2 + 1$, we see that $(v_2 + 1)v_2$ is a 21-pattern of w. Moreover, it can be seen that v_2 is the largest among the smaller elements of the newly formed 21-patterns. From the fact that $f(s) \leq f(u) + 1 \leq v_2$, we deduce that $f(w) = \max\{v_2, f(s)\} = v_2$. Hence from (2.8) and (2.9) we have

$$x = 2n + 4 - \max\{w_1 + 1, f(w) + 1\} = 2n + 3 - v_2,$$

$$y = 2n + 4 - e(w) = 2n + 4 - v_1.$$

Since $e(u) < v_1 < v_2$ and $\max\{u_1 + 1, f(u) + 1\} \le v_2 \le 2n + 2$, we obtain

$$1 \le x \le 2n + 3 - \max\{u_1 + 1, f(u) + 1\}, \tag{2.10}$$

$$2n + 5 - v_2 \le y \le 2n + 3 - e(u). \tag{2.11}$$

Since $a = 2n + 2 - \max\{u_1 + 1, f(u) + 1\}$ and $b = 2n + 2 - e(u)$ as given in (2.6) and (2.7), we may rewrite (2.10) and (2.11) as follows

$$1 \le x \le a + 1, \tag{2.12}$$

$$x + 2 \le y \le b + 1. \tag{2.13}$$

It is easily checked that for any pair (x, y) of integers satisfying conditions (2.12) and (2.13), there exists a unique child $w = v_1 \mapsto (v_2 \mapsto u)$ of u with the label (x, y) such that $e(u) < v_1 < v_2$. Consequently, the set of labels of children of u considered in this case is given by

$$\{(x, y) \,|\, 1 \le x \le a + 1 \text{ and } x + 2 \le y \le b + 1\}.$$

Case 2: $v_1 = v_2$. By (2.5), namely, $\max\{u_1 + 1, f(u) + 1\} \le v_2 \le 2n + 2$, we have $v_1 = v_2 > f(u)$. It follows that $f(s) = f(u)$. Clearly, $(v_2+1)(v_2-1)$ is a 21-pattern of w. Moreover, it is obvious that v_2-1 is the largest among the smaller elements in the newly formed 21-patterns. Since $v_2 - 1 \ge f(u) = f(s)$, we have $f(w) = \max\{v_2 - 1, f(s)\} = v_2 - 1$.

By (2.4), namely, $e(u) < v_1 \le v_2$, we deduce that $e(s) = e(u)$. Since $v_1 = v_2 > f(u)$, the insertions of v_1 and v_2 do not create any new 132-pattern. It yields that $e(w) = e(s) = e(u)$. From the labeling rules (2.8) and (2.9) we have

$$x = 2n + 4 - \max\{w_1 + 1, f(w) + 1\} = 2n + 3 - v_2, \tag{2.14}$$

$$y = 2n + 4 - e(w) = 2n + 4 - e(u). \tag{2.15}$$

Since $f(u) \ge e(u)$, using (2.4) and (2.5) we get

$$\max\{u_1 + 1, f(u) + 1\} \le v_1 = v_2 \le 2n + 2.$$

This implies that

$$1 \le x \le 2n + 3 - \max\{u_1 + 1, f(u) + 1\}. \tag{2.16}$$

Combining (2.6) and (2.16), we get $1 \le x \le a + 1$. By the labeling rule (2.7), (2.15) becomes $y = b + 2$. Conversely, for any pair (x, y) of integers satisfying $1 \le x \le a + 1$ and $y = b + 2$, there exists a unique child $w = v_1 \mapsto (v_2 \mapsto u)$ of u with label (x, y) such that $v_1 = v_2$. Hence the set of labels of children of u considered in this case is given by

$$\{(x, y) \,|\, 1 \le x \le a + 1 \text{ and } y = b + 2\}.$$

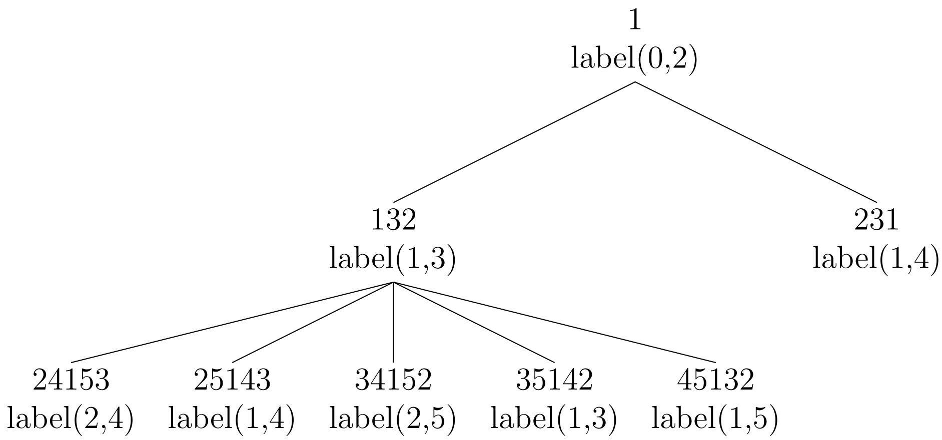

Figure 2.1: The first few levels of the generating tree P_{1243}

Combining Case 1 and Case 2, the set of labels of children of u is given by

$$\{(x, y) \,|\, 1 \leq x \leq a+1,\; x+2 \leq y \leq b+2\}.$$

as required. ∎

Using Theorem 2.4, we obtain the generating tree P_{1243} given in Theorem 2.1. Figure 2.1 gives the first few levels of the generating tree P_{1243}.

