ON EDGE-TRANSITIVE TETRAVALENT GRAPHS OF SQUARE-FREE ORDER

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ABSTRACT. In this paper, a classification is given for tetravalent graphs of square-free order which are vertex-transitive and edge-transitive. It is shown that such graphs are either Cayley graphs or covers of some graphs arisen from simple groups A_7, J_1 and $\mathrm{PSL}(2, p)$.

1. INTRODUCTION

We denote by $\Gamma = (V, E)$ a simple graph with vertex set V and edge set E. Then the cardinality $|V|$ is called the *order* of Γ. A graph $\Gamma = (V, E)$ is called *vertex-transitive* or *edge-transitive* if the automorphism group $\mathsf{Aut}\Gamma$ acts transitively on V and E, respectively. Recall that an *arc* in a graph Γ is an ordered pair of adjacent vertices. Then a graph Γ is called *arc-transitive* if $\mathsf{Aut}\Gamma$ acts transitively on the set of arcs of Γ. A graph Γ is called *edge-regular* or *arc-regular* if $\mathsf{Aut}\Gamma$ acts regularly on the edge set or arc set of Γ, respectively.

This paper is one of a series of articles devoted to studying the class of edge-transitive graphs of square-free order. The study of such graph has a long history. For example, Chao [4] gave a classification of edge-transitive graphs of prime order and proved that those resulting graphs are also arc-transitive; Cheng and Oxley [5] showed that every vertex- and edge-transitive graphs of order twice a prime is isomorphic to one of a list of well-defined arc-transitive graphs. Thereafter, a lot of interesting results have appeared in this topic, especially, for those graphs of order being a product of two primes, see for instance [1, 17, 18, 19, 21, 22].

In [16] we gave a characterization for the class of edge-transitive graphs of square-free order, which says that the basic members in this class consist of a few special families of graphs and a finite number of sporadic graphs. This motivate us to classify edge-transitive graphs of square-free order and of small valency. In a recent paper [15], we classified cubic arc-transitive graphs of square-free order. In the present paper, we shall give a classification of connected tetravalent graphs of square-free order which are vertex-transitive and edge-transitive.

We fist explain some notation and concepts on groups and graphs. For two groups A and B, denote by $A{\times}B$, $A.B$ and $A{:}B$ the direct product, an extension and a semi-direct product of A by B, respectively; for an positive integer m, denote by $\mathbb{Z}_m$ and D_{2m} the cyclic group of order m and the dihedral group of order $2m$, respectively. For a finite group X, the *socle* of X, denoted by $\mathrm{soc}(X)$, is the subgroup generated by all minimal normal subgroups of X. A group X is said to be *almost simple* if its socle $\mathrm{soc}(X)$ is a non-abelian simple group.

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Let $\Gamma = (V, E)$ be a graph. Then Γ is called *vertex-primitive* if $\mathsf{Aut}\Gamma$ is a primitive permutation group on V. The *standard double cover* of the graph Γ is defined to be the bipartite graph with vertex set $V \times \mathbb{Z}_2$ such that two vertices $(\alpha, 0)$ and $(\beta, 1)$ are adjacent if and only if α and β are adjacent in Γ.

Our main result is stated as follows.

Theorem 1.1. *Let $\Gamma = (V, E)$ be a connected tetravalent graph of square-free order. Assume that Γ is both vertex-transitive and edge-transitive. Then one of the following statements holds.*

(1) *Γ is a Cayley graph, that is, $\mathsf{Aut}\Gamma$ contains a regular subgroup.*
(2) *$\mathsf{Aut}\Gamma \cong \mathbb{Z}_m{:}(\mathbb{Z}_n \times \mathbb{Z}_4)$ with $m > 1$, $n > 1$ and $|V| = 2mn$, and Γ is constructed as in Construction* 3.1*.*
(3) *$\mathsf{Aut}\Gamma = \mathrm{S}_7$, and Γ is isomorphic either the odd graph $\mathbf{O}_4$ of valency* 4 *or the graph in Example* 5.1*.*
(4) *$\mathsf{Aut}\Gamma = \mathrm{J}_1$ or $\mathbb{Z}_3 \times \mathrm{J}_1$, and Γ is isomorphic to a graph given in Example* 5.2*.*
(5) *$\mathsf{Aut}\Gamma = \mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$ for a prime $p \geq 5$, and*
   (i) *$p \equiv \pm 3 \pmod 8$, and Γ is edge-regular or arc-regular; or*
   (ii) *Γ is isomorphic to a graph given in Examples* 5.3*,* 5.4*,* 5.5 *and* 5.6*; or*
   (iii) *Γ is vertex-primitive.*
(6) *$\mathsf{Aut}\Gamma = \mathbb{Z}_2 \times \mathrm{PSL}(2, p)$, $\mathbb{Z}_3{:}\mathrm{PGL}(2, p)$, $\mathrm{D}_6 \times \mathrm{PSL}(2, p)$ or $\mathrm{D}_6 \times \mathrm{PGL}(2, p)$ for a prime $p \geq 5$, and Γ is isomorphic to a graph given in Example* 5.7*.*
(7) *$\mathsf{Aut}\Gamma = \mathbb{Z}_2 \times \mathrm{PSL}(2, p)$ or $\mathbb{Z}_2 \times \mathrm{PGL}(2, p)$ for a prime $p \geq 5$, and Γ is either arc-regular or isomorphic to a graph give in Examples* 5.8 *and* 5.9*.*
(8) *$\mathsf{Aut}\Gamma = \mathbb{Z}_l{:}\mathrm{PGL}(2, p)$ or $\mathbb{Z}_{2l}{:}\mathrm{PGL}(2, p)$ for a prime $p \geq 5$ and a square-free integer $l > 1$ coprime to $p(p^2 - 1)$, and Γ is isomorphic to a graph given in Example* 5.10 *or* 5.12*, respectively.*
(9) *$\mathsf{Aut}\Gamma = \mathbb{Z}_l \times \mathrm{PSL}(2, p)$ or $\mathrm{D}_{2l} \times \mathrm{PSL}(2, p)$ for a prime $p \geq 5$ and a square-free integer $l > 1$ coprime to $p(p^2 - 1)$, and Γ is isomorphic to a graph given in Example* 5.11 *or* 5.13*, respectively..*
(10) *Γ is isomorphic the standard double cover of a graph which is of odd order and described as in one of parts* (3), (5), (6) *and* (8)*.*

Remark on Theorem 1.1. The graphs satisfying (1) were classified in [14], and the graphs in item (iii) of part (5) can be read out from [13].

2. Preliminaries

Let $\Gamma = (V, E)$ be a graph and $G \leq \mathsf{Aut}\Gamma$. The graph Γ is said to be *G-vertex-transitive* or *G-edge-transitive* if G acts transitively on V or E, respectively. Let $\alpha \in V$. Denote by G_α and $\Gamma(\alpha)$ the stabilizer of α in G and the set of the neighbors of α in Γ, respectively. For $\beta \in \Gamma(\alpha)$, denote by $G_{\alpha\beta}$ the arc-stabilizer $G_\alpha \cap G_\beta$ of (α, β). Suppose that Γ is both G-vertex-transitive and G-edge-transitive. Then

(i) G_α is transitive on $\Gamma(\alpha)$, so $|\Gamma(\alpha)| = |G_\alpha : G_{\alpha\beta}|$; or
(ii) G_α has exactly two orbits on $\Gamma(\alpha)$, and $|\Gamma(\alpha)| = 2|G_\alpha : G_{\alpha\beta}|$.

In these two cases, Γ is called *G-arc-transitive* and *G-half-transitive*, respectively. If Γ is G-arc-transitive, then there exists $g \in G \setminus G_\alpha$ such that $(\alpha, \beta)^g = (\beta, \alpha)$; obviously, this g can be chosen to be a 2-element in $\mathbf{N}_G(G_{\alpha\beta})$ with $g^2 \in G_{\alpha\beta}$. Set $H = G_\alpha$.

$[G:H] := \{Hx \mid x \in G\}$. The *coset graph* $\mathsf{Cos}(G,H,HgH)$ is defined on $[G:H]$ with edge set $\{\{Hx,Hy\} \mid yx^{-1} \in H\{g,g^{-1}\}H\}$, where $g \in G \setminus H$ is such that $\alpha^g \in \Gamma(\alpha)$. Then the group G can be viewed as a group of automorphisms of $\mathsf{Cos}(G,H,HgH)$ acting on $[G:H]$ by the right multiplication, and the mapping $\alpha^x \mapsto Hx$, $\forall x \in G$ is an isomorphism from Γ to $\mathsf{Cos}(G,H,HgH)$.

Lemma 2.1. *Let $\Gamma = \mathsf{Cos}(G,H,HgH)$ be a coset graph. Then*

(i) *Γ is G-vertex-transitive and G-edge-transitive, and Γ is connected if and only if $\langle H,g\rangle = G$;*

(ii) *Γ is G-arc-transitive if and only if $H\{g,g^{-1}\}H = HxH$ for some 2-element $x \in \mathbf{N}_G(H \cap H^g) \setminus H$ with $x^2 \in H \cap H^g$.*

Let $\Gamma = (V,E)$ be a graph and $G \leq \mathsf{Aut}\Gamma$. Note that, for $\alpha \in V$, the stabilizer G_α fixes $\Gamma(\alpha)$ set-wise. Then G_α induces a permutation group $G_\alpha^{\Gamma(\alpha)}$ (on $\Gamma(\alpha)$). Let $G_\alpha^{[1]}$ be the kernel of this action. Then $G_\alpha^{\Gamma(\alpha)} \cong G_\alpha/G_\alpha^{[1]}$.

Let N be a normal subgroup of G, denoted by $N \trianglelefteq G$. Then N_α is a normal subgroup of G_α. One extreme case is that N_α acts transitively on $\Gamma(\alpha)$. It is easily shown that the following lemma holds for connected arc-transitive graphs.

Lemma 2.2. *Let $\Gamma = (V,E)$ be a connected G-vertex-transitive graph, $\alpha \in V$ and $N \trianglelefteq G \leq \mathsf{Aut}\Gamma$. If N_α is transitive on $\Gamma(\alpha)$, then Γ is N-edge-transitive; in particular, either Γ is N-arc-transitive or N has exactly two orbits on V.*

For the case where N is a semiregular on V with two orbits, by [12, Lemma 2.4], we have the following result.

Lemma 2.3. *Let $\Gamma = (V,E)$ be a connected bipartite graph, $\alpha \in V$ and $N \trianglelefteq G \leq \mathsf{Aut}\Gamma$. If N is regular on both the bipartition subsets of Γ, then $G_\alpha \cong G_\alpha^{\Gamma(\alpha)}$.*

By [6, Lemma 2.1], we have the following result.

Lemma 2.4. *Let $\Gamma = (V,E)$ be a connected G-vertex-transitive graph, $\alpha \in V$ and $N \trianglelefteq G \leq \mathsf{Aut}\Gamma$. Then each prime divisor of $|N_\alpha|$ divides $|N_\alpha^{\Gamma(\alpha)}|$ and, for $\beta \in \Gamma(\alpha)$, each prime divisor of $|N_{\alpha\beta}|$ is less than $|\Gamma(\alpha)|$. In particular, $N_\alpha^{\Gamma(\alpha)} \neq 1$ if $N_\alpha \neq 1$.*

Lemma 2.5. *Let $\Gamma = (V,E)$ be a connected G-vertex-transitive graph, $\alpha \in V$ and $N \trianglelefteq G \leq \mathsf{Aut}\Gamma$. If Γ is G-edge-transitive then $|N_\alpha : N_{\alpha\beta}|$ is a constant, where $\{\alpha,\beta\}$ runs over E. If $N_\alpha^{\Gamma(\alpha)}$ is semiregular on $\Gamma(\alpha)$, then $N_\alpha \cong N_\alpha^{\Gamma(\alpha)}$.*

Proof. The first part of this lemma follows from [14, Lemma 3.1].

Assume that $N_\alpha^{\Gamma(\alpha)}$ is semiregular on $\Gamma(\alpha)$. Let $\beta \in \Gamma(\alpha)$. Then $\beta = \alpha^x$ for some $x \in G$. Since $N \lhd G$, it is easily shown that $N_\beta = N_\alpha^x$ and $N_\beta^{[1]} = (N_\alpha^{[1]})^x$. It follows that $N_\beta^{\Gamma(\beta)}$ and $N_\alpha^{\Gamma(\alpha)}$ are permutation isomorphic. In particular, $N_\alpha^{\Gamma(\alpha)}$ is semiregular on $\Gamma(\alpha)$ if and only if $N_\beta^{\Gamma(\beta)}$ is semiregular on $\Gamma(\beta)$, which yields that $N_\alpha^{[1]}$ acts trivially on $\Gamma(\beta)$, and so $N_\alpha^{[1]} = N_\beta^{[1]}$. Since Γ is connected, $N_\alpha^{[1]}$ fixes each vertex of Γ, hence $N_\alpha^{[1]} = 1$. Then the lemma follows. □

Let $\Gamma = (V,E)$ be a graph. For a positive integer s, an s-arc in Γ is a sequence of $s+1$ vertices $\alpha_0, \alpha_1, \ldots, \alpha_s$ such that α_i is adjacent to α_{i+1} and $\alpha_i \neq \alpha_{i+2}$. For

$G \le \mathsf{Aut}\Gamma$, the graph Γ is said to be (G,s)-*arc-transitive* if G acts transitively on V and on the set of s-arcs of Γ, and (G,s)-*transitive* if further G is intransitive on the set of $(s+1)$-arcs of Γ. The vertex stabilizer for s-arc-transitive graphs of valency 4 is known, refer to [23].

Lemma 2.6. *Let $\Gamma=(V,E)$ be a connected (G,s)-transitive graph of valency 4. Then, for $\alpha\in V$, the stabilizer X_α and s are listed in the following table.*

s	2	3	4	7
G_α	A_4, S_4	$\mathbb{Z}_3\times\mathrm{A}_4$, $(\mathbb{Z}_3\times\mathrm{A}_4).\mathbb{Z}_2$, $\mathrm{S}_3\times\mathrm{S}_4$	$\mathbb{Z}_3^2{:}\mathrm{GL}(2,3)$	$[3^5]{:}\mathrm{GL}(2,3)$

We end this section by a useful observation on permutation groups.

Lemma 2.7. *Let $G=N{:}X$ be a permutation group on V, and let B be an N-orbit. Assume that N is regular on B. Then $(NY)_\alpha\cong Y_B$ for $\alpha\in B$ and $Y\le X$.*

Proof. Let U be the NY-orbit containing B. Then $\{B^g\mid g\in NY\}$ is an NY-invariant partition of U. It follows that $(NY)_\alpha\le(NY)_B$, and so $(NY)_\alpha=((NY)_B)_\alpha$. Since N is transitive on B, we have $(NY)_B=N(NY)_\alpha$. Then $N(NY)_\alpha=(NY)_B=NY\cap G_B=N(Y\cap G_B)=NY_B$. Thus $(NY)_\alpha\cong N(NY)_\alpha/N=NY_B/N\cong Y_B$. □

3. The soluble case

In this section, we treat vertex-transitive and edge-transitive tetravalent graphs which have soluble automorphism groups. We first construct a family of such graphs.

Construction 3.1. Let $F=\langle a\rangle\cong\mathbb{Z}_m$ with m odd and square-free. Assume that $\mathsf{Aut}(F)$ has an element y of order 4. Let $b\in\mathsf{Aut}(F)$ be of order n with n odd square-free and coprime to m. Consider the semi-direct product $G=\langle a\rangle{:}(\langle b\rangle\times\langle y\rangle)$. Let $H=\langle y^2\rangle$, and $g=aby$. If $\langle H,g\rangle=\langle aby,y^2\rangle=G$, then $\Gamma=\mathsf{Cos}(G,H,HgH)$ is a connected vertex-transitive and edge-transitive graph of valency 4.

Lemma 3.2. *Let $\Gamma=(V,E)$ be as in Construction* 3.1. *If $n=1$, then $\mathsf{Aut}\Gamma$ contains two subgroups isomorphic to D_{2m} and $\mathrm{D}_{2m}{:}\mathbb{Z}_4$ which acts regularly on the vertices and arcs of Γ, respectively.*

Proof. Assume that $n=1$ and $a^y=a^r$. Then r is coprime to m, $r^4\equiv 1\pmod{m}$ and $r^2\not\equiv 1\pmod{m}$. Note that Γ is bipartite and $\langle a\rangle$ is semiragular on each of the biparts of Γ. Then $V=\{Ha^i\mid 0\le i\le m-1\}\cup\{Ha^iy\mid 0\le i\le l-1\}$, and $H\{g,g^{-1}\}H=\{y^sa^t\mid s=1,-1;\ t=-1,r,-r^2,r^3\}$. Note that Ha^i and Ha^jy are adjacent if and only if $ya^{rj-i}=(a^jy)a^{-i}\in H\{g,g^{-1}\}H$. Then Ha^i and Ha^jy are adjacent if and only if $rj-i$, modulo m, lies in $\{-1,r,-r^2,r^3\}$. Since $rj-i\equiv r(-r^3)i-(-r)j\pmod{m}$, we know that Ha^i and Ha^jy are adjacent if and only if Ha^{-rj} and $Ha^{-r^3i}y$ are adjacent in Γ. Define a map $\tau: Ha^i\mapsto Ha^{-r^3i}y$, $Ha^jy\mapsto Ha^{-rj}$. Then $\tau\in\mathsf{Aut}\Gamma$ by the above argument. It is easily shown τ is an involution and that $R:=\langle a,\tau\rangle$ is transitive on V. Computation shows that $(Ha^i)^{\tau a\tau}=Ha^{i-1}$ and $(Ha^iy)^{\tau a\tau}=Ha^iya^{-1}$, and so $\tau a\tau=a^{-1}$ and $\langle a,\tau\rangle\cong\mathrm{D}_{2m}$. Then R is regular on V. Further computation indicates that $\tau y=y\tau$. Thus $R{:}\langle\tau y\rangle\cong\mathrm{D}_{2m}{:}\mathbb{Z}_4$ is regular on the arcs of Γ. □

Theorem 3.3. *Let $\Gamma = (V, E)$ be a connected tetravalent graph of square-free order, and $G \leq \mathsf{Aut}\Gamma$. Assume that G is soluble and Γ is both G-vertex-transitive and G-edge-transitive. Then one of the following holds.*

(1) $\mathsf{Aut}\Gamma$ *contains a regular subgroup;*
(2) $G \cong \mathbb{Z}_m{:}(\mathbb{Z}_n{\times}\mathbb{Z}_4)$ *with $m, n > 1$ and $|V| = 2mn$, G is regular on E and Γ described as in Construction* 3.1.

