CONGRUENCES FOR 9-REGULAR PARTITIONS MODULO 3

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Abstract. In view of the modular equation of fifth order, we give a simple proof of Keith's conjecture which are some infinite families of congruences modulo 3 for the 9-regular partition function. Meanwhile, we derive some new congruences modulo 3 for the 9-regular partition function.

1. Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. If ℓ is a positive integer, then a partition is called an ℓ-regular partition if there is no part divisible by ℓ. The generating function for the number of ℓ-regular partitions of n, denoted by $b_\ell(n)$, is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}.$$

Recently, arithmetic properties of ℓ-regular partition functions have received a great deal of attention (see, for example, [2, 4–7, 10, 12, 14–16, 18, 19]). Xia and Yao [18] obtained that for all integers $n \geq 0$ and $k \geq 0$,

$$b_9\left(2^{6k+7}n + \frac{2^{6k+6}-1}{3}\right) \equiv 0 \pmod 2.$$

Later, Keith [13] posed the following conjecture which was proved by Xia and Yao in [19].

Theorem 1.1. *[13] For $k = 0, 2, 3, 4$, $a \geq 1$ and $n \geq 0$,*

$$b_9\left(5^{2\alpha}n + \frac{(3k+2)5^{2\alpha-1}-1}{3}\right) \equiv 0 \pmod 3.$$

In this paper, applying the modular equation of fifth order, we give another proof of Theorem 1.1. Furthermore, we derive the following congruences related to $b_9(n)$.

Theorem 1.2. *For $k = 0, 1, 3, 4, 5, 6$, $\alpha \geq 1$ and $n \geq 0$,*

$$b_9(7^{3\alpha}n + \frac{(3k+1)7^{3\alpha-1}-1}{3}) \equiv 0 \pmod 3.$$

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Theorem 1.3. *For prime* $p \equiv -1 \pmod 6$, $k = 1, \ldots, p-1$, $\alpha \geq 1$ *and* $n \geq 0$,

$$b_9\left(2p^{2\alpha}n + \frac{(6k+p)p^{2\alpha-1}-1}{3}\right) \equiv 0 \pmod 3.$$

We follow the standard q-series notation [9].

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \quad \text{and} \quad (a_1, a_2, \ldots, a_m; q)_\infty = \prod_{j=1}^{m}(a_j;q)_\infty, \quad |q|<1.$$

Let $f(a,b)$ be Ramanujan's general theta function given by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab|<1.$$

Jacobi's triple product identity can be stated in Ramanujan's notation as follows

$$f(a,b) = (-a,-b,ab;ab)_\infty.$$

Thus,

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty.$$

The Legendre symbol is defined by

$$\left(\frac{a}{p}\right) := \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod p, \\ -1, & \text{if } a \text{ is a quadratic non-residue modulo } p, \\ 0, & \text{if } a \equiv 0 \pmod p. \end{cases}$$

This paper is organized as follows. In Section 2, we prove Keith's conjecture. In Section 3, with the aid of the modular equation of seventh order, we obtain some infinite families of congruences modulo 3 for $b_9(n)$. In Section 4, using a p-dissection identity for $f(-q)$, we get more congruences for $b_9(n)$.

2. Proof of Keith's conjecture

Ramanujan [3, Theorem 7.4.1] discovered that

$$\frac{(q^{10},q^{15};q^{25})_\infty}{(q^5,q^{20};q^{25})_\infty} - q - q^2\frac{(q^5,q^{20};q^{25})_\infty}{(q^{10},q^{15};q^{25})_\infty} = \frac{(q;q)_\infty}{(q^{25};q^{25})_\infty}. \tag{2.1}$$

Setting

$$\eta := \frac{f(-q)}{qf(-q^{25})} \quad \text{and} \quad R := \frac{(q^{10},q^{15};q^{25})_\infty}{(q^5,q^{20};q^{25})_\infty},$$

we rewrite (2.1) as

$$\eta = q^{-1}R - 1 - qR^{-1}. \tag{2.2}$$

Hirschhorn and Hunt [11] defined an operator, H, which acts on series of (positive and negative) powers of a single variable, and simply picks out those terms in which the power is congruent to 0 modulo 5. Based on (2.2), they [11] gave the following lemmas.