Comparing the above description of the generating tree P_{1243} and the generating tree for $A_{2n+1}(2143)$ as given by Lewis [7], we arrive at the assertion that there is a bijection between $A_{2n+1}(1243)$ and $A_{2n+1}(2143)$. This proves relation (1.11).

The construction of the generating tree P_{1243} can be easily adapted to give the generating tree Q_{1243}. The following theorem provides a similar characterization of the set of 1243-avoiding alternating permutations in A_{2n+2} that are generated by an alternating permutation u in $A_{2n}(1243)$.

Theorem 2.5 *For $n \geq 1$, given a permutation $u = u_1u_2 \cdots u_{2n} \in A_{2n}(1243)$, the set of children of u consists of sequences of the form $w = v_1 \mapsto (v_2 \mapsto u)$, where*

$$e(u) < v_1 \leq v_2, \tag{2.17}$$

$$max\{u_1+1, f(u)+1\} \leq v_2 \leq 2n+1. \tag{2.18}$$

Based on the above characterization, we assign a label (a, b) to an alternating permutation $u = u_1u_2 \cdots u_{2n} \in A_{2n}(1243)$, where

$$a = 2n+1-\max\{u_1+1, f(u)+1\},$$
$$b = 2n+1-e(u).$$

It is easy to check that the succession rules of Q_{1243} are exactly the same as the succession rules of P_{1243}. Note that $12 \in A_2(1243)$ has label $(1,3)$. So we obtain the generating tree Q_{1243} as given in Theorem 2.2.

It is clear that the generating tree Q_{1243} is isomorphic to the generating tree for $A_{2n}(1234)$ given in (2.3) via the correspondence $(a,b) \to (a+1,b)$. This gives another proof of relation (1.6), which was proved by Bóna [2] by a direct bijection.

3 A restriction of the generating tree P_{1243}

In this section, we give a proof of relation (1.12) conjectured by Lewis. We notice that the generating tree P_{1243} for $A_{2n+1}(1243)$ is isomorphic to the generating tree for the set of shifted standard Young tableaux of shape $(n+2, n+1, n)$ as given by Lewis [7]. By restricting this isomorphism to certain labels of these two generating trees, we obtain a bijection between a subset of $A_{2n+1}(1243)$ and a subset of $SHSYT(n+2, n+1, n)$, which leads to the following assertion.

Theorem 3.1 *For $n \geq 2$, we have $|A_{2n}(4312)| = |SHSYT(n+2, n, n-2)|$.*

By the formula for the number of shifted standard Young tableaux of a given shape, we have $|SHSYT(n+2, n, n-2)| = C_n^{(3)}$, where $C_n^{(3)}$ is the n-th 3-dimensional Catalan number as given by (1.1). Lewis [7] proved that $|A_{2n}(1234)| = C_n^{(3)}$. So by Theorem 3.1, we obtain $|A_{2n}(4312)| = |A_{2n}(1234|$ for $n \geq 2$. Note that this relation also holds for $n = 1$. This proves the relation (1.12).

Let us recall some notation and terminology on partitions and shifted standard Young tableaux. A sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of positive integers is said to be a partition of n if $n = \lambda_1 + \lambda_2 + \cdots + \lambda_m$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$, where each λ_i is called a part of λ. A Young diagram of shape λ is defined to be a left-justified array of n boxes with λ_1 boxes in the first row, λ_2 boxes in the second row and so on. If λ is a partition with distinct parts, then the shifted Young diagram of shape λ is an array of cells with m rows, where each row is indented by one cell to the right with respect to the previous row, and there are λ_i cells in row i.

A standard Young tableau of shape λ is a Young diagram of λ whose boxes are filled with the numbers $1, 2, \ldots, n$ such that the entries are increasing along each row and each column. A shifted standard Young tableau of shape λ is a filling of a shifted Young diagram with the numbers $1, 2, \ldots, n$ such that the entries are increasing along each row and each column. We denote by $SHSYT(\lambda)$ the set of shifted standard Young tableaux of shape λ. Figure 3.1 gives examples of a standard Young tableau and a shifted standard Young tableau.

As shown by Lewis [7], $A_{2n}(1234)$ is enumerated by the n-th 3-dimensional Catalan number $C_n^{(3)}$. To prove relation (1.12), it suffices to demonstrate that $A_{2n}(4312)$ is

1	3	7	9
2	5		
4	6		
8			

1	2	4	8	10
	3	5	6	
		7	9	
			11	

Figure 3.1: A standard Young tableau of shape $(4, 2, 2, 1)$ and a shifted standard Young tableau of shape $(5, 3, 2, 1)$

also counted by $C_n^{(3)}$. In light of the correspondence between 4312-avoiding alternating permutations and 1243-avoiding down-up permutations via complementation, we proceed to consider the generating tree for $A_{2n+1}(1243)$.

It turns out that the generating tree P_{1243} is isomorphic to the following generating tree for $SHSYT(n + 2, n + 1, n)$ obtained by Lewis [7]:

$$\begin{cases} root\colon & (1, 2), \\ rule\colon & (a, b) \mapsto \{(x, y) \,|\, 2 \le x \le a + 1 \text{ and } x + 1 \le y \le b + 2\}. \end{cases} \tag{3.1}$$

The above generating tree is based on the following labeling scheme. Assume that $T \in SHSYT(n + 2, n + 1, n)$, and let $T(i, j)$ denote the entry of T in the i-th row and the j-th column. We associate T with a label (a, b), where

$$a = 3n + 4 - T(2, n + 2), \tag{3.2}$$
$$b = 3n + 4 - T(1, n + 2). \tag{3.3}$$

The isomorphism can be easily established by mapping a label (a, b) in (3.1) to a label $(a - 1, b)$ in (2.1). Thus for $n \ge 0$, we have

$$|A_{2n+1}(1243)| = |SHSYT(n + 2, n + 1, n)|. \tag{3.4}$$

The above isomorphism between the generating tree P_{1243} and the generating tree for $SHSYT(n + 2, n + 1, n)$ can be restricted to certain labels. Let P_n be the set of alternating permutations in $A_{2n+1}(1243)$ with labels of the form $(1, b)$ and Q_n be the set of shifted standard Young tableaux in $SHSYT(n + 2, n + 1, n)$ with labels of the form $(2, b)$. By the isomorphism between the two generating trees, we see that $|P_n| = |Q_n|$.