Proof. For a prime divisor p of $|G|$, denote by $\mathbf{O}_p(G)$ the largest normal p-subgroup of G. By Lemma 2.5, $|(\mathbf{O}_p(G))_\alpha : (\mathbf{O}_p(G))_{\alpha\beta}|$ is a divisor of 4, where $\{\alpha, \beta\} \in E$. It follows that either $p = 2$ or $\mathbf{O}_p(G)$ is semiregular on V. Thus $|\mathbf{O}_p(G)| \leq p$ if $p \geq 3$.

Suppose that $N := \mathbf{O}_2(G)$ has order divisible by 4. Then N is not semiregular on V, and it follows that, for any two N-orbits B and C, the subgraph $[B \cup C]$ induced by $B \cup C$ either contains no edge or is isomorphic to $\mathsf{K}_{2,2}$. It follows that Γ is the lexicographic product of the empty graph $2\mathsf{K}_1$ by an n-cycle, where n is the number of N-orbits. It is easily shown that $\mathsf{Aut}\Gamma \cong \mathbb{Z}_2^n{:}\mathrm{D}_{2n}$ contains two regular subgroups isomorphic to $\mathbb{Z}_{2n}$ and D_{2n}, respectively. So part (1) occurs.

Now assume that $\mathbf{O}_p(G) = 1$ or $\mathbb{Z}_p$ for each prime divisor p of $|G|$. Let F be the Fitting subgroup of G, the largest nilpotent normal subgroup of G. Then $F \neq 1$ as G is soluble, and F is cyclic. It follows that F is semiregular on V. Since G is soluble, the centralizer $\mathbf{C}_G(F) \leq F$, and so $\mathbf{C}_G(F) = F$. Then $G/F = \mathbf{N}_G(F)/\mathbf{C}_G(F)$ is isomorphic to a subgroup of $\mathsf{Aut}(F)$, which is abelian. Thus G/F is abelian. For a vertex α, we have $G_\alpha \cong FG_\alpha/F \leq G/F$; in particular, G_α is an abelian 2-group.

Assume that F has l orbits on V. Then $|G| = l|F||G_\alpha|$. If l is odd, then G contains a normal regular subgroup $F{:}\mathbb{Z}_l$, so part (1) occurs. Thus we assume further that $|F|$ is odd and $l = 2n$ is even. Since $|G : G_\alpha| = 2n|F|$ is square-free, $|F|$ is coprime to $2n|G_\alpha|$. Since G is soluble, G has a Hall subgroup H of order $2n|G_\alpha|$. Then $G = F{:}H$, and H is abelian as $H \cong G/F$. Thus $H = N{\times}P$, where $N \cong \mathbb{Z}_n$ and P is a Sylow 2-subgroup of G with $G_\alpha \leq P$ and $|P : G_\alpha| = 2$. Then $F{:}N$ is a normal semiregular subgroup of G, and it has exactly two orbits on V. Since G is transitive on E, we know that Γ is a bipartite graph with two parts being the FN-orbits on V. Thus By Lemma 2.3, G_α is faithful on $\Gamma(\alpha)$, and so $G_\alpha \cong \mathbb{Z}_2$, $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$.

Suppose that Γ is G-arc-transitive. Then $G_{\alpha\beta} = 1$ for $\beta \in \Gamma(\alpha)$. Let $g \in G$ with $(\alpha, \beta)^g = (\beta, \alpha)$. Then $g^2 \in G_{\alpha\beta} = 1$. Thus G has a regular subgroup $(FN){:}\langle g\rangle$, so part (1) of Theorem 1.1 occurs.

Suppose next that Γ is G-half-transitive. It follows from Lemma 2.5 that $G_\alpha \not\cong \mathbb{Z}_4$. Then $G_\alpha \cong \mathbb{Z}_2$ or $\mathbb{Z}_2^2$. Recall that P is a Sylow 2-subgroup of G with $G_\alpha \leq P$ and $|P : G_\alpha| = 2$. If $P \cong \mathbb{Z}_2^i$ for $i = 2$ or 3, then G has a regular subgroup $(FN){:}\langle g\rangle$ for some involution $g \in P$. Thus we assume further that $P \cong \mathbb{Z}_4$ or $\mathbb{Z}_2{\times}\mathbb{Z}_4$.

Set $F \cong \mathbb{Z}_m$. Then $G = FNP \cong \mathbb{Z}_m{:}(\mathbb{Z}_n{\times}P)$. Write $\Gamma = \mathsf{Cos}(G, G_\alpha\{g, g^{-1}\}G_\alpha)$, where $g = aby$ for $a \in F$, $b \in N$ and $y \in P$. Then $G_\alpha g G_\alpha \neq G_\alpha g^{-1} G_\alpha$, $|G : (G_\alpha \cap G_\alpha^g)| = 2$ and, since Γ is connected, $G = \langle g, G_\alpha\rangle \leq \langle a\rangle{:}\langle b, y, G_\alpha\rangle = \langle a\rangle{:}(\langle b\rangle{\times}\langle y, G_\alpha\rangle)$. It follows that $F = \langle a\rangle$, $N = \langle b\rangle$ and y has order 4. Set $G_\alpha \cap G_\alpha^g = \langle x\rangle$. Then $x^2 = 1$. Note that $G_\alpha \cap G_\alpha^g = G_\alpha \cap G_\alpha^{aby} = (G_\alpha \cap G_\alpha^a)^{by} = G_\alpha \cap G_\alpha^a$. Then $\langle x, x^{a^{-1}}\rangle \leq G_\alpha$. If $x \neq x^{a^{-1}}$ then $G_\alpha = \langle x, x^{a^{-1}}\rangle \cong \mathbb{Z}_2^2$, but $1 \neq xx^{a^{-1}} = a^x a^{-1} \in F$, a contradiction. Thus $x \in \mathbf{C}_G(a^{-1}) = \mathbf{C}_G(F) = F$. It implies that $x = 1$, so $G_\alpha \cap G_\alpha^g = 1$. Then $G_\alpha \cong \mathbb{Z}_2$, yielding $|P| = 4$ and $P \cong \mathbb{Z}_4$. Thus Γ is described as in Construction 3.1. Then, by Lemma 3.2, one of (1) and (2) of this theorem follows. □

4. Insoluble automorphism groups

In this section, we assume that $\Gamma = (V, E)$ is a connected tetravalent graph of square-free order, and that a subgroup $G \leq \mathsf{Aut}\Gamma$ acts transitively on V.

Let $N \lhd G$ be an intransitive normal subgroup. Consider the *normal quotient graph* Γ_N, which is the graph defined on $V_N = \{\alpha^N \mid \alpha \in V\}$ with edge set $\{\{\alpha^N, \beta^N\} \mid \{\alpha, \beta\} \in E\}$. Then Γ_N has valency 1, 2 or 4. If Γ and Γ_N has the same valency, then it is easily shown that N is a semiregular subgroup of G and itself is the kernel of G acting on V_N; in this case, Γ is called a *normal cover* of Γ_N with respect to G and N.

Lemma 4.1. *Let N be an intransitive normal subgroup of G. Assume that Γ is a normal cover of Γ_N. Then $G = N{:}X$ for some $X \leq G$ with $N \cap X = 1$.*

Proof. The lemma is trivial for $N = 1$. Thus we assume that $N \neq 1$.

Since Γ is a normal cover of Γ_N and Γ is connected, N is semiregular on V; in particular, $|N|$ is a divisor of $|V|$, so $|N|$ is square-free. Let p be the largest prime divisor of $|N|$. Then N has a unique Sylow p-subgroup, say P. Thus P is a characteristic subgroup of N, and so $P \lhd G$.

Note that each N-orbits on V is the union of some P-orbits. Since Γ and Γ_N has the same valency, it is easily shown that Γ is a normal cover of Γ_P and that Γ_P is a normal cover of $(\Gamma_P)_{N/P} \cong \Gamma_N$. Then, by induction, we may assume that $G/P = (N/P){:}(Y/P)$ for a subgroup $Y \leq G$ with $Y \cap N = P$.

Clearly, Y acts transitively on V_P, and so Y is transitive on V. Consider the action of Y on V_P. Then, for $B \in V_P$, we have $|V_P| = |Y : Y_B|$. Noting that P is semiregular, each P-orbit on V has length p. Thus $\frac{|V|}{p} = |V_P| = |Y : Y_B|$ is coprime to p as $|V|$ is square-free. Then Y_B contains a Sylow p-subgroup of Y. Since $P \leq Y_B$ is transitive on B, we have $Y_B = PY_\alpha = P{:}Y_\alpha$ for $\alpha \in B$. It follows that Y_B and hence Y has a Sylow p-subgroup $P{:}Q$, where Q is a Sylow p-subgroup of Y_α. Then, by Gaschtz' Theorem (see [2, 10.4]), the extension $Y = P.(Y/P)$ splits over P. Thus $Y = P{:}X$ for $X < Y$ with $X \cap P = 1$. Then $G = NY = NX$ and $X \cap N = X \cap (Y \cap N) = X \cap P = 1$, and the result follows. □

Lemma 4.2. *Assume that Γ is G-vertex-transitive and G-edge-transitive. Let C be the largest soluble normal subgroup of G. Then $G = C{:}X$ for $X \leq G$, and either $C = G$ or X is almost simple with socle centralizing C.*

Proof. Assume $C \neq G$. Let K be the kernel of G acting on the set of C-orbits on V. Let B be a C-orbit and $\alpha \in B$. Then $K = C{:}K_\alpha$. Since $K_\alpha \leq G_\alpha$ is soluble, K is soluble. Thus $K = C$ by the choice of C, and so $G/C = K/C$ is insoluble, hence $\mathsf{Aut}\Gamma_C$ is insoluble. Then Γ_C is of valency 4, so Γ is a normal cover of Γ_C. By Lemma 4.1, $G = C{:}X$ for some $X \leq G$. Identify X with a subgroup of $\mathsf{Aut}\Gamma_C$. Then Γ_C is X-vertex-transitive and X-edge-transitive.

By the choice of C, each minimal normal subgroup of $X \cong G/C$ is a direct product of isomorphic nonabelian simple groups. Then, since Γ_C has square-free order, the order of X is not divisible by p^2 for any prime $p > 3$. It implies that each minimal normal subgroup of X is nonabelian simple. Suppose that X has two distinct minimal normal subgroup, say N_1 and N_2. Then $N_1N_2 = N_1 \times N_2$. For $i = 1, 2$, since N_i is nonabelian simple, N_i is not semiregular on V, either the quotient graph $(\Gamma_C)_{N_i}$ is a cycle or N_i has at most two orbits on V_C. It follows that N_2 fixes set-wise each

N_1-orbit on V_C. Thus $X_\Delta \geq N_1{\times}N_2$, where Δ is an N_1-orbit on V_C containing B. It implies that $|X_B|$ is divisible by $|N_2|$. Thus X_B is not a $\{2,3\}$-group, which contradicts that $\varGamma_C$ is of valency 4. Therefore, X is almost simple.

Since C has square-free order, $\mathsf{Aut}(C)$ is soluble. Then the quotient $G/\mathbf{C}_G(C) = \mathbf{N}_G(C)/\mathbf{C}_G(C)$ is soluble as it is isomorphic to a subgroup of $\mathsf{Aut}(C)$. It follows that $\mathrm{soc}(X) \leq \mathbf{C}_G(C)$, and then our lemma follows. □

We next determine G when G is almost simple. Let $\alpha \in V$. Then G_α is a $\{2,3\}$-group by Lemma 2.4. Since $|V| = |G : G_\alpha|$ is square-free, $|G|$ is not divisible by p^2 for any prime $p \geq 5$. Moreover, either

(1) G_α is a 2-group, and so $|G|$ is not divisible by 9; or
(2) $\varGamma$ is $(G,2)$-arc-transitive and, by Lemma 2.6, $|G|$ is not divisible by 2^6.

In particular, $|G|$ is not divisible by $2^6 3^2$ and $2^2 3^8$.

Lemma 4.3. *Assume that $\varGamma$ is G-vertex-transitive and G-edge-transitive, and that G is an almost simple group. Then* $\mathrm{soc}(G)$ *is one of the following simple groups:* A_5, A_6, A_7, M_{11}, J_1, $\mathrm{PSL}(2,p)$, $\mathrm{PSL}(2,2^f)$, $\mathrm{PSL}(2,3^2)$, $\mathrm{PSL}(2,3^3)$, $\mathrm{PSL}(2,3^4)$, $\mathrm{PSL}(2,3^5)$, $\mathrm{PSL}(2,3^6)$, $\mathrm{PSL}(2,3^7)$, $\mathrm{PSL}(3,2)$, $\mathrm{PSL}(3,3)$ *and* $\mathrm{Sz}(2,2^f)$, *where $p \geq 5$ is a prime.*

Proof. Let $T = \mathrm{soc}(G)$. If $T = \mathrm{A}_n$, then $n < 8$; otherwise 25 or $2^6 3^2$ divides $|T|$. Similarly, if T is a sporadic simple group then $T = \mathrm{M}_{11}$ or J_1.

To finish the proof, we assume that $T \neq \mathrm{PSL}(2,p)$ and T is a simple group of Lie type defined over $\mathrm{GF}(p^f)$, where p is a prime. Then $p \in \{2,3\}$ as p^2 divides $|T|$.

Assume that $p = 3$. Since $|G|$ is not divisible by $2^2 3^8$, we conclude that T is one of $\mathrm{PSL}(2,3^f)$ (with $f \leq 7$), $\mathrm{PSL}(3,3)$, $\mathrm{PSU}(3,3)$, $\mathrm{PSL}(3,9)$, $\mathrm{PSL}(4,3)$, $\mathrm{PSU}(3,9)$, $\mathrm{PSU}(4,3)$, $\mathrm{PSp}(4,3)$, $\Omega(5,3)$, $\mathrm{P}\Omega^+(6,3)$, $\mathrm{P}\Omega^-(6,3)$ and $\mathrm{G}_2(3)$. The last 9 groups are excluded as their orders are divided by 25 or $2^6 3^2$. By the Atlas [7], $\mathrm{PSU}(3,3)$ has no a $\{2,3\}$-subgroup of square-free index. Thus $T = \mathrm{PSL}(2,3^f)$ or $\mathrm{PSL}(3,3)$.

Now let $p = 2$. Then T is one of $\mathrm{PSL}(2,2^f)$, $\mathrm{PSL}(3,2^f)$, $\mathrm{PSU}(3,2^f)$ and $\mathrm{Sz}(2,2^f)$; otherwise, $|T|$ has a divisor $2^6(2^f+1)^2$, which implies that $|T|$ is divisible by $2^6 r^2$, where $r \geq 3$ is a prime. Assume that $T = \mathrm{PSL}(3,2^f)$. Then $|T|$ has a divisor $\frac{(2^f-1)^2}{(3,2^f-1)}$, yielding $2^f - 1 = 3^e$ for some integer e. It follows that $f = 1$ or 2. The group $\mathrm{PSL}(3,4)$ is excluded as its order has a divisor $2^6 3^2$. Thus $T = \mathrm{PSL}(3,2)$.

Suppose that $T = \mathrm{PSU}(3,2^f)$. Then $|T|$ has a divisor $\frac{(2^f+1)^2}{(3,2^f+1)}$, yielding $2^f + 1 = 3^e$ for some integer e. It follows that $f = 1$ or 3. However, $\mathrm{PSU}(3,2)$ is not simple and $\mathrm{PSU}(3,8)$ has order divisible by $2^6 3^2$, a contradiction. Thus the lemma follows. □

Recall that, for a connected G-arc-transitive graph $\varGamma = (V,E)$ and $\{\alpha,\beta\} \in E$, there is $g \in \mathbf{N}_G(G_{\alpha\beta})$ with $\langle g, G_\alpha\rangle = G$. Then several groups in Lemma 4.3 are excluded.

Lemma 4.4. *Assume that $\varGamma$ is G-vertex-transitive and G-edge-transitive. Then* $\mathrm{soc}(G) \neq \mathrm{A}_6$, M_{11}.

Proof. Suppose that $T := \mathrm{soc}(G) = \mathrm{A}_6$ or M_{11}. Then $2^3 3^2$ divides $|G|$, and so $2^2 3$ divides $|G_\alpha|$. By the Atlas [7] and Lemma 2.6, we know that $G_\alpha \cong \mathrm{S}_4$.

Assume that $T = \mathrm{M}_{11}$. Then $G = T$ and $\varGamma$ is $(T,2)$-arc-transitive. Further, checking by the GAP, all subgroups isomorphic to S_4 are conjugate in T. Thus we may assume that T_α is contained in a maximal subgroup $M \cong \mathrm{S}_5$. Since Γ is tetravalent, $T_{\alpha\beta} = \mathrm{S}_3$ for $\beta \in \Gamma(\alpha)$. Checking the subgroups of M_{11} in the Atlas [7], we get $\mathbf{N}_T(T_{\alpha\beta}) \cong \mathrm{D}_{12}$, so $N_T(T_{\alpha\beta}) = \mathbf{N}_M(T_{\alpha\beta})$. Thus there is no an element $g \in \mathbf{N}_T(T_{\alpha\beta})$ with $\langle g, T_\alpha\rangle = T$, a contradiction.