Lemma 2.1. *[11] We have*

$$H(\eta)=-1, H(\eta^2)=-1, H(\eta^3)=5, \text{ and } H(\eta^4)=-5.$$

Let

$$S:=\frac{f^6(-q^5)}{q^5 f^6(-q^{25})}.$$

Lemma 2.2. *[1, Equation (10.3.1)] [11](the Modular Equation of Fifth Order)*

$$S=\eta^5+5\eta^4+15\eta^3+25\eta^2+25\eta.$$

Proof of Theorem 1.1. Applying Lemma 2.1 and Lemma 2.2 yields

$$H(\eta^8)=-125.$$

Since

$$\begin{aligned}\sum_{n=0}^{\infty} b_9(n)q^n &\equiv f^8(-q) \pmod 3 \\ &= q^8 f^8(-q^{25})\eta^8,\end{aligned} \tag{2.3}$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_9(5n+3)q^{5n+3} &\equiv q^8 f^8(-q^{25})H(\eta^8) \pmod 3 \\ &= -125 q^8 f^8(-q^{25}) \\ &\equiv q^8 f^8(-q^{25}) \pmod 3\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} b_9(5n+3)q^n \equiv q f^8(-q^5) \pmod 3. \tag{2.4}$$

Then

$$\sum_{n=0}^{\infty} b_9(5(5n+1)+3)q^n = \sum_{n=0}^{\infty} b_9(25n+8)q^n \equiv f^8(-q) \pmod 3. \tag{2.5}$$

From (2.3) and (2.5) it follows that

$$b_9(25n+8) \equiv b_9(n) \pmod 3.$$

By induction on α, we derive that for all integers $n \geq 0$ and $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} b_9\left(5^{2\alpha}n+\frac{5^{2\alpha}-1}{3}\right)q^n \equiv f^8(-q) \pmod 3.$$

From (2.4) it follows that

$$\sum_{n=0}^{\infty} b_9\left(5^{2\alpha}(5n+3)+\frac{5^{2\alpha}-1}{3}\right)q^n \equiv q f^8(-q^5) \pmod 3.$$

That is,

$$\sum_{n=0}^{\infty} b_9\left(5^{2\alpha+1}n+\frac{2\cdot 5^{2\alpha+1}-1}{3}\right)q^n \equiv q f^8(-q^5) \pmod 3. \tag{2.6}$$

Since there are no terms on the right of (2.6) in which the powers of q are congruent to $0,2,3,4$ modulo 5,

$$b_9\left(5^{2\alpha+1}(5n+k)+\frac{2\cdot 5^{2\alpha+1}-1}{3}\right)\equiv 0 \pmod 3$$

for $k=0,2,3,4$.

That is, for $k=0,2,3,4$ and $\alpha\geq 0$,

$$b_9\left(5^{2\alpha+2}n+\frac{(3k+2)5^{2\alpha+1}-1}{3}\right)\equiv 0 \pmod 3.$$

This proves Theorem 1.1. □

3. New congruences for $b_9(n)$

In order to prove Theorem 1.2, we need the following lemmas.

Lemma 3.1. *[8, Lemma 3.14](the Modular Equation of Seventh Order)*

$$T^2=(7\xi^3+35\xi^2+49\xi)T+\xi^7+7\xi^6+21\xi^5+49\xi^4+147\xi^3+343\xi^2+343\xi,$$

where

$$\xi=\frac{f(-q)}{q^2f(-q^{49})} \quad \text{and} \quad T=\frac{f^4(-q^7)}{q^7f^4(-q^{49})}.$$

Lemma 3.2. *[8, Lemma 3.12]*

$$\begin{aligned}&H(\xi)=-1, && H(\xi^2)=1, && H(\xi^3)=-7,\\ &H(\xi^4)=-4T-7, && H(\xi^5)=10T+49, && H(\xi^6)=49,\end{aligned}$$

where the operator H acts on a series of powers of q and picks out those terms in which the power of q is congruent to 0 modulo 7.