To prove Theorem 3.1, we shall show that for $n \ge 1$,

$$|P_n| = |A_{2n-1}(1243)| + |A_{2n}(4312)|, \tag{3.5}$$

and for $n \ge 2$,

$$|Q_n| = |SHSYT(n + 1, n, n - 1)| + |SHSYT(n + 2, n, n - 2)|. \tag{3.6}$$

Substituting (3.4) with n replaced by $n-1$ into (3.5), we find that $n \geq 1$,

$$|P_n| = |SHSYT(n+1, n, n-1)| + |A_{2n}(4312)|. \tag{3.7}$$

Since $|P_n| = |Q_n|$ for $n \geq 1$, comparing (3.6) and (3.7) yields $|A_{2n}(4312)| = |SHSYT(n+2, n, n-2)|$ for $n \geq 2$, as asserted in Theorem 3.1.

To prove (3.5), we need a characterization of alternating permutations in P_n.

Lemma 3.2 *For $n \geq 0$, an alternating permutation $u = u_1 u_2 \cdots u_{2n+1} \in A_{2n+1}(1243)$ is in P_n if and only if $u_2 = 2n+1$, that is,*

$$P_n = \{u \,|\, u = u_1 u_2 \cdots u_{2n+1} \in A_{2n+1}(1243),\ u_2 = 2n+1\}. \tag{3.8}$$

Proof. Assume that $u = u_1 u_2 \cdots u_{2n+1}$ is an alternating permutation in P_n. By the definition of P_n, we see that u has a label of the form $(1, b)$. Using the labeling scheme of P_{1243}, we have $2n+2-\max\{u_1+1, f(u)+1\} = 1$. It follows that $u_1 = 2n$ or $f(u) = 2n$. When $u_1 = 2n$, we have $u_2 = 2n+1$ since $u_1 < u_2 \leq 2n+1$. When $f(u) = 2n$, by the definition of $f(u)$, we find that $2n+1$ precedes $2n$ in u since $(2n+1)(2n)$ is the only 21-pattern in u with $2n$ being the smaller element. We claim that $u_2 = 2n+1$. Assume to the contrary that $u_2 < 2n+1$. Then $u_1 u_2 (2n+1)(2n)$ forms a 1243-pattern of u, a contradiction. Hence the claim is valid. So we have shown that $u_2 = 2n+1$.

Conversely, assume that $u \in A_{2n+1}(1243)$ and $u_2 = 2n+1$. We proceed to show that $u \in P_n$, namely, $2n+2-\max\{u_1+1, f(u)+1\} = 1$. Here are two cases. If $u_1 = 2n$, by the definition of $f(u)$, we have $f(u) < 2n$. It follows that $2n+2-\max\{u_1+1, f(u)+1\} = 1$. If $u_1 \neq 2n$, from the assumption that $u_2 = 2n+1$ we see that $2n$ appears after $(2n+1)$ in u, that is, $(2n+1)(2n)$ forms a 21-pattern of u. Hence we have $f(u) = 2n$. It can be checked that in this case we also have $2n+2-\max\{u_1+1, f(u)+1\} = 1$. This completes the proof. ∎

The following correspondence implies relation (3.5).

Theorem 3.3 *For $n \geq 1$, there is a bijection between P_n and $A_{2n}(4312) \cup A_{2n-1}(1243)$.*

Proof. We divide P_n into two subsets P_n' and P_n'', where

$$P_n' = \{u \,|\, u = u_1 u_2 \cdots u_{2n+1} \in A_{2n+1}(1243), u_2 = 2n+1 \text{ and } u_1 > u_3\},$$
$$P_n'' = \{u \,|\, u = u_1 u_2 \cdots u_{2n+1} \in A_{2n+1}(1243), u_2 = 2n+1 \text{ and } u_1 < u_3\}.$$

We proceed to show that there is a bijection between P_n' and $A_{2n}(4312)$ and there is a bijection between P_n'' and $A_{2n-1}(1243)$.

First, we define a map $\varphi\colon P_n' \to A_{2n}(4312)$. Given $v = v_1 v_2 \cdots v_{2n+1} \in P_n'$, let $\varphi(v) = \pi^c$, where $\pi = v_1 v_3 v_4 \cdots v_{2n+1}$. Clearly, we have $\pi \in A_{2n}'(1243)$. It follows that $\varphi(v) = \pi^c \in A_{2n}(4312)$.

To prove that φ is a bijection, we construct the inverse of φ. Define a map $\phi\colon A_{2n}(4312) \to P'_n$. Given $w = w_1 w_2 \cdots w_{2n} \in A_{2n}(4312)$, let

$$\phi(w) = \tau = (2n+1-w_1)(2n+1)(2n+1-w_2)\cdots(2n+1-w_{2n}).$$

We claim that τ is 1243-avoiding. Since w is 4312-avoiding, by complementation we see that $\tau_1\tau_3\tau_4\cdots\tau_{2n+1}$ is 1243-avoiding. Note that $\tau_2 = 2n+1$ does not occur in any 1243-pattern of τ. So the claim is verified. Evidently, τ is alternating, and hence we have $\tau \in A_{2n+1}(1243)$. Since w is alternating, we have $w_1 < w_2$, and so $\tau_1 > \tau_3$. It follows that $\tau \in P'_n$. Moreover, it can be checked that ϕ is the inverse of φ. So we conclude that φ is a bijection between P'_n and $A_{2n}(4312)$.