Assume that $T = \mathrm{A}_6$. Then $|V| = 15$ or 30. Suppose that $T_\alpha \cong \mathrm{A}_4$. Then T is transitive on V, so Γ is $(T,2)$-arc-transitive. For $\beta \in \Gamma(\alpha)$, we have $T_{\alpha\beta} \cong \mathbb{Z}_3$. It is easily shown that $\mathbf{N}_T(T_{\alpha\beta}) \cong \mathrm{S}_3$. Let M be a maximal subgroup of T with $T_\alpha < M$. Then $M \cong \mathrm{A}_5$ or S_4, and so $\mathbf{N}_M(T_{\alpha\beta}) \cong \mathrm{S}_3$. Thus $\mathbf{N}_T(T_{\alpha\beta}) = \mathbf{N}_M(T_{\alpha\beta})$, so there is no $g \in \mathbf{N}_T(T_{\alpha\beta})$ with $\langle g, T_\alpha\rangle = T$, a contradiction. Suppose that $T_\alpha = G_\alpha \cong \mathrm{S}_4$. Then $G = T$ or $T.\mathbb{Z}_2$, and $G_{\alpha\beta} \cong \mathrm{S}_3$ for $\beta \in \Gamma(\alpha)$. Checking the maximal subgroups of G in the Atlas [7], we conclude that either $\mathbf{N}_G(G_{\alpha\beta}) = G_{\alpha\beta}$, or $G = \mathrm{S}_6$ and both $\mathbf{N}_G(G_{\alpha\beta})$ and G_α are contained in a maximal subgroup isomorphic to $\mathrm{S}_4\times\mathbb{Z}_2$. Thus $\langle g, G_\alpha\rangle \neq G$ for any $g \in \mathbf{N}_G(G_{\alpha\beta})$, again a contradiction. □

Lemma 4.5. *Assume that Γ is G-vertex-transitive and G-edge-transitive. If* $\mathrm{soc}(G) = \mathrm{PSL}(2, p^f)$ *with $f \geq 2$ and $p = 2$ or 3, then* $\mathrm{soc}(G) \cong \mathrm{A}_5$.

Proof. Assume that $T := \mathrm{soc}(G) = \mathrm{PSL}(2, p^f)$ for $f \geq 2$ and $p = 2$ or 3. Since T is normal in G, all T-orbits on V have the same length $|T : T_\alpha|$, where $\alpha \in V$. Then $|T : T_\alpha|$ is square-free. Thus p^{f-1} is divisor of $|T_\alpha|$.

Suppose that $f > 3$. Then, checking the subgroups of T (see [10, II.8.27], for example), we know that $T_\alpha \cong \mathbb{Z}_p^e{:}\mathbb{Z}_t$, where $e = f - 1$ or f, and t is a divisor of $p^f - 1$. In particular, $e \geq 3$ and T_α has a unique Sylow p-subgroup. For an arbitrary $\beta \in \Gamma(\alpha)$, by Lemma 2.5, $|T : T_{\alpha\beta}|$ is a divisor of 4, so p is divisor of $|T_{\alpha\beta}| = |T_\alpha \cap T_\alpha|$. Let P_1 and P_2 be Sylow p-subgroups of T such that P_1 contains the Sylow p-subgroup of T_α and P_2 contains the Sylow p-subgroup of T_β. Then, by [10, II.8.5], we conclude that $P := P_1 = P_2$. Thus the stabilizers P_α and P_β are the Sylow p-subgroups of T_α and T_β, respectively. Let $\gamma \in \Gamma(\beta)$. Since G is transitive on E, we have $|T_{\alpha\beta}| = |T_{\beta\gamma}|$. A similar argument implies that P_γ is the Sylow p-subgroup of T_γ. It follows from the connectedness of Γ that P_δ is the Sylow p-subgroup of T_δ for any $\delta \in V$. Then P contains a subgroup $Q = \langle P_\delta \mid \delta \in V\rangle \neq 1$. For $x \in G$, we have $P_\delta^x \leq T_\delta^x = T \cap G_\delta^x = T_{\delta^x}$, so P_δ^x is the the Sylow p-subgroup of T_δ, hence $P_\delta^x = P_{\delta^x}$. It follows that Q is a normal subgroup of G, which is impossible.

Therefore, $f = 2$ or 3. By the Atlas [7], neither $\mathrm{PSL}(2,8)$ nor $\mathrm{PSL}(2,27)$ has $\{2,3\}$-subgroups of square-free index. Thus $T = \mathrm{PSL}(2,p^2) \cong \mathrm{A}_5$ by Lemma 4.4. □

By [20], any two distinct Sylow 2-subgroups of $\mathrm{Sz}(2^f)$ intersect trivially. Then a similar argument as in Lemma 4.5 implies the next lemma.

Lemma 4.6. *Assume that G is transitive on both V and E. Then* $\mathrm{soc}(G) \neq \mathrm{Sz}(2^f)$.

Note that $\mathrm{A}_5 \cong \mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$ and $\mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$. By Lemmas 4.2 to 4.6, we have the following Theorem.

Theorem 4.7. *Let $\Gamma = (V,E)$ be a connected tetravalent graph of square-free order. Assume that Γ is G-vertex-transitive and G-edge-transitive, where $G \leq \mathsf{Aut}\Gamma$. If G is insoluble then $G = C{:}X$,* $\mathrm{soc}(X)$ *is normal in G and* $\mathrm{soc}(X) = \mathrm{A}_7$, J_1, $\mathrm{PSL}(3,3)$ *or* $\mathrm{PSL}(2,p)$, *where $p \geq 5$ is a prime.*

5. Examples

In this section we construct the graphs involved in Theorem 1.1. We always assume that p is a prime no less than 5.

5.1. **Graphs constructed from almost simple groups.** The first two examples give arc-transitive graphs associated with the symmetric group S_7 and the first Janko group J_1, respectively.

Example 5.1. Let $G = \mathrm{S}_7$, $P = \langle (1\,2)(3\,4), (1\,3)(2\,4)\rangle$, $K = \langle (234)(567), (34)(56)\rangle$ and $H = P{:}K$. Then $\mathbf{N}_G(K) = K{:}\langle \pi\rangle$, where $\pi = (2\,5)(3\,7)(4\,6)$. It is easily shown that $\langle H, \pi\rangle = G$. Thus $\mathsf{Cos}(G; H, H\pi H)$ is a connected 2-arc-transitive graph of valency 4 and order 210. □

Example 5.2. Let $G = \mathrm{J}_1$. By the information for G given in the Atlas [7], all subgroups isomorphic to A_4 are conjugate, and all subgroups of order 4 are conjugate. Take a subgroup H isomorphic to A_4. Let Q be the Sylow 2-subgroup of H, and let P be a Sylow 3-subgroup of H. Then $Q \cong \mathbb{Z}_2^2$, $P \cong \mathbb{Z}_3$ and $\mathbf{N}_G(P) \cong \mathrm{D}_6{\times}\mathrm{D}_{10}$.

(1) Computation shows that $\mathbf{N}_G(P)$ contains exactly 8 involutions g with $\langle g, H\rangle = G$ (confirmed by GAP). For such an involution g, the coset graph $\mathsf{Cos}(G, H, HgH)$ is connected, $(G, 2)$-arc-transitive and of valency 4.

(2) There are exactly 1184 involutions g in G such that $\langle g, Q\rangle = G$ (confirmed by GAP). For such an involution g, the coset graph $\mathsf{Cos}(G, Q, QgQ)$ is connected, G-arc-transitive and of valency 4.

(3) Computation shows that G has exactly 6 involutions g such that $\langle g, H\rangle = G$ and g centralizes some element of order 3 in H (confirmed by GAP). Let g be such an involution. Take an element $b \in H$ of order 3 with $gb = bg$. Then b induce an automorphism $\widetilde{b}$ of $\varGamma = \mathsf{Cos}(G, Q, QgQ)$ acting on $[G : Q]$ by left multiplication. Recall that G is viewed as a subgroup of $\mathsf{Aut}\varGamma$ which acts on $[G : Q]$ by the right multiplication. Clearly, $b \neq \widetilde{b}$, and $\widetilde{b}$ centralizes G. It is easily shown that $b^{-1}\widetilde{b}$ has order 3 and fixes the vertex Q. Thus $\mathsf{Aut}\varGamma \geq \langle G, \widetilde{b}\rangle = G{\times}\langle\widetilde{b}\rangle \cong \mathrm{J}_1{\times}\mathbb{Z}_3$, and $\varGamma$ is a 2-arc-transitive graph. □

We now construct some graphs associated with the simple group $\mathrm{PSL}(2, p)$. Let $G = \mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$, and let $\varGamma = (V, E)$ be a connected graph of valency 4 such that G acts transitively on both V and E. If $\varGamma$ is $(G, 2)$-arc-transitive then, by [9], we may construct easily $\varGamma$ as a coset graph. If G_α is maximal in G for some $\alpha \in V$, that is, G is primitive on V, then $\varGamma$ is explicitly known by [13]. In the following four examples we list some graphs which are not vertex-primitive.

Example 5.3. Let $p \equiv \pm 1 \pmod 3$ and $p \equiv \mp 1 \pmod 8$. Let $G = \mathrm{PGL}(2, p)$, $\mathrm{S}_4 \cong H < \mathrm{soc}(G)$ and $\mathrm{S}_3 \cong K < H$. Then $\mathbf{N}_G(K) \cong \mathrm{S}_3{\times}\mathbb{Z}_2$. Write $\mathbf{N}_G(K) = K{\times}\langle o\rangle$. Then $\varGamma = \mathsf{Cos}(G, H, HoH)$ is a connected $(G, 2)$-arc-transitive graph of valency 4. If $p = 7$, then $\varGamma$ is the non-incidence graph of the projective plane $\mathrm{PG}(2, 2)$. □

Example 5.4. Let $\epsilon = \pm 1$ such that $p + \epsilon$ is divisible by 3.

(1) Let $G = \mathrm{PSL}(2, p)$ with $p \equiv \pm 3 \pmod 8$ and $p \equiv \pm 1 \pmod{10}$. Then G has one conjugacy class of subgroups isomorphic to A_4 and two conjugacy classes of subgroups isomorphic to A_5. Take M_1, $M_2 < G$ with $M_1 \cong M_2 \cong \mathrm{A}_5$ and $H := M_1 \cap M_2 \cong \mathrm{A}_4$.

Let $K < H$ with $K \cong \mathbb{Z}_3$. Then $\mathbf{N}_{M_1}(K) \cong \mathbf{N}_{M_2}(K) \cong \mathrm{D}_6$. Set $\mathbf{N}_{M_i}(K) = K{:}\langle b_i \rangle$ for $i = 1, 2$. It is easily shown that $\mathbf{N}_{M_1}(K) \cup \mathbf{N}_{M_2}(K)$ contains 6 involutions, which form two distinct cosets Kb_1 and Kb_2. Moreover, b_1, $b_2 \in \mathbf{N}_G(K) \cong \mathrm{D}_{p+\epsilon}$. Set $\mathbf{C}_G(K) = \langle a \rangle$. Then $\mathbf{N}_G(K) = \langle a, b_1 \rangle = \langle a, b_2 \rangle$. Write $b_2 = a^r b_1$ for some $1 \le r \le \frac{p+\epsilon}{2}$. Then $\langle a^r \rangle \not\le K = \langle a^{\frac{p+\epsilon}{6}} \rangle$. Replacing b_1 by $a^{\frac{p+\epsilon}{6}} b_1$ or $a^{\frac{p+\epsilon}{3}} b_1$ if necessarily, we assume that $1 \le r < \frac{p+\epsilon}{6}$. Then, for each j with $1 \le j < \frac{r}{2}$ or $r < j < \frac{r}{2} + \frac{p+\epsilon}{12}$, the coset graph $\Gamma_j = \mathsf{Cos}(G, H, Ha^j b_1 H)$ is connected, $(G, 2)$-arc-transitive and of odd order.

(2) Let $G = \mathrm{PSL}(2, p)$ with $p \equiv \pm 1 \pmod 8$. Then G has a maximal subgroup $M \cong \mathrm{S}_4$. Let $\mathrm{A}_4 \cong H < M$ and $\mathbb{Z}_3 \cong K < H$. Then $\mathbf{N}_G(K) \cong \mathrm{D}_{p+\epsilon}$. Set $M = H{:}\langle b \rangle$, where b is an involution normalizes K. Write $\mathbf{N}_G(K) = \langle a \rangle{:}\langle b \rangle$, where a has order $\frac{p+\epsilon}{2}$. For each $1 \le j < \frac{p+\epsilon}{12}$, define $\Gamma_j = \mathsf{Cos}(G, H, Ha^j bH)$. Then Γ_j is $(G, 2)$-arc-transitive.

If $p \not\equiv \pm 1 \pmod{10}$ then it is easily shown that each Γ_j is connected.

Assume that $p \equiv \pm 1 \pmod{10}$. In this case, G has two conjugacy classes of subgroups isomorphic to A_4 and two conjugacy classes of subgroups isomorphic to A_5. Computation shows that $H \cong \mathrm{A}_4$ is contained exactly two subgroups isomorphic A_5. Let $H < M_1 \cong \mathrm{A}_5$. Then $H < M_2 := M_1^b$. Set $\mathbf{N}_{M_1}(K) = K{:}\langle b_1 \rangle$ and $b_2 = b_1^b$. Then $\mathbf{N}_{M_2}(K) = K{:}\langle b_2 \rangle$ and b_1, $b_2 \in \mathbf{N}_G(K)$. Choosing a suitable b_1, we may set $b_1 = a^r b$ for some $1 \le r < \frac{p+\epsilon}{6}$. For $1 \le j < \frac{p+\epsilon}{12}$, the graph Γ_j is connected if and only if $a^j b \notin \mathbf{N}_{M_1}(K) \cup \mathbf{N}_{M_2}(K)$, that is, $j \ne r$.

(3) Let $G = \mathrm{PGL}(2, p)$ for $p \equiv \pm 3 \pmod 8$. Then G has a maximal subgroup $M \cong \mathrm{S}_4$. Let $\mathrm{A}_4 \cong H < M$ and $\mathbb{Z}_3 \cong K < H$. Set $M = H{:}\langle z \rangle$, where z is an involution normalizes K. Then $\mathbf{N}_G(K) \cong \mathrm{D}_{2(p+\epsilon)}$. Write $\mathbf{N}_G(K) = \langle a \rangle{:}\langle z \rangle$, where a has order $p + \epsilon$. For each $1 \le j < \frac{p+\epsilon}{6}$, the graph $\Gamma_j = \mathsf{Cos}(G, H, Ha^j zH)$ is a connected $(G, 2)$-arc-transitive graph. $\Box$

Example 5.5. Let $\epsilon = \pm 1$ such that $p + \epsilon$ is divisible by 4. Let G be an almost simple group with socle $T = \mathrm{PSL}(2, p)$. Suppose that G has a subgroup isomorphic to D_8. Let $x \in G$ be of order 4 and $y \in G$ be an involution with $x^y = x^{-1}$. Then $x^2 \in \mathbf{C}_G(y) = \mathbf{N}_G(\langle y \rangle)$. Set $H = \langle x, y \rangle$ and write $\mathbf{C}_G(y) = \langle a \rangle{:}\langle x^2 \rangle$.

(1) Let $G = \mathrm{PSL}(2, p)$ with $p \equiv \pm 7, \pm 9 \text{ or } \pm 15 \pmod{32}$. Then $a \in G$ is of order $\frac{p+\epsilon}{2}$, $y = a^{\frac{p+\epsilon}{4}}$ and $\mathbb{Z}_2^2 \cong \langle x^2, y \rangle \lhd \langle x, y, a^{\frac{p+\epsilon}{8}} \rangle \cong \mathrm{S}_4$. For each $i \ne \frac{p+\epsilon}{8}$ with $1 \le i < \frac{p+\epsilon}{4}$, the graph $\mathsf{Cos}(G, H, Ha^i H)$ is connected and G-arc-transitive.

(2) Let $G = \mathrm{PGL}(2, p)$ with $p \equiv \pm 7 \pmod{16}$. Then $x \in T$, and $\mathbf{C}_G(y) \cong \mathrm{D}_{2(p\pm\epsilon)}$. If $y \in T$ then, for each odd i with $1 \le i < \frac{p+\epsilon}{2}$, the graph $\mathsf{Cos}(G, H, Ha^i H)$ is connected, bipartite and G-arc-transitive. If $y \notin T$ then, for each even i with $1 < j < \frac{p-\epsilon}{2}$, the graph $\mathsf{Cos}(G, H, Ha^j H)$ is of even order, connected and G-arc-transitive.

(3) Let $G = \mathrm{PGL}(2, p)$ with $p \equiv \pm 3 \pmod 8$. If $y \in T$ then, for each $i \ne \frac{p+\epsilon}{4}$ with $1 \le i < \frac{p+\epsilon}{2}$, the graph $\mathsf{Cos}(G, H, Ha^i H)$ is connected and G-arc-transitive. If $y \in G \setminus T$ then, for each j with $1 \le j < \frac{p-\epsilon}{2}$, the graph $\mathsf{Cos}(G, H, Ha^j H)$ is connected and G-arc-transitive. $\Box$

Example 5.6. Let $G = \mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$, and $\mathbb{Z}_2^2 \cong K < T := \mathrm{soc}(G)$. Suppose that K is contained in a subgroup $H \cong \mathrm{D}_{16}$ of G. Then $\mathbf{N}_G(K) \cong \mathrm{S}_4$. Write $\mathbf{N}_G(K) = K{:}(\langle y \rangle{:}\langle z \rangle)$ with $\langle y \rangle{:}\langle z \rangle \cong \mathrm{S}_3$. If $H < T$ then $\mathsf{Cos}(T, H, HyzH)$ is a connected T-arc-transitive graph of valency 4; if $H \not\leq T$ then $G = \mathrm{PGL}(2, p)$ and $\mathsf{Cos}(G, H, HyzH)$ is a connected G-arc-transitive graph of valency 4. $\square$

5.2. **Examples of normal covers.** Now we construct some graphs which are normal covers of graphs admitting $\mathrm{PSL}(2, p)$.

Example 5.7. Let $\epsilon = \pm 1$ such that $p+\epsilon$ is divisible by 3. Let $T = \mathrm{PSL}(2, p)$, $\mathbb{Z}_2^2 \cong P < T$ and $x \in T$ such that $\langle P, x\rangle = P{:}\langle x\rangle \cong \mathrm{A}_4$. Then $\mathbf{C}_T(x) \cong \mathbb{Z}_{\frac{p+\epsilon}{2}}$. Set $\mathbf{C}_T(x) = \langle a\rangle$. Then $\langle x\rangle = \langle a^{\frac{p+\epsilon}{6}}\rangle$.

(1) Assume that $p+\epsilon$ is divisible 12. Let $C = \langle y\rangle \cong \mathbb{Z}_3$ and $G = C{\times}T$. Take $H = P{:}\langle xy\rangle$ and $K = \langle xy\rangle$. Then $H \cong \mathrm{A}_4$ and $\mathbf{N}_G(K) = \mathbf{C}_T(x){\times}\langle y\rangle$ contains a uniqe involution $a^{\frac{p+\epsilon}{4}}$. It is easy to show that $\Gamma = \mathsf{Cos}(G, H, Ha^{\frac{p+\epsilon}{4}}H)$ is a connected $(G, 2)$-arc-transitive graph of valency 4.