Proof of Theorem 1.2. From Lemma 3.1 and Lemma 3.2, we get

$$H(\xi^8)=20T^2-343\equiv 2T^2+2 \pmod 3.$$

Since

$$\begin{aligned}\sum_{n=0}^{\infty}b_9(n)q^n&\equiv f^8(-q) \pmod 3 \qquad (3.1)\\ &=q^{16}f^8(-q^{49})\xi^8,\end{aligned}$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}b_9(7n+2)q^{7n+2}&\equiv q^{16}f^8(-q^{49})H(\xi^8) \pmod 3\\ &\equiv q^{16}f^8(-q^{49})\left(2q^{-14}\frac{f^8(-q^7)}{f^8(-q^{49})}+2\right) \pmod 3\\ &=2q^2f^8(-q^7)+2q^{16}f^8(-q^{49})\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} b_9(7n+2)q^n \equiv 2f^8(-q) + 2q^2 f^8(-q^7) \pmod{3}. \tag{3.2}$$

In view of (3.1) and (3.2), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} b_9\left(7(7n+2)+2\right)q^n &= \sum_{n=0}^{\infty} b_9(49n+16)q^n \\ &\equiv 2\left(2f^8(-q) + 2q^2 f^8(-q^7)\right) + 2f^8(-q) \pmod{3} \\ &\equiv q^2 f^8(-q^7) \pmod{3}. \end{aligned}\tag{3.3}$$

Then

$$\sum_{n=0}^{\infty} b_9\left(49(7n+2)+16\right)q^n = \sum_{n=0}^{\infty} b_9(343n+114)q^n \equiv f^8(-q) \pmod{3}. \tag{3.4}$$

From (3.1) and (3.4) it can be seen that

$$b_9(343n+114) \equiv b_9(n) \pmod{3}.$$

By induction on α, we derive that for $\alpha \geq 0$ and $n \geq 0$

$$\sum_{n=0}^{\infty} b_9\left(7^{3\alpha}n + \frac{7^{3\alpha}-1}{3}\right)q^n \equiv f^8(-q) \pmod{3}. \tag{3.5}$$

Invoking (3.3) and (3.5), we get

$$\sum_{n=0}^{\infty} b_9\left(7^{3\alpha}(49n+16) + \frac{7^{3\alpha}-1}{3}\right)q^n \equiv q^2 f^8(-q^7) \pmod{3}.$$

That is,

$$\sum_{n=0}^{\infty} b_9\left(7^{3\alpha+2}n + \frac{7^{3\alpha+2}-1}{3}\right)q^n \equiv q^2 f^8(-q^7) \pmod{3}. \tag{3.6}$$

Since there are no terms on the right of (3.6) in which the powers of q are congruent to $0, 1, 3, 4, 5, 6$ modulo 7,

$$b_9\left(7^{3\alpha+2}(7n+k) + \frac{7^{3\alpha+2}-1}{3}\right) \equiv 0 \pmod{3}$$

for $k = 0, 1, 3, 4, 5, 6$.

That is, for $k = 0, 1, 3, 4, 5, 6$ and $\alpha \geq 0$,

$$b_9\left(7^{3\alpha+3}n + \frac{(3k+1)7^{3\alpha+2}-1}{3}\right) \equiv 0 \pmod{3}.$$

This proves Theorem 1.2. □

4. More congruences for $b_9(n)$

By means of the following lemma given by the authors in [6], we derive some new congruences for $b_9(n)$.