We next construct a bijection between P''_n and $A_{2n-1}(1243)$. Given an alternating permutation $u = u_1u_2\cdots u_{2n+1}$ in P''_n, define $\psi(u) = st(r)$, where $r = u_1u_3u_5u_6\cdots u_{2n}u_{2n+1}$ and $st(r)$ is the permutation of $[2n-1]$ which is order isomorphic to r.

We aim to show that $\psi(u) \in A_{2n-1}(1243)$. Since $u \in P''_n$, we find that $u_1 < u_3 < u_4$ and $u_2 = 2n+1$. We claim that $u_3 + 1 = u_4$. Otherwise, $u_1u_3u_4(u_3+1)$ would form a 1243-pattern of u, contradicting the fact u is 1243-avoiding. Since $u_4 > u_5$, we deduce that $u_3 > u_5$. It follows that $\psi(u)$ is an alternating permutation of length $2n-1$. It is clear that $\psi(u)$ is 1243-avoiding. So we deduce that $\psi(u) \in A_{2n-1}(1243)$.

To prove that ψ is a bijection, we describe the inverse of ψ. Given $q = q_1q_2\cdots q_{2n-1}$ in $A_{2n-1}(1243)$, define $\theta(q) = p$, where $p = p_1p_2\cdots p_{2n+1}$ is obtained from q by inserting $2n+1$ after q_1 and inserting q_2+1 after q_2, and increasing each element of q which is not less than q_2+1 by 1. For example, for $q = 34152 \in A_5(1243)$, we have $p = \theta(q) = 3745162$.

We need to show that $p = \theta(q)$ is an alternating permutation in P''_n. By the construction of p, we have $p_1 = q_1$, $p_2 = 2n+1$, $p_3 = q_2$, $p_4 = q_2+1$ and $p_5 = q_3$. Now it is not difficult to check that p is alternating.

Next we show that p is 1243-avoiding. Assume to the contrary that $p_tp_ip_jp_k$ forms a 1243-pattern of p, where $t < i < j < k$. Since q is 1243-avoiding, from the construction of p, we see that $p_tp_ip_jp_k$ contains either p_2 or p_4. Since $p_2 = 2n+1$ does not occur in any 1243-pattern, p_4 appears in $p_tp_ip_jp_k$. Moreover, p_3 appears in $p_tp_ip_jp_k$. If not, since $p_3+1 = p_4$, then there is a 1243-pattern which does not contain p_4 by replacing p_4 with p_3 in $p_tp_ip_jp_k$, which contradicts the fact that p_4 appears in $p_tp_ip_jp_k$. Thus $p_tp_ip_jp_k$ contains both p_3 and p_4.

To prove that p is 1243-avoiding, it is sufficient for us to show that $p_tp_ip_jp_k$ does not contain both p_3 and p_4. Assume that both p_3 and p_4 appear in $p_tp_ip_jp_k$. Since $p_3 < p_4$, by the assumption that $p_tp_ip_jp_k$ forms a 1243-pattern, we have either $p_tp_i = p_3p_4$ or $p_ip_j = p_3p_4$. If $p_tp_i = p_3p_4$, that is, $p_3p_4p_jp_k$ is a 1243-pattern of p, then $p_1p_3p_jp_k$ forms a 1243-pattern since $p_1 < p_3 < p_4$, contradicting the assertion that p_4 must appear in any 1243-pattern of p as shown before. Hence we have $p_tp_i \neq p_3p_4$. It follows that $p_ip_j = p_3p_4$, namely, $p_tp_3p_4p_k$ is a 1243-pattern of p. However, this is not possible since $p_3+1 = p_4$. So the claim is verified, that is, p is 1243-avoiding.

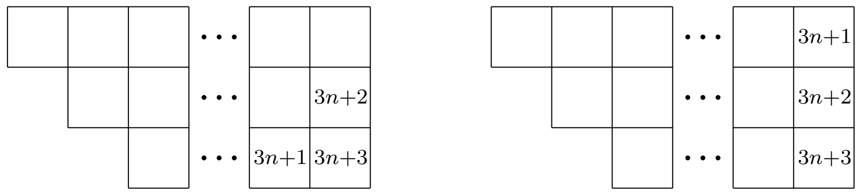

Figure 3.2: The two cases when $T(2, n+2) = 3n+2$

It is easy to check that $p \in P''_n$. By the definition of P''_n, we only need to verify that $p_2 = 2n+1$ and $p_1 < p_3$. But this is obvious from the construction of p. Thus θ is a map from $A_{2n-1}(1243)$ to P''_n. Moreover, it is not difficult to verify that $\theta = \psi^{-1}$. Hence ψ is a bijection between P''_n and $A_{2n-1}(1243)$. ∎

The following theorem gives a bijection for relation (3.6). Recall that Q_n is the set of shifted standard Young tableaux in $SHSYT(n+2, n+1, n)$ with labels of the form $(2, b)$.

Theorem 3.4 *For $n \geq 2$, there is a bijection between Q_n and*

$$SHSYT(n+1, n, n-1) \cup SHSYT(n+2, n, n-2).$$

Proof. By the definition of Q_n and the labeling scheme of the generating tree for $SHSYT(n+2, n+1, n)$, we have

$$Q_n = \{T \mid T \in SHSYT(n+2, n+1, n),\, T(2, n+2) = 3n+2\}. \tag{3.9}$$

Let

$$Q'_n = \{T \mid T \in Q_n,\ T(3, n+1) = 3n+1\},$$
$$Q''_n = \{T \mid T \in Q_n,\ T(1, n+2) = 3n+1\}.$$

Clearly, $Q'_n \cap Q''_n = \emptyset$. We claim that

$$Q_n = Q'_n \cup Q''_n.$$

In other words, for any $T \in Q_n$ we have either $T(3, n+1) = 3n+1$ or $T(1, n+2) = 3n+1$.