Take involutions $\sigma \in \mathsf{Aut}(C)$ and $\tau \in \mathrm{PGL}(2, p) \setminus T$ such that $x^\tau = x^{-1}$ and $P{:}\langle x, \tau\rangle \cong \mathrm{S}_4$. Then τ normalizes $\langle x\rangle$ and centralizes $a^{\frac{p+\epsilon}{4}}$. Thus $\sigma\tau$ centralizes $a^{\frac{p+\epsilon}{4}}$. Clearly, $\sigma\tau$ normalizes H. Define $\theta : Hg \mapsto Hg^{\sigma\tau}$, $g \in G$. Then $\mathsf{Aut}\Gamma \geq \langle \theta, G\rangle \cong (\mathbb{Z}_3{\times}\mathrm{PSL}(2, p)){:}\mathbb{Z}_2$ with $\langle y, \theta\rangle \cong \mathrm{D}_6$ and $\langle T, \theta\rangle \cong \mathrm{PGL}(2, p)$.

(2) Assume that $p \equiv \pm 3 \pmod 8$. Let $C = \langle y\rangle \cong \mathbb{Z}_3$ and $G = (C{\times}T){:}\langle\theta\rangle$ such that $y^\theta = y^{-1}$, $x^\theta = x^{-1}$, $\langle P, x, \theta\rangle = P{:}\langle x, \theta\rangle \cong \mathrm{S}_4$ and $\langle T, \theta\rangle = T{:}\langle\theta\rangle \cong \mathrm{PGL}(2, p)$. Take $H = P{:}\langle xy\rangle$ and $K = \langle xy\rangle$. Then $\mathbf{N}_G(K) = (\langle a\rangle{\times}\langle y\rangle){:}\langle\theta\rangle = \langle xy\rangle{:}(\langle a\rangle{:}\langle\theta\rangle) \cong \mathbb{Z}_3{:}\mathrm{D}_{p+\epsilon}$. It is easily shown that $G = \langle a^i\theta, H\rangle$ if and only if $\langle a^i, P\rangle = T$. For $1 \leq i < \frac{p+\epsilon}{2}$ with $i \notin \{\frac{p+\epsilon}{6}, \frac{p+\epsilon}{4}, \frac{p+\epsilon}{3}\}$, define $\Gamma_i = \mathsf{Cos}(G, H, Ha^i\theta H)$. Then Γ_i is a connected $(G, 2)$-arc-transitive bipartite graph of valency 4.

(3) Assume that $p \equiv \pm 1 \pmod 8$ and $p+\epsilon$ is divisible by 12. Let $G = C{\times}T$, where $C = \langle y\rangle \cong \mathbb{Z}_2$. Take an involution $b \in T$ with $x^b = x^{-1}$ and $\langle P, x, b\rangle = (P{:}\langle x\rangle){:}\langle b\rangle \cong \mathrm{S}_4$. Set $H = \langle P, x\rangle{:}\langle by\rangle$ and $K = \langle x, by\rangle$. Then $H \cong \mathrm{S}_4$, $K \cong \mathrm{S}_3$ and $\mathbf{N}_G(K) = \langle a^{\frac{p+\epsilon}{4}}\rangle{\times}\langle x, b\rangle{\times}\langle y\rangle$. It is easily shown that both $\mathsf{Cos}(G, H, Ha^{\frac{p+\epsilon}{4}}H)$ and $\mathsf{Cos}(G, H, Ha^{\frac{p+\epsilon}{4}}yH)$ are connected $(G, 2)$-arc-transitive graphs of valency 4.

(4) Assume that $p \equiv \pm 1 \pmod 8$ and $p+\epsilon$ is divisible by 12. Let $G = (\langle y\rangle{:}\langle y_1\rangle){\times}T \cong \mathrm{D}_6{\times}\mathrm{PSL}(2, p)$. Take an involution $b \in T$ with $x^b = x^{-1}$. Set $H = (\langle P\rangle{:}\langle xy\rangle){:}\langle by_1\rangle$ and $K = \langle xy, by_1\rangle$. Then $H \cong \mathrm{S}_4$, $K \cong \mathrm{S}_3$ and $\mathbf{N}_G(K) = \langle a^{\frac{p+\epsilon}{4}}\rangle{\times}K$. It is easily shown that $\mathsf{Cos}(G, H, Ha^{\frac{p+\epsilon}{4}}H)$ is a connected $(G, 2)$-arc-transitive graph.

(5) Assume that $p \equiv \pm 3 \pmod 8$ and $p+\epsilon$ is not divisible by 4. Let $z \in \mathrm{PGL}(2, p) \setminus \mathrm{PSL}(2, p)$ be an involution with $x^z = x^{-1}$ and $P^z = P$. Let $G = (\langle y\rangle{:}\langle y_1\rangle){\times}(T{:}\langle z\rangle) \cong \mathrm{D}_6{\times}\mathrm{PGL}(2, p)$. Take $H = (P{:}\langle xy\rangle){:}\langle y_1 z\rangle$ and $K = \langle xy\rangle{:}\langle y_1 z\rangle$. Then $H \cong \mathrm{S}_4$, $K \cong \mathrm{S}_3$ and $\mathbf{N}_G(K) = \langle o\rangle{\times}K$, where o is the unique involution in $\mathbf{C}_{T:\langle z\rangle}(x) \cong \mathrm{D}_{2(p+\epsilon)}$. It is easily shown that $\mathsf{Cos}(G, H, HoH)$ is a connected $(G, 2)$-arc-transitive graph.

(6) Assume that $p \equiv \pm 3 \pmod 8$. Let $G = \langle y\rangle{:}\langle y_1\rangle{\times}T \cong \mathrm{D}_6{\times}\mathrm{PSL}(2, p)$. Take $H = P{:}\langle xy\rangle \cong \mathrm{A}_4$ and $K = \langle xy\rangle$. Take an involution $b \in T$ with $x^b = x^{-1}$. Then $\mathbf{N}_G(K) = (\langle a\rangle{\times}\langle y\rangle){:}\langle by_1\rangle = \langle xy\rangle(\langle a\rangle{:}\langle by_1\rangle)$. For each $1 \leq i < \frac{p+2+\epsilon}{4}$, the coset graph $\mathsf{Cos}(G, H, Ha^i by_1 H)$ is a connected $(G, 2)$-arc-transitive graph. $\square$

Example 5.8. Let $X = \mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$ such that X has a Sylow 2-subgroup isomorphic to D_8. Let $x \in X$ be of order 4 and $z \in X$ be an involution such that

$x^z = x^{-1}$. Then $x^2 \in \mathbf{C}_X(z)$. Write $\mathbf{C}_X(z) = \langle a, x^2\rangle$ with $a^{x^2} = a^{-1}$. Let $\epsilon = \pm 1$ such that $p+\epsilon$ is divisible by 4. Let $G = \langle y\rangle \times X$, where y has order 2.

(1) Let $X = \mathrm{PGL}(2,p)$ with $p \equiv \pm 3 \pmod 8$. Then a has order $p \pm \epsilon$, and the following graphs are connected and G-arc-transitive.

(i) $\mathsf{Cos}(G, H, Hy^j a^i H)$, where $H = \langle xy, z\rangle$, $z \notin T$, $j = 0, 1$ and i is even with $1 \le i < \frac{p-\epsilon}{2}$.

(ii) $\mathsf{Cos}(G, H, Ha^i H)$, where $H = \langle x, yz\rangle$ and either $1 \le i < \frac{p-\epsilon}{2}$ for $z \notin \mathrm{PSL}(2,p)$, or $i \ne \frac{p+\epsilon}{4}$ with $1 \le i < \frac{p+\epsilon}{2}$ for $z \in \mathrm{PSL}(2,p)$.

(iii) $\mathsf{Cos}(G, H, Ha^i H)$, where $H = \langle xy, yz\rangle$, $z \in \mathrm{PSL}(2,p)$ and $i \ne \frac{p+\epsilon}{4}$ with $1 \le i < \frac{p+\epsilon}{2}$.

(2) Let $X = \mathrm{PSL}(2,p)$ with $p \equiv \pm 7 \pmod{16}$. Then $\mathbf{C}_X(z) \cong \mathrm{D}_{p+\epsilon}$, $z = a^{\frac{p+\epsilon}{4}}$. For $i \ne \frac{p+\epsilon}{8}$ with $1 \le i < \frac{p+\epsilon}{4}$, the following graphs are connected and G-arc-transitive.

(iv) $\mathsf{Cos}(G, H, Ha^i H)$, where $H = \langle xy, z\rangle$, $\langle x, yz\rangle$ or $\langle xy, yz\rangle$.

(v) $\mathsf{Cos}(G, H, Hya^i H)$, where $H = \langle xy, z\rangle$. □

Example 5.9. Let $T = \mathrm{PSL}(2,p)$ with $p \equiv \pm 15 \pmod{32}$. Then each Sylow 2-subgroup of T is isomorphic to D_{16}. Let $\mathrm{D}_8 \cong P < T$ and $\mathbb{Z}_2^2 \cong K < T$. Then $P < \mathbf{N}_T(K) \cong \mathrm{S}_4$. Write $\mathbf{N}_T(K) = K{:}\langle a, b\rangle$, where a has order 3 and $b \in P$ is an involution with $a^b = a^{-1}$. Take an involution $z \in T$ such that $\langle P, z\rangle = P{:}\langle z\rangle \cong \mathrm{D}_{16}$.

Let $G = \langle y\rangle \times T$, where y has order 2. Then $\mathbf{N}_G(K) = \langle y\rangle \times (K{:}\langle a, b\rangle)$. Set $H = P{:}\langle yz\rangle$. Then, for $g \in \mathbf{N}_G(K)\setminus H$, we have $HgH = HaH$ or $HayH$. It is easily shown that $\mathsf{Cos}(G, H, HaH)$ and $\mathsf{Cos}(G, H, HayH)$ are connected and G-arc-transitive. □

Example 5.10. Let $p \equiv \pm 3 \pmod 8$ and $\epsilon = \pm 1$ such that $p+\epsilon$ is divided by 4. Let $T = \mathrm{PSL}(2,p)$, $X = \mathrm{PGL}(2,p)$ and $z \in X \setminus T$ be an involution. Let $C = \langle c\rangle \cong \mathbb{Z}_l$, where $l > 1$ is coprime to $|T|$. Define a semidirect product $G = C{:}X$ such that $c^z = c^{-1}$ and $CT = C \times T$.

(1) Take an involution $o \in T$ such that $oz = zo$. Set $H = \langle o, z\rangle$. For each $x \in T$ with $x^z = x^{-1}$ and $\langle x, o\rangle = T$, the graph $\mathsf{Cos}(G, H, HcxH)$ is a connected G-arc-transitive graph of valency 4. (It is easily shown there is at least such an x.)

(2) Let $H \cong \mathrm{D}_8$ be a Sylow 2-subgroup of X containing z. Take an involution $o \in H \cap T$ which is not in the center of H. Then $\mathbf{C}_X(o) \cong \mathrm{D}_{2(p+\epsilon)}$. Set $\mathbf{C}_X(o) = \langle a\rangle{:}\langle b\rangle$, where $b \in H \cap T$ and a has order $p+\epsilon$. Then, for each odd $i \ne \frac{p+\epsilon}{4}$ with $1 \le i < \frac{p+\epsilon}{2}$, the graph $\mathsf{Cos}(G, H, Hca^i H)$ is a connected G-arc-transitive graph of valency 4. □

Example 5.11. Let $p \equiv \pm 3 \pmod 8$. Let $T = \mathrm{PSL}(2,p)$ and $o \in T$ be an involution. Let $C = \langle c\rangle \cong \mathbb{Z}_l$ with $l > 1$ coprime to $|T|$. Set $G = C \times T$ and $H = \langle o\rangle$. Take an element $x \in T$ with $\langle x, o\rangle = T$ such that $x^\sigma \ne x^{-1}$ for each automorphism σ of T which fixes o. (It is easily shown there is at least such an x.) Then $\mathsf{Cos}(G, H, H\{cx, c^{-1}x^{-1}\}H)$ is connected, G-half-transitive and of valency 4. □

Example 5.12. Let $X = \mathrm{PGL}(2,p)$ with $p \equiv \pm 3 \pmod 8$. Let $x \in X$ be of order 4 and $z \in T := \mathrm{soc}(X)$ be an involution such that $x^z = x^{-1}$. Then $x^2 \in \mathbf{C}_X(z) \cong \mathrm{D}_{2(p+\epsilon)}$, where $\epsilon = \pm 1$ such that $p+\epsilon$ is divisible by 4. Write $\mathbf{C}_X(z) = \langle a, x^2\rangle$ with $a^{x^2} = a^{-1}$. Then $X = T{:}\langle ax^2\rangle$. Let $C = \langle c, y\rangle \cong \mathbb{Z}_{2l}$, where $l > 1$ is coprime to $|T|$,

c has order l and y is an inovlution. Define a semidirect product $G = (C \times T){:}\langle ax^2 \rangle$ such that $c^{ax^2} = c^{-1}$ and $y^{ax^2} = y$.

Set $H = \langle x, yz \rangle$ or $\langle xy, yz \rangle$. Then $\mathsf{Cos}(G, H, Hc^k a^i H)$ is connected and G-arc-transitive, where k is coprime to l, $i \neq \frac{p+\epsilon}{4}$ and $1 \leq i < \frac{p+\epsilon}{2}$. □

Example 5.13. Let $X = \mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$, and let $C = \langle c, y \rangle \cong \mathrm{D}_{2l}$ with $c^y = c^{-1}$, where $l > 1$ is coprime to $|p(p^2 - 1)|$. Set $G = C \times X$. Suppose that X has a Sylow 2-subgroup $P \cong \mathbb{Z}_2^2$, D_8 or D_{16}. Write $P = Q{:}\langle z \rangle$, where z is an involution.

Set $H = Q{:}\langle yz \rangle$, and take $K < Q$ with $|Q : K| = 2$. For each j coprime to l and a 2-element $x \in \mathbf{N}_X(K)$ with $x^2 \in K$ and $\langle x, P \rangle = X$, the coset graph $\mathsf{Cos}(G, H, Hc^j yxH)$ is connected, G-arc-transitive and of valency 4. □

6. The almost simple case

Let $\Gamma = (V, E)$ be a connected tetravalent graph of square-free order, and $G \leq \mathsf{Aut}\Gamma$. Assume that G is almost simple and Γ is G-vertex-transitive and G-edge-transitive. By Theorem 4.7, we have $T := \mathrm{soc}(G) = \mathrm{soc}(G) = \mathrm{A}_7$, J_1, $\mathrm{PSL}(3, 3)$ or $\mathrm{PSL}(2, p)$. We next determine the possible associated graphs.

Lemma 6.1. *If $T = \mathrm{PSL}(3, 3)$, then Γ is the incidence graph of the projective plane $\mathrm{PG}(2, 3)$, and $\mathsf{Aut}\Gamma = G = T.\mathbb{Z}_2$ has a regular subgroup isomorphic to D_{26}.*

Proof. Let $T = \mathrm{PSL}(3, 3)$. Then, by Lemma 2.6 and the information given in the Atlas [7], we know that $G = T.\mathbb{Z}_2$ and $T_\alpha = \mathbb{Z}_3^2{:}2\mathrm{S}_4$. By [11], the lemma follows. □

Lemma 6.2. *If $T = \mathrm{A}_7$ or J_1, then Γ satisfies one line of Table 1.*

G	$\lvert V \rvert$	Γ
$\mathrm{A}_7, \mathrm{S}_7$	35	Odd graph $\mathbf{O}_4$
S_7	70	Standard double cover of $\mathbf{O}_4$
S_7	210	Example 5.1
J_1		Example 5.2 (1), (2)

Table 1

Proof. Assume first that $T = \mathrm{J}_1$. Then $G = T$ and, by Lemma 2.6 and the information given in the Atlas [7], $T_\alpha \cong \mathbb{Z}_2^2$ or A_4. If $T_\alpha = \mathrm{A}_4$, then Γ is one of the graphs given in Example 5.2 (1). Thus we assume that $T_\alpha = \mathbb{Z}_2^2$.

Suppose that Γ is not T-arc-transitive. Let $\{\alpha, \alpha^x\}$ be an edge of Γ, where $x \in G$. Then $\langle G_\alpha, x \rangle = G$, and $G_\alpha \cap (G_\alpha)^x = G_\alpha \cap G_{\alpha^x} = \langle h^x \rangle$ for an involution h in G_α. If $h^x = h$, then $\langle h \rangle \lhd \langle x, G_\alpha \rangle = G$, a contradiction. Thus $G_\alpha = \langle h, h^x \rangle$ and $G_{\alpha^x} = \langle h^x, h^{x^2} \rangle$. Let Y be the centralizer of h^x in T. Then $h, h^x, h^{x^2} \in Y \cong \mathbb{Z}_2 \times \mathrm{A}_5$. Thus $h^x, h^{x^2} \in Y^x$, and so $G_{\alpha^x} \leq Y \cap Y^x$. By the argument in Example 5.2, we know that $Y = Y^x$, yielding $x \in Y$ as Y is maximal in T. Then $\langle G_\alpha, x \rangle = \langle h, h^x, x \rangle \leq Y$, a contradiction. Thus Γ is T-arc-transitive, and then Γ is isomorphic to one of the graphs given in Example 5.2 (2).

Let $T = \mathrm{A}_7$ in the following. Then $|T_\alpha|$ is divided by 12, and hence Γ is $(G, 2)$-arc-transitive. It is easily shown that $T_\alpha \cong \mathrm{A}_4$, S_4, $\mathrm{A}_4{\times}\mathbb{Z}_3$ or $(\mathrm{A}_4{\times}\mathbb{Z}_3){:}\mathbb{Z}_2$.

Assume that $T_\alpha \cong (\mathrm{A}_4{\times}\mathbb{Z}_3){:}\mathbb{Z}_2$. Then the vertices in each T-orbit on V can be viewed as 3-subsets of $\Pi := \{1, 2, 3, 4, 5, 6, 7\}$. Thus either T is transitive on V and Γ is isomorphic to the odd graph $\mathbf{O}_4$ of order 35, or $G = \mathrm{S}_7$ and Γ is the standard double cover of $\mathbf{O}_4$.