Lemma 4.1. *[6, Theorem 2.2] For any prime $p \geq 5$, we have*

$$f(-q) = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}).$$

Further, for $-(p-1)/2 \leq k \leq (p-1)/2$ and $k \neq (\pm p - 1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p},$$

where

$$\frac{\pm p - 1}{6} := \begin{cases} \frac{p-1}{6}, & p \equiv 1 \pmod{6}, \\ \frac{-p-1}{6}, & p \equiv -1 \pmod{6}. \end{cases}$$

Proof of Theorem 1.3. Xia and Yao [17, Lemma 3.5] provided that

$$\sum_{n=0}^{\infty} b_9(n) q^n = \frac{f^3(-q^{12}) f(-q^{18})}{f^2(-q^2) f(-q^6) f(-q^{36})} + q \frac{f^2(-q^4) f(-q^6) f(-q^{36})}{f^3(-q^2) f(-q^{12})}.$$

Then we find

$$\sum_{n=0}^{\infty} b_9(2n) q^n \equiv f^4(-q) \equiv f(-q) f(-q^3) \pmod{3}. \tag{4.1}$$

Due to Lemma 4.1, we discuss congruence properties modulo p for the following form

$$\frac{3k^2+k}{2} + 3 \cdot \frac{3m^2+m}{2},$$

where $-(p-1)/2 \leq k, m \leq (p-1)/2$. Notice that when $k = m = (-p-1)/6$, we have

$$\frac{3k^2+k}{2} + 3 \cdot \frac{3m^2+m}{2} = \frac{p^2-1}{6}.$$

If we have

$$\frac{3k^2+k}{2} + 3 \cdot \frac{3m^2+m}{2} \equiv \frac{p^2-1}{6} \pmod{p}, \tag{4.2}$$

then

$$(6k+1)^2 + 3(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-3}{p}\right) = -1$ for $p \equiv -1 \pmod 6$, we have the only one solution $k = m = (-p-1)/6$ for (4.2). So there are no other k and m such that $(3k^2+k)/2+3\cdot(3m^2+m)/2$ and $(p^2-1)/6$ are in the same residue class modulo p. Therefore, we get

$$\sum_{n=0}^{\infty} b_9\left(2\left(pn + \frac{p^2-1}{6}\right)\right) q^n = \sum_{n=0}^{\infty} b_9\left(2pn + \frac{p^2-1}{3}\right) q^n$$
$$\equiv \left((-1)^{\frac{-p-1}{6}}\right)^2 f(-q^p) f(-q^{3p}) \pmod{3}$$

$$= f(-q^p)f(-q^{3p}). \tag{4.3}$$

Then

$$\sum_{n=0}^{\infty} b_9\left(2p^2n + \frac{p^2-1}{3}\right)q^n \equiv f(-q)f(-q^3) \pmod 3. \tag{4.4}$$

From (4.1) and (4.4) it follows that

$$b_9\left(2p^2n + \frac{p^2-1}{3}\right) \equiv b_9(2n) \pmod 3.$$

By induction on α, it is easy to establish that for $\alpha \geq 0$ and $n \geq 0$

$$\sum_{n=0}^{\infty} b_9\left(2p^{2\alpha}n + \frac{p^{2\alpha}-1}{3}\right)q^n \equiv f(-q)f(-q^3) \pmod 3. \tag{4.5}$$

From (4.3) and (4.5) it can be seen that

$$\sum_{n=0}^{\infty} b_9\left(2p^{2\alpha}\left(pn + \frac{p^2-1}{6}\right) + \frac{p^{2\alpha}-1}{3}\right)q^n \equiv f(-q^p)f(-q^{3p}) \pmod 3.$$

That is,

$$\sum_{n=0}^{\infty} b_9\left(2p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{3}\right)q^n \equiv f(-q^p)f(-q^{3p}) \pmod 3. \tag{4.6}$$

Since there are no terms on the right of (4.6) in which the powers of q are congruent to $1, \ldots, p-1$ modulo p,

$$b_9\left(2p^{2\alpha+1}(pn+k) + \frac{p^{2\alpha+2}-1}{3}\right) \equiv 0 \pmod 3$$

for $k = 1, \ldots, p-1$.

That is, for $k = 1, \ldots, p-1$ and $\alpha \geq 0$,

$$b_9\left(2p^{2\alpha+2}n + \frac{(6k+p)p^{2\alpha+1}-1}{3}\right) \equiv 0 \pmod 3.$$

This proves Theorem 1.3. □

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