Let T be a shifted standard Young tableau in Q_n. By (3.9) we have $T(2, n+2) = 3n+2$. Since T is filled with $1, 2, \ldots, 3n+3$ and T contains the cell $(3, n+2)$, we have $T(3, n+2) = 3n+3$. Moreover, it can be seen that the element $3n+1$ appears in the cell $(3, n+1)$ or in the cell $(1, n+2)$. This proves $Q_n = Q'_n \cup Q''_n$. Figure 3.2 gives an illustration of the two cases when $T(2, n+2) = 3n+2$.

We now define a map χ from Q_n to the set $SHSYT(n+1, n, n-1) \cup SHSYT(n+2, n, n-2)$. Let T be a shifted standard Young tableau in Q_n. If $T \in Q'_n$, then let $\chi(T) = T_1$, where T_1 is obtained from T by deleting the boxes $T(2, n+2)$, $T(3, n+1)$ and $T(3, n+2)$. If $T \in Q''_n$, then let $\chi(T) = T_2$, where T_2 is obtained from T by deleting the boxes $T(1, n+2)$, $T(2, n+2)$ and $T(3, n+2)$. It is easily verified that χ is a bijection between Q_n and $SHSYT(n+1, n, n-1) \cup SHSYT(n+2, n, n-2)$, since we can recover the shifted standard Young tableau T from T_1 or T_2 depending on the shape of T_1 or T_2. Thus χ is the required bijection. ∎

4 A generating tree for $A_{2n}(4312)$

In this section, we construct a generating tree Q_{4312} for $A_{2n}(4312)$. While this generating tree is not isomorphic to the generating tree for $A_{2n}(1234)$ given by Lewis, it leads to an alternative proof of relation (1.11), namely, $|A_{2n+1}(1243)| = |A_{2n+1}(2143)|$ for $n \geq 0$. To be more specific, by deleting the leaves of the generating tree Q_{4312} and changing every label (a, b) to $(a-1, b)$, we are led to the generating tree for $A_{2n}(3412)$ as given by Lewis [7]. By restricting this correspondence to certain labels, we obtain relation (1.11).

Theorem 4.1 *The generating tree Q_{4312} for $\{A_{2n}(4312)\}_{n\geq 1}$ is given by*

$$\begin{cases} \mathit{root}: & (2,3), \\ \mathit{rule}: & (a,b) \mapsto \{(x,y) \,|\, 2 \leq x \leq a+1 \text{ and } a+2 \leq y \leq b+2\} \\ & \qquad\qquad \cup \left\{ \binom{b-a+1}{2} \text{ occurrences of } (0,0) \right\}. \end{cases}$$

The construction of the generating tree Q_{4312} is analogous to the construction of the generating tree P_{1243} given in Section 2. First, we introduce two statistics on sequences of positive integers. For $n \geq 0$, let $s = s_1 s_2 \cdots s_n$ be a sequence of positive integers. Define

$$g(s) = \min\{n+1,\ s_i \,|\, \text{there exist } j \text{ and } k \text{ such that } i < j < k \text{ and } s_j < s_k < s_i\},$$

$$h(s) = \min\{n+1,\ s_j \,|\, \text{there exists } i \text{ such that } i < j \text{ and } s_i < s_j\}.$$

For example, given a permutation $s = 65128743 \in S_8$, we have $g(s) = 5$ and $h(s) = 2$. In fact, for any permutation $u \in S_n$, $g(u) = n+1$ if u is 312-avoiding; otherwise $g(u)$ is the smallest among the largest elements of 312-patterns of u. Similarly, $h(u) = n+1$ if u is 12-avoiding; otherwise $h(u)$ is the smallest among the larger elements of 12-patterns of u. Since each 312-pattern contains a 12-pattern, we have $h(u) \leq g(u)$. Moreover, $h(u) = g(u)$ if and only if $u = n \cdots 21$.

Let u be an alternating permutation in $A_{2n}(4312)$. An alternating permutation w in $A_{2n+2}(4312)$ is said to be a child of u, or generated by u if w is of the form $w = v_1 \mapsto (v_2 \mapsto u)$.

Theorem 4.2 *For $n \geq 1$, given a permutation $u = u_1u_2\cdots u_{2n} \in A_{2n}(4312)$, the set of children of u consists of sequences of the form $w = v_1 \mapsto (v_2 \mapsto u)$, where*

$$1 \leq v_1 \leq v_2, \tag{4.1}$$

$$u_1 + 1 \leq v_2 \leq g(u). \tag{4.2}$$

Proof. Suppose that $w = w_1w_2\cdots w_{2n+2}$ is a child of u, that is, w is an alternating permutation in $A_{2n+2}(4312)$ and it is of the form $v_1 \mapsto (v_2 \mapsto u)$. We proceed to prove (4.1) and (4.2). Since w is alternating on $[2n+2]$, we have $1 \leq v_1 \leq v_2 \leq 2n+1$ and $v_2 \geq u_1 + 1$. Moreover, we claim that $v_2 \leq g(u)$. Otherwise, there exist $i < j < k$ such that $u_j < u_k < u_i$ and $v_2 > u_i$. This implies that $w_2w_{i+2}w_{j+2}w_{k+2}$ forms a 4312-pattern of w, contradicting the fact that w is 4312-avoiding. So the claim is verified and we are led to the relations (4.1) and (4.2).