Now we deal with the other cases. We may set $T_\alpha = P{:}T_{\alpha\beta}$, where $\beta \in \Gamma(\alpha)$ and $P \cong \mathbb{Z}_2^2$. Consider the natural action of A_7 on Π. Then P is conjugate to $\langle(12)(34), (13)(24)\rangle$. Without loss of generality, we let $P = \langle(12)(34), (13)(24)\rangle$. Then $\mathbf{N}_T(P) = P{:}\langle(123), (567), (34)(67)\rangle$.

Assume that $T_\alpha \cong \mathrm{A}_4$ or $\mathrm{A}_4{\times}\mathbb{Z}_3$. Then $|T : T_\alpha|$ is even, and it follows that T is transitive on V. Thus Γ is $(T, 2)$-arc-transitive. Write $\Gamma = \mathsf{Cos}(T, T_\alpha, T_\alpha x T_\alpha)$ for a 2-element $x \in \mathbf{N}_T(T_{\alpha\beta})$ with $x^2 \in T_{\alpha\beta}$ and $\langle x, T_\alpha\rangle = T$. Then x is an involution. Since x is an even permutation, x is a product of two transpositions. Noting $T_{\alpha\beta}$ is a Sylow 3-subgroup of T_α, we may choose $T_{\alpha\beta} = \langle(123)\rangle$, $\langle(123)(567)\rangle$ or $\langle(123), (567)\rangle$. Suppose that $T_{\alpha\beta} = \langle(123)\rangle$. Then $\mathbf{N}_T(T_{\alpha\beta}) = \langle(123)\rangle{:}\langle(45)(67), (23)(45)\rangle$, and x is conjugate to $(45)(67)$ or $(23)(45)$ under $T_{\alpha\beta}$. But, for such an x, the group $\langle x, T_\alpha\rangle$ is intransitive on Π, and so $\langle x, T_\alpha\rangle \neq T$, a contradiction. If $T_{\alpha\beta} = \langle(123)(567)\rangle$ or $\langle(123), (567)\rangle$ then, noting that x fixes each $T_{\alpha\beta}$-orbit on Π, x is conjugate to $(2\,3)(6\,7)$ under $T_{\alpha\beta}$, which gives a similar contradiction as above.

Assume that $T_\alpha \cong \mathrm{S}_4$. Then $T_{\alpha\beta} \cong \mathrm{S}_3$, and we may take $T_{\alpha\beta} = \langle(234), (34)(56)\rangle$ or $\langle(234)(567), (34)(56)\rangle$. Suppose that $T_{\alpha\beta} = \langle(234), (34)(56)\rangle$. Then $\mathbf{N}_T(T_{\alpha\beta}) = T_{\alpha\beta}{\times}\langle(1\,7)(5\,6)\rangle$. It is easily shown that, for $x \in \mathbf{N}_T(T_{\alpha\beta})$, the group $\langle T_\alpha, x\rangle$ fixes $\{5, 6\}$ set-wise; in particular, $\langle T_\alpha, x\rangle \neq T$. It follows that T is intransitive the vertices of Γ. Then $G = \mathrm{S}_7$ and $G_\alpha = T_\alpha$, and hence $G_{\alpha\beta} = T_{\alpha\beta}$. Computation shows that $\mathbf{N}_G(T_{\alpha\beta}) = T_{\alpha\beta}{:}\langle(1\,7), (2\,3)\rangle$. Then $\langle G_\alpha, x\rangle \neq G$ for any $x \in \mathbf{N}_G(T_{\alpha\beta})$, a contradiction. Thus $T_{\alpha\beta} = \langle(234)(567), (34)(56)\rangle$. Then $\mathbf{N}_T(T_{\alpha\beta}) = T_{\alpha\beta}$, it implies that T is intransitive on V. Hence $G = \mathrm{S}_7$, $G_\alpha = T_\alpha$ and $G_{\alpha\beta} = T_{\alpha\beta}$. Then $\mathbf{N}_G(T_{\alpha\beta}) = T_{\alpha\beta}{:}\langle\pi\rangle$, where $\pi = (2\,5)(3\,7)(4\,6)$. It is easily shown that $\langle G_\alpha, \pi\rangle = G$. It implies that Γ is isomorphic to the graph constructed in Example 5.1. □

Next we deal with the case where $T = \mathrm{PSL}(2, p)$. Let $\alpha \in V$. Note that G_α is a $\{2, 3\}$-group and the subgroups of $\mathrm{PGL}(2, p)$ are all known, see [3] for example. Then G_α is isomorphic to one of $\mathbb{Z}_2^2$, $\mathbb{Z}_{2^s}$, D_{2^t}, A_4 and S_4, where $s \geq 1$ and $t \geq 3$.

Lemma 6.3. *Assume that $T = \mathrm{PSL}(2, p)$. If $\Gamma = (V, E)$ is not $(G, 2)$-arc-transitive, then one of the following statements holds.*

(1) *$G_\alpha \cong \mathbb{Z}_2$ and Γ is G-half-transitive, or $G_\alpha \cong \mathbb{Z}_4$ and Γ is G-arc-transitive;*

(2) *$G_\alpha \cong \mathbb{Z}_2^2$, either Γ is G-arc-transitive or one of the following occurs:*

   (i) *$\mathbf{C}_{\mathsf{Aut}\Gamma}(G)$ contains an involution θ such that Γ is $\langle\theta, G\rangle$-arc-transitive;*

   (ii) *$G = \mathrm{PSL}(2, p)$ with $p \equiv \pm 3 \pmod 8$, there exists $X \leq \mathsf{Aut}\Gamma$ such that $G < X \cong \mathrm{PGL}(2, p)$ and Γ is X-arc-transitive.*

(3) *$G_\alpha \cong \mathrm{D}_8$, either Γ is isomorphic to one of the graphs in Example 5.5, or $\mathbf{C}_{\mathsf{Aut}\Gamma}(G)$ contains an involution θ such that Γ is $\langle\theta, G\rangle$-arc-transitive.*

(4) *Γ is G-arc-transitive and isomorphic to one of the two graphs in Example 5.6.*

Proof. Assume that Γ is not $(G, 2)$-arc-transitive. Let $\alpha \in V$. Then $G_\alpha \cong \mathbb{Z}_2^2$, $\mathbb{Z}_{2^s}$ or D_{2^t}, where $s \geq 1$ and $t \geq 3$.

Case 1. Assume that G_α is abelian. If $G_\alpha \cong \mathbb{Z}_{2^s}$ then, by Lemma 2.5, $G_\alpha \cong \mathrm{G}_\alpha^{\Gamma(\alpha)} \cong \mathbb{Z}_2$ or $\mathbb{Z}_4$, so part (1) follows. Thus we assume that $G_\alpha \cong \mathbb{Z}_2^2$ in the following.

Suppose that Γ is G-half-transitive. Then $G_{\alpha\beta} \cong \mathbb{Z}_2$ for $\beta \in \Gamma(\alpha)$. Set $G_{\alpha\beta} = \langle o \rangle$ and $\beta = \alpha^x$. Then $o \in G_\beta = G_\alpha^x$, and so $o^{x^{-1}} \in G_\alpha$. Since Γ is connected, we have $G = \langle G_\alpha, x \rangle$. If x centralizes o then o lies in the center of G, which is impossible. Thus $o^{x^{-1}} \neq o$, so $G_\alpha = \langle o, o^{x^{-1}} \rangle$ and $G_\beta = \langle o, o^x \rangle$. Then $G_\alpha, G_\beta < \mathbf{C}_G(x) \cong \mathrm{D}_{l(p\pm 1)}$, where $l = |G : T|$. Set $\mathbf{C}_G(o) = \langle a \rangle{:}\langle o^{x^{-1}} \rangle$. Noting that all subgroups isomorphic to $\mathbf{C}_G(o)$ are conjugate in G, it is easily shown that two subgroups isomorphic to $\mathbb{Z}_2^2$ of $\mathbf{C}_G(o)$ are conjugate in $\mathbf{C}_G(o)$ if and only if they are conjugate in G. Thus $G_\alpha^{a^i} = G_\beta = G_\alpha^x$ for some i, and so $x \in \mathbf{N}_G(G_\alpha)a^i \setminus \langle a \rangle$.

Assume that $p \equiv \pm 1 \,(\mathsf{mod}\ 8)$. Then $G = T$, $\mathbf{N}_G(G_\alpha) \cong \mathrm{S}_4$ and $\mathbf{N}_{\mathbf{C}_G(o)}(G_\alpha) \cong \mathrm{D}_8$. Thus we may write $\mathbf{N}_G(G_\alpha) = G_\alpha{:}(\langle y \rangle{:}\langle z \rangle)$, where y has order 3 and $z \in \mathbf{C}_T(o)$ is an involution normalizing G_α and $\langle y \rangle$. Computation shows that $\alpha^x = \alpha^{y^j a^k}$ for some integers j and k. Define $\theta : \alpha^g \mapsto \alpha^{y^j z g}$, $g \in G$. Then θ is an involution in $\mathbf{C}_{\mathsf{Aut}\Gamma}(G)$. It is easily shown that Γ is $(G{\times}\langle\theta\rangle)$-arc-transitive.

Assume that $p \equiv \pm 3 \,(\mathsf{mod}\ 8)$ and $G = \mathrm{PGL}(2,p)$. If $G_\alpha < T$ then $\mathbf{N}_G(G_\alpha) \cong \mathrm{S}_4$ and $\mathbf{N}_{\mathbf{C}_G(o)}(G_\alpha) \cong \mathrm{D}_8$; a similar argument as above implies that there is an involution $\theta \in \mathsf{Aut}\Gamma$ such that Γ is $(G{\times}\langle\theta\rangle)$-arc-transitive. Suppose that $G_\alpha \not\leq T$. Then $\mathbf{N}_G(G_\alpha) \cong \mathrm{D}_8$ and $\mathbf{N}_{\mathbf{C}_G(o)}(G_\alpha) = G_\alpha$. Write $\mathbf{N}_G(G_\alpha) = G_\alpha{:}\langle z \rangle$ for an involution $z \in \mathrm{PSL}(2,p)$. Then $\alpha^x = \alpha^{z a^i}$. Note that G_α and $\mathrm{PSL}(2,p)$ contain only one involution $oo^{x^{-1}}$ in common. This implies that $oo^{x^{-1}}$ lies in the center of $\mathbf{N}_G(G_\alpha)$. Define $\theta : \alpha^g \mapsto \alpha^{oo^{x^{-1}} z g}$, $g \in G$. Then θ is an involution in $\mathbf{C}_{\mathsf{Aut}\Gamma}(G)$, and Γ is $(G{\times}\langle\theta\rangle)$-arc-transitive.

Let $p \equiv \pm 3 \,(\mathsf{mod}\ 8)$ and $G = \mathrm{PSL}(2,p)$. Then $\mathbf{N}_G(G_\alpha) = G_\alpha{:}\langle y \rangle \cong \mathrm{A}_4$, where $y \in T$ has order 3. Thus $\alpha^x = \alpha^{y^j a^i}$ for some integer j. Noting that $\mathbf{N}_{\mathrm{PGL}(2,p)}(G_\alpha) \cong \mathrm{S}_4$, there is an involution $\sigma \in \mathbf{C}_{\mathrm{PGL}(2,p)}(o) \setminus T$ such that $G_\alpha^\sigma = G_\alpha$ and $y^\sigma = y^{-1}$. Define $\rho : \alpha^g \mapsto \alpha^{y^j g^\sigma}$, $g \in G$. Then $\rho \in \mathsf{Aut}\Gamma$, $\langle G, \rho \rangle \cong \mathrm{PGL}(2,p)$ and Γ is $\langle T, \rho \rangle$-arc-transitive. Then part (2) follows.

Case 2. Assume that $G_\alpha \cong \mathrm{D}_{2^t}$ for $t \geq 3$. Let $\beta \in \Gamma(\alpha)$.

Suppose that $G_{\alpha\beta}$ contains a cyclic subgroup C of order no less than 3. Then C is the unique subgroup of order $|C|$ in both G_α and G_β. For an arbitrary edge $\{\gamma, \delta\}$, since G is transitive on E, there is $x \in G$ with $\{\gamma, \delta\} = \{\alpha, \beta\}^x$, so $G_{\gamma\delta} = G_{\alpha\beta}^x$. Then C^x is the unique subgroup of order $|C|$ in both G_γ and G_δ. So $C \leq G_\gamma$ for $\gamma \in \Gamma(\alpha) \cup \Gamma(\beta)$. Since Γ is connected, C fixes each vertex of Γ, and so $C = 1$ as $C \leq \mathsf{Aut}\Gamma$, a contradiction. Thus $|G_{\alpha\beta}|$ is a divisor of 4, hence $G_\alpha \cong \mathrm{D}_8$ or D_{16}.

Assume that $G_\alpha \cong \mathrm{D}_8$ and Γ is G-arc-transitive. Then G_α is transitive on $\Gamma(\alpha)$. Set $G_\alpha = \langle x \rangle{:}\langle y \rangle$, where x has order 4 and y is an involution with $x^y = x^{-1}$. By Lemma 2.5, we know that $G_\alpha^{[1]} = 1$. It follows that $G_{\alpha\beta}$ dose not lies in the center of G_α. Thus we may choose a suitable y such that $G_{\alpha\beta} = \langle y \rangle$. Write Γ as a coset $\mathsf{Cos}(G, G_\alpha, G_\alpha g G_\alpha)$ for $g \in \mathbf{N}_G\langle y \rangle = \mathbf{C}_G(y)$. Then Γ is constructed as in Example 5.5.

Assume that $G_\alpha \cong \mathrm{D}_8$ and Γ is G-half-transitive. Then $G_{\alpha\beta} \cong \mathbb{Z}_2^2$. Hence $G_{\alpha\beta}$ is normal in $M := \langle G_\alpha, G_\beta \rangle$, yielding $\mathbf{N}_G(G_{\alpha\beta}) = M \cong \mathrm{S}_4$. Let $y \in M$ be an involution such that $G_{\alpha\beta}{:}\langle y\rangle$ is the Sylow 2-subgroup of M other than G_α and G_β. Then $G_\beta = G_\alpha^y$. Let $x \in G$ with $\beta = \alpha^x$. Then $G_\alpha^y = G_\beta = G_\alpha^x$, so $xy \in \mathbf{N}_G(G_\alpha)$. If $xy \in G_\alpha$ then $\langle x, G_\alpha\rangle \leq \langle M, G_\alpha\rangle = M$, which contradicts the connectedness of Γ. Thus $xy \notin \mathbf{N}_G(G_\alpha)$, and so $\mathbf{N}_G(G_\alpha) \neq G_\alpha$, It follows that $\mathbf{N}_G(G_\alpha) \cong \mathrm{D}_{16}$ is a Sylow 2-subgroup of G as $|G : G_\alpha|$ is square-free. Write $\mathbf{N}_G(G_\alpha) = G_\alpha{:}\langle z\rangle$ for some involution z. Then $xy = hz$ for some $h \in G_\alpha$, so $G_\alpha x G_\alpha = G_\alpha xyyG_\alpha = G_\alpha zyG_\alpha$ and $(G_\alpha xG_\alpha)^z = (G_\alpha zyG_\alpha)^z = G_\alpha(zy)^{-1}G_\alpha = G_\alpha x^{-1}hG_\alpha = G_\alpha x^{-1}G_\alpha$. Define $\theta : \alpha^g \mapsto \alpha^{zg}$, $g \in G$. Then θ is an involution in $\mathbf{C}_{\mathsf{Aut}\Gamma}(G)$, and Γ is $(G\times\langle\theta\rangle)$-arc-transitive. Thus part (3) of this lemma follows.

Assume that $G_\alpha \cong \mathrm{D}_{16}$. Then $G_{\alpha\beta} \cong \mathbb{Z}_2^2$ and Γ is G-arc-transitive. If $G_{\alpha\beta} \not\leq T$ then $\mathbf{N}_G(G_{\alpha\beta}) \cong \mathrm{D}_8$, and so $\mathbf{N}_G(G_{\alpha\beta}) \leq G_\alpha$, which is impossible. Thus $G_{\alpha\beta} \leq T$ and $T > \mathbf{N}_G(G_{\alpha\beta}) \cong \mathrm{S}_4$. Write $\mathbf{N}_{G_\alpha}(G_{\alpha\beta}) = G_{\alpha\beta}{:}\langle z\rangle$ and $\mathbf{N}_G(G_{\alpha\beta}) = G_{\alpha\beta}{:}(\langle y\rangle{:}\langle z\rangle)$. Then, for $x \in \mathbf{N}_G(G_{\alpha\beta})$ with $\langle x, G_\alpha\rangle = G$, we have $G_\alpha xG_\alpha = G_\alpha y^{\pm1}G_\alpha$, and either $G = T$ or $G_\alpha \not\leq T$. Noting $G_\alpha yG_\alpha = G_\alpha zyzG_\alpha = G_\alpha y^{-1}G_\alpha$ as $z \in G_\alpha$, it implies that Γ is isomorphic to one of the graphs in Example 5.6. Thus (4) occurs. □

Lemma 6.4. *Assume that $T = \mathrm{soc}(G) = \mathrm{PSL}(2,p)$ and $\alpha \in V$. If Γ is $(G,2)$-arc-transitive, then one of the following statements holds.*

1. $\mathsf{Aut}\Gamma = \mathrm{PSL}(2,p)$, $7 \neq p \equiv \pm1 \pmod 8$, *$\Gamma$ is unique and of order* $\frac{p(p^2-1)}{48}$*;*
2. $\mathsf{Aut}\Gamma = \mathrm{PGL}(2,p)$, $p \equiv \pm3 \pmod 8$, *Γ is unique and of order* $\frac{p(p^2-1)}{24}$*;*
3. $\mathsf{Aut}\Gamma = \mathrm{PSL}(2,p)$, $5 \neq p \equiv \pm3 \pmod 8$ *and* $p \not\equiv 1 \pmod{10}$, *Γ is of order* $\frac{p(p^2-1)}{24}$ *and isomorphic to one of* $[\frac{p+\varepsilon}{12}]$ *graphs, where* $\epsilon = \pm1$ *such that* $p+\epsilon$ *is divisible by* 3*;*
4. $G = \mathrm{PGL}(2,p)$, $\mathrm{S}_4 \cong G_\alpha < T$ *and Γ is constructed as in Example* 5.3*;*
5. $G_\alpha = T_\alpha \cong \mathrm{A}_4$ *and Γ is isomorphic to one of the graphs in Example* 5.4*;*
6. $G_\alpha = T_\alpha \cong \mathrm{A}_4$ *and* $\mathbf{C}_{\mathsf{Aut}\Gamma}(G)$ *contains an involution.*

Proof. Assume that Γ is $(G,2)$-arc-transitive. Then $G_\alpha \cong \mathrm{A}_4$ or S_4. If G_α is maximal in G, then, by [13], one of parts (1)-(3) occurs. Thus we assume further that G_α is not maximal in G. Then either $G_\alpha = T_\alpha \cong \mathrm{A}_4$, or $\mathrm{S}_4 \cong G_\alpha < T$ and $G = \mathrm{PGL}(2,p)$. Let $\epsilon = \pm1$ with $p+\epsilon$ divisible by 3.