Conversely, we assume that $w = v_1 \mapsto (v_2 \mapsto u)$, where v_1 and v_2 are integers satisfying (4.1) and (4.2). We need to show that $w \in A_{2n+2}(4312)$. Clearly, w is alternating since $v_2 \geq u_1 + 1$ and $1 \leq v_1 \leq v_2$.

It remains to check that w is 4312-avoiding. Assume to the contrary that $w_tw_iw_jw_k$ is a 4312-pattern of w, where $t < i < j < k$. We claim that we can always choose $t = 2$. It is easily seen that $w_tw_iw_jw_k$ contains either w_1 or w_2. Moreover, if $w_1w_iw_jw_k$ is a 4312-pattern, then so is $w_2w_iw_jw_k$ since $w_1 < w_2$. So the claim holds. Now we may assume that $w_2w_iw_jw_k$ is a 4312-pattern. Clearly, $v_2 > u_{i-2}$. Since $u_{i-2} \geq g(u)$, we get $v_2 > g(u)$, contradicting condition (4.2). Hence $w_2w_iw_jw_k$ does not form a 4312-pattern. This implies that w is 4312-avoiding. So we conclude that $w \in A_{2n+2}(4312)$. This completes the proof. ∎

Notice that some alternating permutations in $A_{2n}(4312)$ do not have any children. Such permutations are leaves of the generating tree. Permutations having at least one child are internal vertices of the generating tree. For example, the alternating permutation $3412 \in A_4(4312)$ is a leaf and the alternating permutation $132645 \in A_6(4312)$ is an internal vertex, since $23154867 \in A_8(4312)$ is a child.

Theorem 4.3 *For $n \geq 1$ and $u = u_1u_2\cdots u_{2n} \in A_{2n}(4312)$, u is an internal vertex in the generating tree Q_{4312} if and only if $h(u) = u_1 + 1$.*

Proof. Assume that u is an internal vertex. We proceed to show that $h(u) = u_1 + 1$. First, we claim that $u_1 \leq h(u)$. Otherwise, we may assume that $u_1 > h(u)$. By the definition of $h(u)$, we see that there exist $i < j$ such that $u_i < u_j$ and $u_1 > u_j$. Hence $u_1u_iu_j$ forms a 312-pattern. By the definition of $g(u)$, we get $g(u) \leq u_1$. In view of Theorem 4.2, we see that if u has a child, then it is of the form $v_1 \mapsto (v_2 \mapsto u)$ satisfying conditions (4.1) and (4.2), namely, $1 \leq v_1 \leq v_2$ and $u_1 + 1 \leq v_2 \leq g(u)$. Since $g(u) \leq u_1$, u has no child, which contradicts the assumption that u is an internal vertex. So the claim is verified. Again, by the definition of $h(u)$, it can be checked that $u_1 \neq h(u)$.

It follows that $u_1 < h(u)$. On the other hand, since $u_1(u_1+1)$ is a 12-pattern, it can be checked that $h(u) \leq u_1 + 1$. Thus we deduce that $u_1 < h(u) \leq u_1 + 1$, namely, $h(u) = u_1 + 1$.

Conversely, we assume that $h(u) = u_1 + 1$. We wish to show that u is an internal vertex. Since $h(u) \leq g(u)$, we obtain $u_1 + 1 \leq g(u)$. By Theorem 4.2, we see that if u has a child, then it is of the form $v_1 \mapsto (v_2 \mapsto u)$ subject to conditions (4.1) and (4.2), namely, $1 \leq v_1 \leq v_2$ and $u_1 + 1 \leq v_2 \leq g(u)$. Since $u_1 + 1 \leq g(u)$, we see that the set of children of u is nonempty. So u is an internal vertex. Thus we reach the conclusion that u is an internal vertex if and only if $h(u) = u_1 + 1$. ∎

To construct the generating tree Q_{4312} for $A_{2n}(4312)$, we give a labeling scheme for alternating permutations in $A_{2n}(4312)$. For $n \geq 1$, let $u = u_1 u_2 \cdots u_{2n}$ be an alternating permutation in $A_{2n}(4312)$. The label (a, b) of u is defined as follows

$$(a,b) = \begin{cases} (0,0), & \text{if } u \text{ is a leaf.} \\ (h(u), g(u)), & \text{if } u \text{ is an interal vertex.} \end{cases} \tag{4.3}$$

For example, let $u = 46253817 \in A_8(4312)$. Since $h(u) \neq u_1 + 1$, by Theorem 4.3, we see that u is a leaf. So the label of u is $(0, 0)$. It is easily seen that $12 \in A_2(4312)$ is an internal vertex with label $(2, 3)$.

The above labeling scheme enables us to give a characterization of the labels of children generated by u.

Theorem 4.4 *Assume that $u = u_1 u_2 \cdots u_{2n}$ is an alternating permutation in $A_{2n}(4312)$ with label (a, b). If u is an interval vertex, then it generates $\binom{b-a+1}{2}$ leaves and the set of labels of the internal vertices generated by u is given by*

$$\{(x, y) \,|\, 2 \leq x \leq a+1,\ a+2 \leq y \leq b+2\}.$$

Proof. Assume that $w = v_1 \mapsto (v_2 \mapsto u)$ is a child of u and let $w = w_1 w_2 \cdots w_{2n+2}$. According to Theorem 4.2, we have $1 \leq v_1 \leq v_2$ and $u_1 + 1 \leq v_2 \leq g(u)$. Since u is an internal vertex, it follows from Theorem 4.3 that $u_1 + 1 = h(u)$. Hence we get $h(u) \leq v_2 \leq g(u)$.