Let $\mathrm{S}_4 \cong G_\alpha < T$ and $G = \mathrm{PGL}(2,p)$. Then $G_{\alpha\beta} \cong \mathrm{S}_3$, and $\mathbf{N}_G(G_{\alpha\beta}) \cong \mathrm{S}_3\times\mathbb{Z}_2$. If $p+\epsilon$ is divisible by 4, then $G_{\alpha\beta}$ is contained in a subgroup $M \cong \mathrm{D}_{p+\epsilon}$ of T, so $\mathbf{N}_G(G_{\alpha\beta}) \geq \mathbf{N}_M(G_{\alpha\beta}) \cong \mathrm{S}_3\times\mathbb{Z}_2$, hence $\mathbf{N}_G(G_{\alpha\beta}) \leq T$, a contradiction. Thus $p+\epsilon$ is not divisible by 4, and Γ is isomorphic to the graph in Example 5.3. Thus (4) occurs.

We assume next that $G_\alpha = T_\alpha \cong \mathrm{A}_4$ and $G_{\alpha\beta} \cong \mathbb{Z}_3$. Let $1 \neq x \in G$ a be 2-element with $(\alpha,\beta)^x = (\beta,\alpha)$. Since Γ is connected, $\langle x, G_\alpha\rangle = G$. Moreover, $x \in \mathbf{N}_G(G_{\alpha\beta})$ and $x^2 \in G_{\alpha\beta}$, and so x is an involution. Since G_α is not maximal in G, we have

(i) $G = \mathrm{PSL}(2,p)$ with $p \equiv \pm3 \pmod 8$ and $p \equiv \pm1 \pmod{10}$; or
(ii) $G = \mathrm{PSL}(2,p)$ with $p \equiv \pm1 \pmod 8$; or
(iii) $G = \mathrm{PGL}(2,p)$ with $p \equiv \pm3 \pmod 8$.

Case (i). Suppose that (i) occurs. Then G has one conjugacy class of subgroups isomorphic to A_4 and two conjugacy classes of subgroups isomorphic to A_5. Thus G_α is contained in exactly two subgroups isomorphic to A_5. Take M_1, $M_2 < G$ with $M_1 \cong M_2 \cong \mathrm{A}_5$ and $G_\alpha = M_1 \cap M_2$. Then $\mathbf{N}_{M_1}(G_{\alpha\beta}) \cong \mathbf{N}_{M_2}(G_{\alpha\beta}) \cong \mathrm{D}_6$. Set $\mathbf{N}_{M_i}(G_{\alpha\beta}) = G_{\alpha\beta}{:}\langle b_i\rangle$ for $i = 1, 2$. It is easily shown that $\mathbf{N}_{M_1}(G_{\alpha\beta}) \cup \mathbf{N}_{M_2}(G_{\alpha\beta})$

contains 6 involutions, which form two distinct cosets $G_{\alpha\beta}b_1$ and $G_{\alpha\beta}b_2$. Note that b_1, $b_2 \in \mathbf{N}_G(G_{\alpha\beta}) \cong \mathrm{D}_{p+\epsilon}$. Write $\mathbf{C}_G(G_{\alpha\beta}) = \langle a\rangle$. Then $\mathbf{N}_G(G_{\alpha\beta}) = \langle a, b_1\rangle = \langle a, b_2\rangle$. Set $b_1 = a^r b_2$ for some $1 \le r \le \frac{p+\epsilon}{2}$. Then $\langle a^r\rangle \not\le G_{\alpha\beta} = \langle a^{\frac{p+\epsilon}{6}}\rangle$. Replacing b_1 by $a^{\frac{p+\epsilon}{6}}b_1$ or $a^{\frac{p+\epsilon}{3}}b_1$ if necessarily, we may choose $1 \le r < \frac{p+\epsilon}{6}$. For an involution $x \in \mathbf{N}_G(G_{\alpha\beta})$ with $\langle x, G_\alpha\rangle = G$, we get

(i.1) $G_\alpha x G_\alpha = G_\alpha a^j b_1 G_\alpha$ for $1 \le j < \frac{p+\epsilon}{6}$ with $j \ne r$; or

(i.2) $G_\alpha x G_\alpha = G_\alpha a^{\frac{p+\epsilon}{4}} G_\alpha$ and 4 is a divisor of $p+\epsilon$.

Take an involution $z \in \mathrm{PGL}(2,p) \setminus G$ with $\langle G_\alpha, z\rangle \cong \mathrm{S}_4$ and $\langle G_{\alpha\beta}, z\rangle \cong \mathrm{S}_3$. Then $z \in \mathbf{N}_{\mathrm{PGL}(2,p)}(G_{\alpha\beta}) \cong \mathrm{D}_{2(p+\epsilon)}$, and so $\mathbf{N}_{\mathrm{PGL}(2,p)}(G_{\alpha\beta}) = \langle a, b_1, z\rangle = \langle a, zb_1, z\rangle = \langle a, zb_1\rangle{:}\langle z\rangle$; in particular, $zb_1 \notin \langle a\rangle$ and $\langle a, zb_1\rangle \cong \mathbb{Z}_{p+\epsilon}$. It is easily shown that $M_1^z = M_2$, and so $\mathbf{N}_{M_2}(G_{\alpha\beta}) = (\mathbf{N}_{M_1}(G_{\alpha\beta}))^z$. Thus we may choose z such that $b_2 = b_1^z$. Then $(zb_1)^2 = b_1^z b_1 = a^r$.

Suppose that $2j \equiv r \,(\mathsf{mod}\ \frac{p+\epsilon}{6})$ for some $1 \le j < \frac{p+\epsilon}{6}$. Then $2j = r + \frac{p+\epsilon}{6}$ by the choice of r. Note that $p+\epsilon$ is not divisible by 8. If $p+\epsilon$ is divisible by 4, then r is even, so a^r is of odd order, hence the order of zb_1 is not divisible by 4, which contradicts the fact that $\langle a, zb_1\rangle \cong \mathbb{Z}_{p+\epsilon}$. Thus $\frac{p+\epsilon}{6}$ is odd, and so r is odd and $j > r$.

For (i.1), we have

$$(G_\alpha x G_\alpha)^z = G_\alpha a^{-j} b_1^z G_\alpha = G_\alpha a^{r-j} b_1 G_\alpha = \begin{cases} G_\alpha a^{r-j} b_1 G_\alpha, & \text{if } 1 \le j < \frac{r}{2}; \\ G_\alpha a^{\frac{p+\epsilon}{6}+r-j} b_1 G_\alpha, & \text{if } r < j < \frac{r}{2} + \frac{p+\epsilon}{12}; \\ G_\alpha a^j b_1 G_\alpha, & \text{if } j = \frac{r}{2} + \frac{p+\epsilon}{12}, \frac{p+\epsilon}{6} \text{ is odd.} \end{cases}$$

Thus $\mathit{\Gamma}$ is one of the graphs in Example 5.4 (1), or $\mathit{\Gamma} \cong \mathsf{Cos}(G, G_\alpha, G_\alpha a^{\frac{r}{2}+\frac{p+\epsilon}{12}} b_1 G_\alpha)$ with odd $\frac{p+\epsilon}{6}$. For the latter case, define $\rho : \alpha^g \mapsto \alpha^{g^z}$, $g \in G$. Then $\rho \in \mathbf{N}_{\mathsf{Aut}\mathit{\Gamma}}(G_\alpha)$, $\alpha^\rho = \alpha$ and $X := \langle G, \rho\rangle \cong \mathrm{PGL}(2,p)$. Moreover, $X_\alpha = \langle G_\alpha, \rho\rangle$ is maximal in X. Thus $\mathit{\Gamma}$ is isomorphic to the graph described in part (2).

For (i.2), $\mathrm{D}_{2(p+\epsilon)} \cong \mathbf{N}_{\mathrm{PGL}(2,p)}(G_{\alpha\beta}) = \langle z, \mathbf{N}_G(G_{\alpha\beta})\rangle$. It implies that $a^{\frac{p+\epsilon}{4}}$ lies in the center of $\mathbf{N}_{\mathrm{PGL}(2,p)}(G_{\alpha\beta})$. Then z induces an automorphism of $\mathit{\Gamma}$ by $\alpha^g \mapsto \alpha^{g^z}$, $g \in G$. Arguing as above, we know that $\mathit{\Gamma}$ is isomorphic to the graph described in part (2).

Case (ii). Suppose that (ii) occurs, that is, $G = \mathrm{PSL}(2,p)$ with $p \equiv \pm 1 \,(\mathsf{mod}\ 8)$. Then $G_\alpha \cong \mathrm{A}_4$ is contained a maximal subgroup $M \cong \mathrm{S}_4$. Set $M = G_\alpha{:}\langle b\rangle$, where b is an involution normalizing $G_{\alpha\beta}$. Then $b \in \mathbf{N}_G(G_{\alpha\beta}) \cong \mathrm{D}_{p+\epsilon}$. Write $\mathbf{N}_G(G_{\alpha\beta}) = \langle a\rangle{:}\langle b\rangle$, where a has order $\frac{p+\epsilon}{2}$. By a similar argument as in Case (i), for an involution $x \in \mathbf{N}_G(G_{\alpha\beta})$ with $\langle x, G_\alpha\rangle = G$, either $G_\alpha x G_\alpha = G_\alpha a^j b G_\alpha$ for some $1 \le j < \frac{p+\epsilon}{6}$, or $G_\alpha x G_\alpha = G_\alpha a^{\frac{p+\epsilon}{4}} G_\alpha$ if further 4 is a divisor of $p+\epsilon$. Moreover,

$$(G_\alpha x G_\alpha)^b = G_\alpha x^b G_\alpha = \begin{cases} G_\alpha a^{\frac{p+\epsilon}{6}-j} b G_\alpha \text{ for } 1 \le j < \frac{p+\epsilon}{6}; \text{ or} \\ G_\alpha a^{\frac{p+\epsilon}{4}} G_\alpha. \end{cases}$$

Assume that $p+\epsilon$ is a divisible by 4 and $G_\alpha x G_\alpha = G_\alpha a^{\frac{p+\epsilon}{4}} G_\alpha$ or $G_\alpha a^{\frac{p+\epsilon}{12}} b G_\alpha$. Define $\theta : \alpha^g \mapsto \alpha^{bg}$, $g \in G$. Then θ is an involution in $\mathbf{C}_{\mathsf{Aut}\mathit{\Gamma}}(G)$. Thus part (6) occurs.

Assume that $G_\alpha x G_\alpha = G_\alpha a^j b G_\alpha$, where $j \ne \frac{p+\epsilon}{12}$ and $1 \le j < \frac{p+\epsilon}{6}$. Define $\sigma : G_\alpha g \mapsto G_\alpha bg$, $g \in G$. Then σ is an isomorphism from $\mathsf{Cos}(G, G_\alpha, G_\alpha a^j b G_\alpha)$ to $\mathsf{Cos}(G, G_\alpha, G_\alpha a^{\frac{p+\epsilon}{6}-j} b G_\alpha)$. Thus $\mathit{\Gamma}$ is isomorphic a graph in Example 5.4 (2).

Case (iii). Suppose that (iii) occurs, that is, $G = \mathrm{PGL}(2,p)$ with $p \equiv \pm 3 \,(\mathsf{mod}\ 8)$. Then $G_\alpha = T_\alpha \cong \mathrm{A}_4$ is contained a maximal subgroup $M \cong \mathrm{S}_4$ of G. Set $M =$

$G_\alpha{:}\langle z\rangle$, where $z \in G \setminus T$ is an involution normalizing $G_{\alpha\beta}$. Then $z \in \mathbf{N}_G(G_{\alpha\beta}) \cong \mathrm{D}_{2(p+\epsilon)}$. Write $\mathbf{N}_G(G_{\alpha\beta}) = \langle a\rangle{:}\langle z\rangle$, where a has order $p+\epsilon$. For an involution $x \in \mathbf{N}_G(G_{\alpha\beta})$ with $\langle x, G_\alpha\rangle = G$, either $G_\alpha x G_\alpha = G_\alpha a^j z G_\alpha$ for some $1 \le j < \frac{p+\epsilon}{3}$, or $G_\alpha x G_\alpha = G_\alpha a^{\frac{p+\epsilon}{2}} G_\alpha$. Note that $(G_\alpha a^j z G_\alpha)^z = G_\alpha a^{-j} z G_\alpha = G_\alpha a^{\frac{p+\epsilon}{3}-j} z G_\alpha$ for $1 \le j < \frac{p+\epsilon}{3}$. It follows that Γ is isomorphic to a graph in Example 5.4 (3) or one of $\mathsf{Cos}(G, G_\alpha, G_\alpha a^{\frac{p+\epsilon}{6}} z G_\alpha)$ and $\mathsf{Cos}(G, G_\alpha, G_\alpha a^{\frac{p+\epsilon}{2}} G_\alpha)$. For the latter, $\mathsf{Aut}\Gamma$ has an involution $\alpha^g \mapsto \alpha^{zg}$, $g \in G$, which centralizes G, and so part (6) occurs. □

7. Normal covers

In this section we give a proof of Theorem 1.1.

Let $\Gamma = (V, E)$ be a connected tetravalent graph of square-free order. Assume that Γ is both vertex-transitive and edge-transitive. Let $G = \mathsf{Aut}\Gamma$. If G is soluble then, by Theorem 3.3, one of of Theorem 1.1 (1) and (2) occurs. If G is almost simple then, by the argument in Section 6, either $\mathrm{soc}(G) = \mathrm{PSL}(3,3)$ and Γ is a Cayley graph or one of parts (3)-(5) of Theorem 1.1 occurs.

By Theorem 4.7, we assume next that $G = C{:}X$, $C \ne 1$, $T := \mathrm{soc}(X) \lhd G$ and $T = \mathrm{A}_7$, J_1, $\mathrm{PSL}(3,3)$ or $\mathrm{PSL}(2,p)$, where $p \ge 5$ is a prime. Let B be a C-orbit on V and $\alpha \in B$. Then $G_\alpha \cong X_B$ by Lemma 2.7. Note that Γ is 2-arc-transitive if and only if Γ_C is $(X,2)$-arc-transitive. We shall characterizes Γ in three lemmas.

Lemma 7.1. *Assume that Γ is 2-arc-transitive. Then one of Theorem* 1.1 (4)*,* (6) *and* (10) *occurs.*

Proof. Since Γ has square-free order, T is not semiregular on V, and so $T_\alpha \ne 1$. By Lemma 2.4, $T_\alpha^{\Gamma(\alpha)} \ne 1$. Since Γ is $(G,2)$-arc-transitive, G_α is 2-transitive on $\Gamma(\alpha)$. Noting that $T_\alpha \lhd G_\alpha$ as $T \lhd G$, it follows that T_α is transitive on $\Gamma(\alpha)$. Then T has at most two orbits on V by Lemma 2.2. Thus T_B has at most two orbits on B

Since $(CT)_B = C{\times}T_B$ and C is regular on B, we conclude that T_α is the kernel of T_B acting on B, and so $T_\alpha \lhd T_B$. Let B' be the T_B-orbit on B containing α. Then either $C_{B'} = C$, or $C_{B'}$ is the unique $2'$-Hall subgroup of C. Moreover, $C_{B'}$ and T_B induce two regular permutation groups on B'. Thus $C_{B'} \cong T_B/T_\alpha$ by [8, Theorem 4.2A]. Then T_B/T_α is isomorphic to C or the $2'$-Hall subgroup of C, and hence $|C| = 2$, 3 or 6 by noting that T_B is a $\{2,3\}$-group. Note that $|C| = |B|$ is coprime to $|V_C| = |X : X_B|$. Applying Lemmas 6.1, 6.2 and 6.4 to Γ_C and X, we get a table as follows, where lines 1-6 arise if $C \cong T_B/T_\alpha$ and lines 7-10 arise otherwise.

line	T	T_B	T_α	C	Γ_C	remark
1	A_7	$(\mathbb{Z}_3{\times}\mathrm{A}_4){:}\mathbb{Z}_2$	$\mathbb{Z}_3{\times}\mathrm{A}_4$	$\mathbb{Z}_2$	$\mathbf{O}_4$	
2		$(\mathbb{Z}_3{\times}\mathrm{A}_4){:}\mathbb{Z}_2$	A_4	D_6	$\mathbf{O}_4$	
3	J_1	A_4	$\mathbb{Z}_2^2$	$\mathbb{Z}_3$	5.2 (1)	$G = \mathbb{Z}_3{\times}\mathrm{J}_1$
4	$\mathrm{PSL}(2,p)$	A_4	$\mathbb{Z}_2^2$	$\mathbb{Z}_3$	6.4 (2), (3), (5), (6)	
5	$(p \ge 5)$	S_4	A_4	$\mathbb{Z}_2$	6.4 (1)	$G = C{\times}T$
6		S_4	$\mathbb{Z}_2^2$	D_6	6.4 (1)	$G = C{\times}T$
7	A_7	$(\mathbb{Z}_3{\times}\mathrm{A}_4){:}\mathbb{Z}_2$	$(\mathbb{Z}_3{\times}\mathrm{A}_4){:}\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{O}_4$	
8	$\mathrm{PSL}(2,p)$	A_4	A_4	$\mathbb{Z}_2$	5.4 (1), 6.4 (2), (3)	
9		S_4	S_4	$\mathbb{Z}_2$	6.4 (1)	$G = C{\times}T$
10		A_4	$\mathbb{Z}_2^2$	$\lvert M\rvert = 6$	5.4 (1), 6.4 (2), (3)	

If line 1 or 2 occurs, then Γ is $(\mathrm{A}_7, 2)$-arc-transitive; however, by Lemma 6.2, there is no such a graph, a contradiction. For line 3, Γ is J_1-arc-transitive, and so Γ isomorphic to one of the graphs in Example 5.2 (3), and so Theorem 1.1 (4) occurs.

Assume that line 4 occurs. Then $(CT)_\alpha \cong T_B \cong \mathrm{A}_4$ by Lemma 2.7, and $(CT)_\alpha = T_\alpha{:}\langle xy\rangle$ such that $\mathrm{A}_4 \cong T_\alpha{:}\langle x\rangle \leq T$ and $C = \langle y\rangle$. If T is transitive on V then Γ is $(CT, 2)$-arc-transitive, it follows that Γ is isomorphic to a graph in Example 5.7 (1); in this case, Theorem 1.1 (6) occurs. Thus we assume further that T has two orbits on V. Then $|T : T_\alpha|$ is odd, and $C{\times}T$ has two orbits on V. Hence $G_\alpha = (CT)_\alpha$ and $X = \mathrm{PGL}(2,p)$. We may choose $\beta \in \Gamma(\alpha)$ such that $G_{\alpha\beta} = \langle xy\rangle$. Let $\theta \in X \setminus T$ with $x^\theta = x^{-1}$. If θ centralizes C, then $\mathbf{N}_G(\langle xy\rangle) = \mathbf{C}_T(x){\times}\langle y\rangle$ contains no 2-element $g \in G$ with $\langle g, G_\alpha\rangle = G$, a contradiction. Thus $\langle C, \theta\rangle \cong \mathrm{D}_6$. Note that $|V_C| = |X : X_B| = \frac{p(p^2-1)}{12}$ is square-free and coprime to $|C| = 3$. Then (p^2-1) is not divisible by 9 and 16; in particular, $p \equiv \pm 3 \,(\mathsf{mod}\ 8)$. Let $\epsilon = \pm 1$ such that $p+\epsilon$ is divisible by 3. Set $\mathbf{C}_T(x) = \langle a\rangle$. Then $\mathbf{N}_G(\langle xy\rangle) = (\langle a\rangle{\times}\langle y\rangle){:}\langle\theta\rangle = \langle xy\rangle{:}(\langle a\rangle{:}\langle\theta\rangle) \cong \mathbb{Z}_3{:}\mathrm{D}_{p+\epsilon}$. It is easily shown that $G = \langle a^i\theta^j, G_\alpha\rangle$ if and only if $\langle a^i, P\rangle = T$ and $j = 1$. Thus either Γ is isomorphic to a graph in Example 5.7 (2), or $\Gamma \cong \mathsf{Cos}(G, G_\alpha, G_\alpha a^{\frac{p+\epsilon}{4}}\theta G_\alpha)$ and $p+\epsilon$ is divisible by 12. The former case yields Theorem 1.1 (6). Suppose that the latter case occurs. Note that $a^{\frac{p+\epsilon}{4}}$ lies in the center of $\mathbf{N}_{T:\langle\theta\rangle}(\langle xy\rangle) = \langle a\rangle{:}\langle\theta\rangle$, and so $(G_\alpha a^{\frac{p+\epsilon}{4}}\theta G_\alpha)^\theta = G_\alpha a^{\frac{p+\epsilon}{4}}\theta G_\alpha$. Define $\rho : \alpha^g \mapsto \alpha^{\theta g}$, $g \in G$. It is easily shown that ρ is an automorphism of Γ. Moreover, ρ centralizes G. Thus $C\langle\rho\rangle$ is normal in $G = \mathsf{Aut}\Gamma$. By the choice of C, we have $\rho \in C \cong \mathbb{Z}_3$, and so $\rho = 1$ as $\rho^2 = 1$. Then $\alpha = \alpha^\rho = \alpha^\theta$, hence $\theta \in G_\alpha = T_\alpha\langle xy\rangle$, yielding $\theta \in T_\alpha$, a contradiction.

Similarly, line 5 or 6 implies that Γ is isomorphic to a graph in Example 5.7 (3) or (4), respectively. Thus Theorem 1.1 (6) follows.

For one of lines 7-9, it is easily shown that Γ is the standard double cover of Γ_C which is one of the odd graph $\mathbf{O}_4$ and the graphs in Example 5.4 (1) and Lemma 6.4 (1)-(3), and so Theorem 1.1 (10) occurs.

Assume finally that line 10 occurs. Then $C \cong \mathbb{Z}_6$ or D_6, $T_\alpha \cong \mathbb{Z}_2^2$, and CT is transitive on V. By Lemma 2.7, $(CT)_\alpha \cong T_B$, and so Γ is $(CT, 2)$-arc-transitive. Set $T_B = T_\alpha{:}\langle x\rangle$ and $C = \langle y\rangle{:}\langle y_1\rangle$, where x and y are of order 3 and y_1 is an involution. Since $(CT)_\alpha \not\leq T$, without loss of generality, we may assume that $(CT)_\alpha = T_\alpha{:}\langle xy\rangle$.

Let $C \cong \mathbb{Z}_6$. Then Γ is the standard double cover of a $(\langle y\rangle{\times}T, 2)$-arc-transitive graph Σ of odd order satisfying line 4 of the above table. Thus, by the foregoing argument, Σ is isomorphic to a graph described in Example 5.7 (1). Thus Theorem 1.1 (10) occurs. Thus we assume next that $C \cong \mathrm{D}_6$.

Suppose that $X = \mathrm{PGL}(2,p)$. Then $G_\alpha \cong X_B \cong \mathrm{S}_4$. Take an involution $z \in X \setminus T$ such that $x^z = x^{-1}$ and $X_B = \langle x, z, T_\alpha\rangle \cong \mathrm{S}_4$. Then $X = \langle T, z\rangle \cong \langle T, y_1 z\rangle$, and it is easily shown that one of z and $y_1 z$ centralizes C. Thus, without loss of generality, we assume that $G = C{\times}X$. Then $\mathbf{N}_G((CT)_\alpha) = (CT)_\alpha{:}\langle y_1, z\rangle$. Since $\mathrm{A}_4 \cong (CT)_\alpha \lhd G_\alpha \cong \mathrm{S}_4$, we conclude that $G_\alpha = (CT)_\alpha{:}\langle z\rangle$ or $(CT)_\alpha{:}\langle y_1 z\rangle$. Suppose that $G_\alpha = (CT)_\alpha{:}\langle z\rangle$. Then $G_{\alpha\beta} = \langle xy\rangle{:}\langle z\rangle$ for some $\beta \in \Gamma(\alpha)$. Computation shows that $\mathbf{N}_G(G_{\alpha\beta}) = \langle o\rangle{\times}G_{\alpha\beta}$, where o is the involution in $\mathbf{C}_X(x) \cong \mathbb{Z}_{p+\epsilon}$. Thus, for any $g \in \mathbf{N}_G(G_{\alpha\beta})$, we have $\langle g, G_\alpha\rangle \le \langle o, xy, z, T_\alpha\rangle \le \langle y\rangle\times X \ne G$, which contradicts the connectedness of Γ. Therefore, $G_\alpha = (CT)_\alpha{:}\langle y_1z\rangle = (T_\alpha{:}\langle xy\rangle){:}\langle y_1z\rangle$ and $G_{\alpha\beta} = \langle xy\rangle{:}\langle y_1z\rangle$ for some $\beta \in \Gamma(\alpha)$. Computation shows that $\mathbf{N}_G(G_{\alpha\beta}) = \langle o\rangle\times G_{\alpha\beta}$. Suppose that $p+\epsilon$ is divisible by 4. Then it is easily shown that $o \in T$. For each $g \in \mathbf{N}_G(G_{\alpha\beta})$, we have $\langle g, G_\alpha\rangle \le \langle o, xy, y_1z, T_\alpha\rangle \le (\langle y\rangle\times T){:}\langle y_1z\rangle \ne G$, a contradiction. Thus $p{+}\epsilon$ is not divisible by 4, and Γ is isomorphic the graph in Example 5.7 (5), and so Theorem 1.1 (6) occurs..

Suppose that $X = T$. Then $G = C\times T$ and $T_\alpha{:}\langle xy\rangle = G_\alpha \cong T_B \cong \mathrm{A}_4$. Thus $G_{\alpha\beta} = \langle xy\rangle$ for some $\beta \in \Gamma(\alpha)$, and $\mathbf{N}_G(G_{\alpha\beta}) = (\mathbf{C}_T(x)\times\langle y\rangle){:}\langle by_1\rangle = \langle xy\rangle{:}(\mathbf{C}_T(x){:}\langle by_1\rangle)$, where $b \in T$ is an involution with $x^b = x^{-1}$. Set $\mathbf{C}_T(x) = \langle a\rangle$. Then a has order $\frac{p+\epsilon}{2}$, where $\epsilon = \pm1$ with $p{+}\epsilon$ divisible by 3. Since Γ is connected and G-arc-transitive, there is a 2-element $h \in \mathbf{N}_G(G_{\alpha\beta})$ with $\beta = \alpha^h$ and $\langle h, G_\alpha\rangle = G$. Then such an element h must be an involution, and $G_\alpha h G_\alpha = G_\alpha a^i b y_1 G_\alpha$ for some $0 \le i < \frac{p+\epsilon}{2}$. Note $G = C\times T < C\times\mathrm{PGL}(2,p)$. Take an involution $z \in \mathrm{PGL}(2,p)\setminus T$ with $x^z = x^{-1}$ and $\langle T_\alpha, x, z\rangle \cong \mathrm{S}_4$. Then z, $b \in \mathbf{N}_{\mathrm{PGL}(2,p)}(\langle x\rangle) \cong \mathrm{D}_{2(p+\epsilon)}$. We may write $\mathbf{N}_{\mathrm{PGL}(2,p)}(\langle x\rangle) = \langle a_0\rangle{:}\langle b\rangle$, where a_0 has order $p+\epsilon$ with $a_0^2 = a$. Then, since $z \notin T$, we may set $z = a_0^s b$ for some odd integer s. Replacing b by $a_0^{1-s}b$ if necessary, we assume further that $z = a_0 b$. Then $(G_\alpha a^i b y_1 G_\alpha)^{y_1z} = G_\alpha a^{1-i} b y_1 G_\alpha$. It follows that $\mathsf{Cos}(G, G_\alpha, G_\alpha a^i b y_1 G_\alpha) \cong \mathsf{Cos}(G, G_\alpha, G_\alpha a^{1-i} b y_1 G_\alpha)$. Thus either Γ is isomorphic a graph in Example 5.7 (6), or $p+\epsilon$ is not divisible by 4 and $\Gamma \cong \mathsf{Cos}(G, G_\alpha, G_\alpha a^{\frac{p+2+\epsilon}{4}} b y_1 G_\alpha)$. For the latter case, Γ has an automorphism $\theta : \alpha^g \mapsto \alpha^{g^{y_1z}}, g \in G$, and so $\mathrm{D}_6\times\mathrm{PSL}(2,p) \cong G = \mathsf{Aut}\Gamma \ge \langle G, \theta\rangle = \langle C, T, y_1\theta\rangle \cong \mathrm{D}_6\times\mathrm{PGL}(2,p)$, a contradiction. Then Γ is isomorphic to a graph in Example 5.7 (6), and so Theorem 1.1 (6) occurs. □

Lemma 7.2. *Assume that $C \cong \mathbb{Z}_2$ and Γ is not 2-arc-transitive. Then one of Theorem 1.1 (7) and (10) occurs.*

Proof. By the assumption, $\mathsf{Aut}\Gamma = G = C\times X$ and the quotient graph Γ_C has odd order. Applying Lemmas 6.1-6.4 to the pair (X, Γ_C), we conclude that $T = \mathrm{soc}(X) = \mathrm{PSL}(2,p)$ and Γ_C is X-arc-transitive, and so Γ is arc-transitive. Moreover, $G_\alpha \cong X_B \cong \mathbb{Z}_2^2$, D_8 or D_{16}, where B is a C-orbit and $\alpha \in B$. If $X_B = X_\alpha$ then $G_\alpha \le X$; in this case, it is easily shown that Γ is isomorphic to the standard double cover of Γ_C, and so Theorem 1.1 (10) occurs. Thus we assume next that $X_B \ne X_\alpha$, that is, X_B is transitive on B. In particular, $|X_B : X_\alpha| = 2$.

Since Γ_C has odd order, T is transitive on the vertices of Γ_C. It implies that $|X : X_B| = |T : T_B|$, and so $|X_B : T_B| = |X : T|$. Set $C = \langle y\rangle$.

Assume first that T_B is intransitive on B. Then $T_B = T_\alpha$. Since $T = \mathrm{soc}(X) = \mathrm{PSL}(2,p)$, we have $X = \mathrm{PGL}(2,p)$ and $X_\alpha = T_B = T_\alpha$. Take an involution $z \in X_B\setminus T$. Then $X_B = T_\alpha{:}\langle z\rangle$ and z interchanges the vertices of Γ contained in B. Thus $yz \in G_\alpha$, and so $G_\alpha = T_\alpha{:}\langle yz\rangle$. Set $X_1 = T{:}\langle yz\rangle$. Then $G_\alpha < X_1 \cong \mathrm{PGL}(2,p)$ and $G = C\times X_1$. It follows that Γ is isomorphic to the standard double cover of an X_1-arc-transitive graph (which is isomorphic to Γ_C). Thus Theorem 1.1 (10) occurs.

Assume that T_B is transitive on B. Then $|X_B : X_\alpha| = |T_B : T_\alpha| = 2$, and both X and T are transitive on V. If $X_B \cong \mathbb{Z}_2^2$, then $X = \mathrm{PSL}(2,p)$ and Γ is arc-regular, and hence Theorem 1.1 (7) occurs. We next deal with the cases: $X_B \cong \mathrm{D}_8$ and $X_B \cong \mathrm{D}_{16}$.

Case 1. Let $X_B \cong \mathrm{D}_8$. We shall show that Γ is isomorphic to a graph in Example 5.8, and thus Theorem 1.1 (7) occurs.

Let $x \in X_B$ be of order 4. Then x or xy is contained in G_α. By Lemma 2.5, we conclude $\langle x\rangle$ is regular on $\Gamma_C(B)$. Let $\beta \in \Gamma(\alpha)$ and $B' \in \Gamma_C(B)$ the C-orbit containing β. Take an involution $z \in X_B$ which fixes B' set-wise. Since Γ is a cover of Γ_C, it is easily shown that either z or yz fixes $B \cup B'$ point-wise. Thus $G_{\alpha\beta} = \langle z\rangle$ or $\langle yz\rangle$. By the choices of x and z, we have $x^z = x^{-1}$, $X_B = \langle x, z\rangle$ and G_α is one of $\langle x\rangle{:}G_{\alpha\beta}$ and $\langle xy\rangle{:}G_{\alpha\beta}$. Recalling that $G_\alpha \neq X_B$, either $G_{\alpha\beta} = \langle z\rangle$ and $G_\alpha = \langle xy, z\rangle$, or $G_{\alpha\beta} = \langle yz\rangle$ and $G_\alpha = \langle x, yz\rangle$ or $\langle xy, yz\rangle$.

Since Γ is connected and arc-transitive, $\Gamma \cong \mathsf{Cos}(G, G_\alpha, G_\alpha g G_\alpha)$ for some 2-element $g \in \mathbf{N}_X(G_{\alpha\beta}) = \langle y\rangle\times\mathbf{C}_X(z)$ with $\langle g, G_\alpha\rangle = G$ and $g^2 \in G_{\alpha\beta} \cong \mathbb{Z}_2$. Noting that $x^2 \in \mathbf{C}_X(z)$ and $\mathbf{C}_X(z)$ is dihedral, write $\mathbf{C}_X(z) = \langle a, x^2\rangle$ with $a^{x^2} = a^{-1}$. Then $g = y^j a^i (x^2)^k$ for some integers i, j and k. Thus $G = \langle g, G_\alpha\rangle \le \langle y, x, z, a^i\rangle$, yielding that $\langle x, z, a^i\rangle = X$. It follows that $a^i \neq z$. If a^i has order 4 then $a^{2i} = z$ and $(a^i)^{x^2} = a^{-i}$, and so $(x^2)^{a^i} = x^2 z \in \langle x^2, z\rangle$, yielding $\langle x^2, z\rangle \lhd \langle x, z, a^i\rangle = X$, a contradiction. Thus a^i is not of order 4. Let $\epsilon = \pm 1$ such that $p + \epsilon$ is divisible by 4.

Let $X = \mathrm{PGL}(2, p)$. Then $p \equiv \pm 3 \pmod 8$ and $|T_\alpha| = 2$. Thus $T_\alpha = \langle x^2\rangle$. Assume that $G_{\alpha\beta} = \langle z\rangle$. Then $G_\alpha = \langle xy, z\rangle$, $z \in X \setminus T$, $\mathbf{C}_X(z) \cong \mathrm{D}_{2(p-\epsilon)}$ and $z = a^{\frac{p-\epsilon}{2}}$. Noting that $G_\alpha g G_\alpha = G_\alpha y^j a^i G_\alpha = G_\alpha y^j a^i z G_\alpha$, we conclude that Γ is isomorphic to a graph in Example 5.8 (1.i). Assume that $G_{\alpha\beta} = \langle yz\rangle$. If $G_\alpha = \langle x, yz\rangle$ then $z = a^{\frac{p-\epsilon}{2}}$ or $z = a^{\frac{p+\epsilon}{2}}$ and $G_\alpha g G_\alpha = G_\alpha y^j a^i G_\alpha = G_\alpha y^j a^i yz G_\alpha$, which implies that Γ is isomorphic to a graph in Example 5.8 (1.ii). Suppose that $G_\alpha = \langle xy, yz\rangle$. Then $xz \in X_\alpha$. Since $T_\alpha = \langle x^2\rangle$, we have $xz \notin T$, and so $z \in T$. It follows that Γ is isomorphic to a graph in Example 5.8 (1.iii).

Let $X = \mathrm{PSL}(2, p)$. Then $p \equiv \pm 7 \pmod{16}$, $\mathbf{C}_X(z) \cong \mathrm{D}_{p+\epsilon}$ and $z = a^{\frac{p+\epsilon}{4}}$. It is easily shown that Γ is isomorphic a graph in Example 5.8 (2).