Let (x, y) be the label of w. To determine the range of (x, y), we consider when w is a leaf. Recall that if w is a leaf, then $(x, y) = (0, 0)$, and if w is an internal vertex, then $(x, y) = (h(w), g(w))$. We shall compute $h(w)$ and $g(w)$ based on v_1, v_2 and the label (a, b).

Let $s = w_3 w_4 \cdots w_{2n+2}$. To determine $h(w)$, it suffices to compare $h(s)$ with the larger element of each new 12-pattern caused by the insertions of v_1 and v_2 into u. The computation of $g(w)$ can be carried out in the same manner. Here are three cases.

Case 1: $h(u) + 1 \leq v_1 \leq v_2$ and $h(u) + 1 \leq v_2 \leq g(u)$. By the construction of w, we see that $w_1 = v_1$ and $w_2 = v_2 + 1$. Since $h(u) + 1 \leq v_1 \leq v_2$, we have $h(s) = h(u)$. Now we consider the newly formed 12-patterns caused by the insertions of v_1 and v_2 into u. Clearly, $v_1(v_1+1)$ forms a 12-pattern of w and it can be verified that v_1+1 is the smallest among the larger elements of the newly formed 12-patterns. Since $h(u) < v_1 + 1$, we deduce that $h(w) = \min\{v_1 + 1, h(s)\} = h(u)$. Notice that $h(w) \neq w_1 + 1$. By Theorem 4.3, we find that w is a leaf. Hence in this case u only generates leaves. Using the labeling scheme for the generating tree Q_{4312}, we obtain that $a = h(u)$ and $b = g(u)$. So the number of leaves generated by u is given by

$$\sum_{v_2=a+1}^{b} (v_2 - a) = 1 + 2 + \cdots + (b - a) = \binom{b-a+1}{2}.$$

Case 2: $1 \leq v_1 \leq h(u)$ and $h(u) + 1 \leq v_2 \leq g(u)$. Clearly, we have $h(s) = h(u) + 1$ and $g(s) = g(u) + 2$. To compute $h(w)$, we consider the newly formed 12-patterns caused by the insertions of v_1 and v_2 into u. First, $v_1(v_1 + 1)$ is a newly formed 12-pattern in w. Moreover, it can be seen that $v_1 + 1$ is the minimum among the larger elements in the newly formed 12-patterns. Notice that $v_1 + 1 \leq h(u) + 1 = h(s)$. So we have $h(w) = \min(h(s), v_1 + 1) = v_1 + 1$. In view of Theorem 4.3, we see w is an internal vertex.

To determine the range of (x, y), it suffices to compute $g(w)$. Let us consider the newly formed 312-patterns in w. By the assumptions of this case, we see that $w_1 = v_1 \leq h(u)$. Thus w_1 does not occur in any 312-pattern of w. Moreover, by the assumption $v_2 \geq h(u) + 1$, we find that $w_2 = v_2 + 1$ is the largest entry of a 312-pattern in w. From the fact that $v_2 + 1 < g(u) + 2 = g(s)$, we obtain that $g(w) = \min(v_2 + 1, g(s)) = v_2 + 1$. Therefore, the label of w is given by $(x, y) = (v_1 + 1, v_2 + 1)$. From the assumptions of this case, we get $2 \leq x \leq a+1$ and $a+2 \leq y \leq b+1$. Moreover, it can be easily checked that for any pair (x, y) of integers satisfying $2 \leq x \leq a + 1$ and $a + 2 \leq y \leq b + 1$, there exists a unique child $w = v_1 \mapsto (v_2 \mapsto u)$ of u with label (x, y) such that $1 \leq v_1 \leq h(u)$ and $h(u) + 1 \leq v_2 \leq g(u)$. This implies that the set of labels of children of u considered in this case is given by

$$\{(x, y) \,|\, 2 \leq x \leq a + 1, \ a + 2 \leq y \leq b + 1\}.$$

Case 3: $1 \leq v_1 \leq h(u)$ and $v_2 = h(u)$. Clearly, $h(s) = h(u)+2$. Notice that $w_1(w_1+1)$ is a 12-pattern of w and $w_1 + 1$ is the minimum of the larger elements in the newly formed 12-patterns caused by the insertions of v_1 and v_2. Since $w_1 = v_1 \leq h(u)$, we find that $h(w) = \min(w_1 + 1, h(s)) = \min(w_1 + 1, h(u) + 2) = w_1 + 1$. According to Theorem 4.3, w is an internal vertex.

It remains to determine $g(w)$. Recall that $h(u) \leq g(u)$. Hence in this case we have $v_1 \leq v_2 \leq g(u)$. It follows that $g(s) = g(u) + 2$. Since $v_1 \leq v_2 = h(u)$, we see that neither w_1 nor w_2 can be the largest entry of a 312-pattern of w. This yields that $g(w) = g(s) = g(u) + 2$. Therefore, the label of w is given by $(x, y) = (v_1 + 1, g(u) + 2)$. From the assumptions $1 \leq v_1 \leq h(u)$ and $v_2 = h(u)$ we deduce that $2 \leq x \leq a + 1$ and

$y = b+2$. It is easily verified that for any pair (x, y) of integers satisfying $2 \leq x \leq a+1$ and $y = b+2$, there exists a unique child $w = v_1 \mapsto (v_2 \mapsto u)$ of u with label (x, y) such that $1 \leq v_1 \leq h(u)$ and $v_2 = h(u)$. Thus in this case the set of labels of children of u is given by

$$\{(x, y) \,|\, 2 \leq x \leq a+1, y = b+2\}.$$

Combining the above three cases, we see that an internal vertex u with label (a, b) generates $\binom{b-a+1}{2}$ leaves and $a(b-a+1)$ internal vertices labeled by (x, y), where $2 \leq x \leq a+1$ and $a+2 \leq y \leq b+2$. ∎