Case 2. Let $X_B \cong \mathrm{D}_{16}$. Then $X_\alpha \cong \mathrm{D}_8$ and $G_\alpha \cong \mathrm{D}_{16}$. Recall that X is transitive on V. Suppose that Γ is X-arc-transitive. Then, by Lemma 2.5, we conclude that the cyclic subgroup of X_α with order 4 must regular on $\Gamma(\alpha)$. It follows that $G_\alpha \cong \mathrm{D}_{16}$ can be written as a product of two subgroups of order 4, which is impossible. Then X_α is not transitive on $\Gamma(\alpha)$. Thus, by Lemma 2.5, $|X_{\alpha\beta}| = 4$ for $\beta \in \Gamma(\alpha)$. Hence $G_{\alpha\beta} = X_{\alpha\beta}$. Note that G_α contains a unique cyclic subgroup of order 4. Again by Lemma 2.5, we conclude that $G_{\alpha\beta} = X_{\alpha\beta} \cong \mathbb{Z}_2^2$. Suppose that $X = \mathrm{PGL}(2, p)$. Recalling that $|X_B : X_\alpha| = |T_B : T_\alpha| = 2$, we know that T_α has order 4. Since T is not semiregular on V, we have $T_\alpha \neq T_{\alpha\beta}$. It follows that $G_{\alpha\beta} = X_{\alpha\beta} \not\le T$. Then $X_\alpha \le \mathbf{N}_X(G_{\alpha\beta}) \cong \mathrm{D}_8$, and so $\mathbf{N}_G(G_{\alpha\beta}) = C\times\mathbf{N}_X(G_{\alpha\beta}) = C\times X_\alpha$. Thus there is no $g \in \mathbf{N}_G(G_{\alpha\beta})$ with $\langle g, G_\alpha\rangle = G$, a contradiction. Then $X = T = \mathrm{PSL}(2, p)$, and so $\mathbf{N}_G(G_{\alpha\beta}) = C\times\mathbf{N}_X(G_{\alpha\beta}) \cong \mathbb{Z}_2\times\mathrm{S}_4$. This implies that Γ is isomorphic a graph in Example 5.9, and so Theorem 1.1 (7) occurs. □

By the foregoing argument, we assume finally that $|C| > 2$ and Γ is not 2-arc-transitive. Applying the argument in Section 6 to the pair (Γ_C, X), we have $T = \mathrm{soc}(X) = \mathrm{J}_1$ or $\mathrm{PSL}(2, p)$. The following lemma will fulfill the proof of Theorem 1.1.

Lemma 7.3. *Assume that $|C| > 2$ and Γ is not 2-arc-transitive. Then one of Theorem 1.1 (8)-(10) occurs.*

Proof. Let B be a C-orbit on V and $\alpha \in B$. Since Γ is not 2-arc-transitive, G_α is a 2-group. Recall $\mathsf{Aut}\Gamma = G = C{:}X$, $G_\alpha \cong X_B$ and C semiregular on V. Since $|V| = |G : G_\alpha| = |C||X : X_B|$ is square-free, $|C|$ and $|X|$ have no common prime divisors other than 2. Since $T = \mathrm{soc}(X) \lhd G$, all T-orbits on V has the same length $|T : T_\alpha|$. Then the number of T-orbits equals to $\frac{|V|}{|T:T_\alpha|} = |C|\frac{|X:X_B|}{|T:T_\alpha|}$, which is no less than 3 as $|C| > 2$. Thus the quotient graph Γ_T is a cycle. Let N be the kernel of G acting on V_T. Then $T \le N$ and G/N is isomorphic to a subgroup of $\mathsf{Aut}\Gamma_T$ which is a dihedral group. Moreover, G/N is transitive on both the vertices and edges of Γ_T. It implies that G/N is either cyclic or isomorphic to $\mathsf{Aut}\Gamma_T$. Note that $N = TN_\alpha$, $T \le X$ and N_α is a 2-group. Then $|C|$ and $|N|$ have no common prime divisors other than 2. In particular, $|C \cap N| \le 2$. Since $C/(C \cap N) \cong N/N \le G/N$ and $|C|$ is square-free, we conclude that C is cyclic or dihedral. We shall discuss in two cases according to the parity of $|C|$.

Case 1. Assume first that $|C|$ is odd. Then $C \cap N = 1$, C is cyclic and X contains a Sylow 2-subgroup of G. Since G_α is a 2-group, let $G_\alpha < X$ by choosing α suitably. Then $G_\alpha \le X_B$. Thus we assume next that $G_\alpha = X_B$ as $G_\alpha \cong X_B$.

Subcase 1.1. Let Γ_C be X-arc-transitive. Then Γ is arc-transitive, and so G acts transitively on the arcs of Γ_T. Thus $G/N \cong \mathsf{Aut}\Gamma_T$ is dihedral. In particular, $N = T$ and $G = CX \ne C{\times}X$. Recalling that $T = \mathrm{soc}(X) = \mathrm{J}_1$ or $\mathrm{PSL}(2,p)$, it follows that $T = \mathrm{PSL}(2,p)$ and $X = \mathrm{PGL}(2,p)$. Set $X = T{:}\langle z\rangle$ for an involution $z \in X \setminus T$.

Take a 2-element $g \in G$ with $(\alpha,\beta)^g = (\beta,\alpha)$ for some $\beta \in \Gamma(\alpha)$. Write $g = cxz^j$ for some $c \in C$, $x \in T$ and $j = 0$ or 1. Then $cc^{(xz^j)^{-1}}(xz^j)^2 = g^2 \in G_{\alpha\beta} < X$, yielding $c^{-1} = c^{xz^j} = c^{z^j}$ and $(xz^j)^2 \in G_{\alpha\beta}$. In particular, $g = cxz$, $c^z = c^{-1}$ and $(xz)^2 \in G_{\alpha\beta}$. Since Γ is connected, $G = \langle g, G_\alpha\rangle \le \langle x, cz, G_\alpha\rangle \cap \langle c, xz, G_\alpha\rangle$. It follows that $\langle c\rangle = C$ and $G_\alpha \not\le T$. Thus we may choose $z \in G_\alpha$.

Recalling that $G_\alpha = X_B$, we have $T_B = T_\alpha$ and $G_\alpha = T_\alpha{:}\langle z\rangle$. Since $CT/N = CT/T \cong C$ is cyclic, CT is transitive on the edges but not on the arcs of Γ_T, it implies that Γ is CT-half-transitive. Then Γ_C is T-half-transitive, and so, by Lemma 6.3, $T_\alpha = T_B \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$ or D_8. Moreover, since Γ is CT-half-transitive, we have $|T_\alpha : T_{\alpha\beta}| = 2$. Then $|G_\alpha : T_{\alpha\beta}| = 4$, hence $G_{\alpha\beta} = T_{\alpha\beta}$.

Suppose that $T_\alpha \cong \mathrm{D}_8$. Then $G_{\alpha\beta} = T_{\alpha\beta} \cong \mathbb{Z}_2^2$, and so $\mathrm{S}_4 \cong \mathbf{N}_X(G_{\alpha\beta}) < T$. Thus $\mathbf{N}_G(G_{\alpha\beta}) = C\mathbf{N}_G(G_{\alpha\beta}) = C{\times}\mathbf{N}_T(T_{\alpha\beta})$, and so $g \notin \mathbf{N}_G(G_{\alpha\beta})$, a contradiction.

Assume that $T_\alpha \cong \mathbb{Z}_2$. Then $G_{\alpha\beta} = T_{\alpha\beta} = 1$, so xz is an involution, and hence $x^z = x^{-1}$. By $G = \langle g, G_\alpha\rangle = \langle cxz, T_\alpha, z\rangle = \langle c, x, T_\alpha, z\rangle$, we know $\langle x, T_\alpha\rangle = T$. Then Γ is isomorphic to a graph given in Example 5.10 (1), and so Theorem 1.1 (8) occurs.

Assume that $T_\alpha \cong \mathbb{Z}_2^2$. Then $G_\alpha \cong \mathrm{D}_8$ and $G_{\alpha\beta} = T_{\alpha\beta} \cong \mathbb{Z}_2$. If $p \equiv \pm 1 \pmod 8$ then $\mathrm{S}_4 \cong N_X(T_\alpha) < T$, and hence $z \in N_X(T_\alpha) < T$, a contradiction. Thus $p \equiv \pm 3 \pmod 8$. Set $G_{\alpha\beta} = \langle o\rangle$ for an involution $o \in T$. Then $T_\alpha < \mathbf{C}_T(o)$, and $g = cxz \in \mathbf{N}_G(G_{\alpha\beta}) = \mathbf{C}_G(o) = C{:}\mathbf{C}_X(o)$, and so $xz \in \mathbf{C}_X(o)$. If $G_\alpha = \mathbf{N}_{\mathbf{C}_X(o)}(T_\alpha)$ then $z \in \mathbf{C}_X(o)$, so $G = \langle cxz, T_\alpha, z\rangle \le C\mathbf{C}_X(o)$, a contradiction. Thus $z \notin \mathbf{C}_X(o)$, and hence Γ is isomorphic to a graph in Example 5.10 (2). Then Theorem 1.1 (8) occurs.

Subcase 1.2. Let Γ_C be X-half-transitive. Then, by the argument in Section 6, we know that $T = \mathrm{PSL}(2,p)$ and $G_\alpha = X_B \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$ or D_8. Suppose that $G \ne C{\times}X$. Then $X = \mathrm{PGL}(2,p)$ and there is an involution $z \in X \setminus T$ such that z dose not centralize C. It follows that $N = T$, and so $G/N \cong C{:}\langle z\rangle$ is not abelian. Then

G/N is dihedral, and so G acts transitively on the arcs of $\varGamma_T$, it implies that $\varGamma$ is arc-transitive, a contradiction. Thus $G = C{\times}X$.

Suppose that $X_B \cong \mathbb{Z}_2^2$ or D_8. By Lemma 6.3 (2) and (3), we conclude that $\mathsf{Aut}\varGamma_C \setminus X$ contains an involution θ which normalizes X and X_B. Define a map $\rho: V \to V$; $\alpha^{c^ix} \mapsto \alpha^{c^{-i}x^\theta}$, $0 \le i \le |C|-1$, $x \in X$. It is easily shown that $\rho \in \mathsf{Aut}\varGamma$ but $\rho \notin G$; however, $G = \mathsf{Aut}\varGamma$, a contradiction.

Let $G_\alpha = X_B \cong \mathbb{Z}_2$. Take $\beta \in \varGamma(\alpha)$ and $g \in G$ with $\beta = \alpha^g$. Then $\varGamma \cong \mathsf{Cos}(G, X_B, X_B\{g, g^{-1}\}X_B)$. Set $g = cx$ with $c \in C$ and $x \in X$. Since $\varGamma$ is connected, we have $G = \langle g, X_B\rangle = \langle c, x, X_B\rangle = \langle c\rangle{\times}\langle x, X_B\rangle$. It implies that $C = \langle c\rangle$ and $\langle x, X_B\rangle = X$. Thus we get a connected X-half-transitive graph $\mathsf{Cos}(X, X_B, X_B\{x, x^{-1}\}X_B)$, which is of valency 4. Then $\varGamma$ is constructed as in Example 5.11, and so Theorem 1.1 (9) occurs.

Case 2. Assume that $|C|$ is even. Then $\varGamma_C$ has odd order, and so X_B is a Sylow 2-subgroup of X. Applying Lemmas 6.1-6.4 to the pair $(\varGamma_C, X)$, we conclude that $T = \mathrm{soc}(X) = \mathrm{PSL}(2,p)$, $G_\alpha \cong X_B \cong \mathbb{Z}_2^2$, D_8 or D_{16}, and $\varGamma_C$ is X-arc-transitive. Then $\varGamma$ is arc-transitive. Since $|C|$ is square-free, C has a unique $2'$-Hall subgroup, say L. Then L is a characteristic subgroup of C, and hence $L \lhd G$. Recall that C is cyclic or dihedral. We set $L = \langle c\rangle \cong \mathbb{Z}_l$, where $l > 1$ is odd and square-free.

Subcase 2.1. Assume that $C \cong \mathbb{Z}_{2l}$ and set $C = L{\times}\langle y\rangle$. Then $G = \langle y\rangle{\times}(L{:}X)$. Consider the quotient graph $\varGamma_{\langle y\rangle}$. Then $\varGamma_{\langle y\rangle}$ is LX-arc-transitive and, by the argument in Case 1, $\varGamma_{\langle y\rangle}$ is isomorphic to a graph in Example 5.10 (2). In particular, $X = \mathrm{PGL}(2,p)$, $p \equiv \pm 3 \,(\mathsf{mod}\ 8)$ and $c^g = c^{-1}$ for each involution $g \in X \setminus T$. If $G_\alpha < LX$ then it is easily shown that $\varGamma$ is isomorphic to the standard double cover of $\varGamma_{\langle y\rangle}$, and then Theorem 1.1 (10) occurs. Thus we assume next that $G_\alpha \not\le LX$.

Let B_1 be the $\langle y\rangle$-orbit containing α. Then $G_\alpha \cong (LX)_{\mathrm{B}_1}$ by Lemma 2.7, and $(LX)_{\mathrm{B}_1}$ is a Sylow 2-subgroup of LX, and so $G_\alpha \cong (LX)_{\mathrm{B}_1} \cong \mathrm{D}_8$. Since X contains a Sylow 2-subgroup of LX, we may assume that $(LX)_{\mathrm{B}_1} < X$. Thus $(LX)_{\mathrm{B}_1} = X_{B_1} \ne G_\alpha$. Let $x \in X_{B_1}$ be of order 4. Then x or xy is contained in G_α. By Lemma 2.5, X_{B_1} is faithful on $\varGamma_{\langle y\rangle}(B_1)$. Thus $\langle x\rangle$ is regular on $\varGamma_{\langle y\rangle}(B_1)$. Let $B_1' \in \varGamma_{\langle y\rangle}(B_1)$, and let $z \in X_{B_1}$ be an involution which fixes B_1' set-wise. Since $\varGamma$ is a cover of $\varGamma_{\langle y\rangle}$, it is easily shown that either z or yz fixes $B_1 \cup B_1'$ point-wise. Let $\beta \in B_1' \cap \varGamma(\alpha)$. Then $G_{\alpha\beta} = \langle z\rangle$ or $\langle yz\rangle$. By the choices of x and z, we have $x^z = x^{-1}$, $X_{B_1} = \langle x, z\rangle$ and $G_\alpha = \langle x\rangle{:}G_{\alpha\beta}$ or $\langle xy\rangle{:}G_{\alpha\beta}$. It follows that either $G_{\alpha\beta} = \langle z\rangle$ and $G_\alpha = \langle xy, z\rangle$, or $G_{\alpha\beta} = \langle yz\rangle$ and $G_\alpha = \langle x, yz\rangle$ or $\langle xy, yz\rangle$.

Suppose that $z \in X \setminus T$. Then $c^z = c^{-1}$. Computation shows that $\mathbf{N}_G(G_{\alpha\beta}) = \mathbf{C}_G(z) = \langle y\rangle{\times}\mathbf{C}_X(z)$. Then there is no $g \in \mathbf{N}_G(G_{\alpha\beta})$ with $\langle g, G_\alpha\rangle = G$, which contradicts that $\varGamma$ is a connected G-arc-transitive graph. Thus $z \in T$.

If $G_{\alpha\beta} = \langle yz\rangle$ then, writing $\varGamma$ as a coset graph, $\varGamma$ is constructed as in Example 5.12, and so Theorem 1.1 (8) occurs. Assume that $G_{\alpha\beta} = \langle z\rangle$ and $G_\alpha = \langle xy, z\rangle$. Set $X_1 = T{:}\langle xyz\rangle$. Then $X_1 \cong \mathrm{PGL}(2,p)$, $G = \langle y\rangle{\times}(LX_1)$ and $G_\alpha < X_1 < LX_1$. It follows that $\varGamma$ is the standard double cover of an LX_1-arc-transitive graph, which is isomorphic to $\varGamma_{\langle y\rangle}$. Thus Theorem 1.1 (10) occurs.

Subcase 2.2. Assume that $C \cong \mathrm{D}_{2l}$. We claim that $G = C{\times}X_1$ for a subgroup X_1 of G. This is clear if $X = T$. Assume that $X = \mathrm{PGL}(2,p)$. Recall that N is the kernel of G acting on T-orbits. Since C is dihedral and $|C \cap N| \le 2$, we have $C \cap N = 1$. If $G = CN$ then the claim hold by taking $X_1 = N$. Suppose that $G \ne CN$. Then

$N = T$ and $G = (C{\times}T){:}\langle z\rangle$ for an arbitrary involution $z \in X \setminus T$. It follows that $G/N = G/T \cong C{:}\langle z\rangle < G$. Since Γ is arc-transitive, Γ_T is G/N-arc-transitive. Then G/N is dihedral, and so either $c^z = c^{-1}$ or z lies in the center of $C\langle z\rangle$. The latter case yields $G = C{\times}X$. Suppose that $c^z = c^{-1}$. Note that the set of involutions in C is invariant under the conjugation of z. We may take an involution $y \in C$ with $y^z = y$. Then yz centralizes C, and our claim holds by taking $X_1 = T{:}\langle yz\rangle$.

Without loss of generality, we assume that $G = C{\times}X$. Then $G_B = C{\times}X_B$. Recall that $|C|$ and $|X|$ has no prime divisors in common other that 2. Considering the action of X_B on B, we conclude that either X_B fixes B point-wise, or $|X_B : X_\alpha| = 2$ for $\alpha \in B$. The former case implies that $X_B \le G_\alpha$, and so $G_\alpha = X_B$ as $G_\alpha \cong X_B$; in this case, there is no a 2-element g with $\langle g, G_\alpha\rangle = G$, a contradiction. Thus $|X_B : X_\alpha| = 2$, and so $|G_\alpha : X_\alpha| = 2$; in this case LX is transitive on V. Clearly, there is no 2-element g in LX with $\langle g, X_\alpha\rangle = LX$. It follows that Γ is not LX-arc-transitive, and so X_α is intransitive on $\Gamma(\alpha)$. By Lemmas 2.4 and 2.5, $|X_\alpha : X_{\alpha\beta}| = 2$ for $\beta \in \Gamma(\alpha)$. Since Γ is arc-transitive, $|G_\alpha : G_{\alpha\beta}| = 4$. It implies that $G_{\alpha\beta} = X_{\alpha\beta}$, and so Γ is isomorphic to a graph constructed in Example 5.13. Theorem 1.1 (9) occurs. This completes the proof. □

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