If we ignore the leaves in the generating tree Q_{4312}, then we are led to the following generating tree

$$\begin{cases} \mathit{root}: & (2,3), \\ \mathit{rule}: & (a,b) \mapsto \{(x,y) \,|\, 2 \leq x \leq a+1 \text{ and } a+2 \leq y \leq b+2\}. \end{cases} \tag{4.4}$$

The above generating tree turns out to be isomorphic to the generating tree for $A_{2n}(3412)$ as given by Lewis [7]

$$\begin{cases} \mathit{root}: & (1,3), \\ \mathit{rule}: & (a,b) \mapsto \{(x,y) \,|\, 1 \leq x \leq a+1 \text{ and } a+3 \leq y \leq b+2\}. \end{cases} \tag{4.5}$$

Clearly, a label (a, b) in (4.4) corresponds to a label $(a-1, b)$ in (4.5). Let U_n be the set of alternating permutations in $A_{2n}(4312)$ with labels of the form $(2, b)$ and V_n be the set of alternating permutations in $A_{2n}(3412)$ with labels of the form $(1, b)$. The isomorphism between the above two generating trees implies $|U_n| = |V_n|$.

The following characterizations of U_n and V_n without using labels will be used to give an alternative proof of relation (1.11).

Theorem 4.5 *For $n \geq 1$, we have*

$$U_n = \{u \,|\, u = u_1u_2 \cdots u_{2n} \in A_{2n}(4312),\ u_1 = 1\}, \tag{4.6}$$

$$V_n = \{u \,|\, u = u_1u_2 \cdots u_{2n} \in A_{2n}(3412),\ u_{2n} = 2n\}. \tag{4.7}$$

Proof. Recall that for a permutation $w \in A_{2n}(3412)$ with label (a, b) in the generating tree defined by Lewis [7], we have $a = d(w)$, where

$$d(w) = 2n - \max\{w_i \,|\, \text{there exists } j \text{ such that } j > i \text{ and } w_i < w_j\}. \tag{4.8}$$

By Theorem 4.3, a permutation $u = u_1u_2 \cdots u_{2n} \in A_{2n}(4312)$ is an internal vertex if and only if $h(u) = u_1 + 1$. Using the labeling schemes for $A_{2n}(4312)$ and $A_{2n}(3412)$ given in (4.3) and (4.8) respectively, we find that U_n and V_n can be described in terms of the functions $h(u)$ and $d(u)$, namely,

$$U_n = \{u \,|\, u = u_1u_2\cdots u_{2n} \in A_{2n}(4312),\ h(u) = u_1 + 1 \text{ and } h(u) = 2\}, \tag{4.9}$$

$$V_n = \{u \,|\, u = u_1u_2\cdots u_{2n} \in A_{2n}(3412),\ d(u) = 1\}. \tag{4.10}$$

We first prove (4.6). Given $u = u_1u_2\cdots u_{2n} \in U_n$, we have $u_1 = 1$. Conversely, assume that $u = u_1u_2\cdots u_{2n}$ is an alternating permutation in $A_{2n}(4312)$ with $u_1 = 1$. Since the subsequence 12 forms a 12-pattern of u, by the definition of $h(u)$, we obtain that $h(u) = 2$. So we have $h(u) = u_1 + 1$. It follows that $u \in U_n$. This yields (4.6).

We now consider (4.7). Assume that $u = u_1u_2\cdots u_{2n}$ is an alternating permutation in V_n. Since $d(u) = 1$, we see that $2n-1$ is the largest among the smaller elements of 12-patterns of u. It follows that $2n-1$ precedes $2n$ in u. If $u_{2n} \neq 2n$, then $(2n-1)(2n)u_{2n-1}u_{2n}$ forms a 3412-pattern of u, which contradicts the fact that u is 3412-avoiding. Thus we have $u_{2n} = 2n$. Conversely, if $u_{2n} = 2n$, then it is easy to check that $d(u) = 1$. This completes the proof. ∎

Recall that $|U_n| = |V_n|$. To prove relation (1.11), it is easy to establish a bijection between U_n and $A_{2n-1}(1243)$ and a bijection between V_n and $A_{2n-1}(3412)$. Then relation (1.11) follows from the fact $|A_{2n-1}(2143)| = |A_{2n-1}(3412)|$.

Define a map $\rho\colon U_n \to A_{2n-1}(1243)$ as follows. Given an alternating permutation $w = w_1w_2\cdots w_{2n}$ in U_n, let $\rho(w) = \pi^c$, where $\pi = (w_2-1)(w_3-1)\cdots(w_{2n}-1)$. It is routine to check that ρ is a bijection.

To define a map $\mu\colon V_n \to A_{2n-1}(3412)$, we assume that u is an alternating permutation in V_n. Let $\mu(u)$ be the alternating permutation obtained from u by deleting the last element. It can be verified that μ is a bijection. This gives an alternative proof of relation (1.11).

Note added in proof: After this work was announced (arXiv:1212.2697), Gowravaram and Jagadeesan [4] proved the following Wilf-equivalence by the approach of shape-Wilf-equivalence as used by Backelin, West and Xin [1]. They showed that for any permutation τ of $\{3, 4, \ldots, k\}$, the patterns 12τ and 21τ are Wilf-equivalent for alternating permutations. After the manuscript of Gowravaram and Jagadeesan appeared on the arXiv, Yan [12] showed that for $k \geq 2$, $n \geq k+1$, and for any permutation α of $\{k+1, k+2, \ldots, n\}$, the patterns $12\cdots k\alpha$ and $k\cdots 21\alpha$ are Wilf-equivalent for alternating permutations.

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