Note on a Turán-type problem on distances*

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Abstract

A new Turán-type problem on distances of graphs was introduced by Tyomkyn and Uzzell. In this paper, we focus on the case of distance two. We show that for any positive integer n, a graph G on n vertices without three vertices pairwise at distance 2 has at most $(n-1)^2/4+1$ pairs of vertices at distance 2, if G has a vertex $v \in V(G)$ whose neighbors are covered by at most two cliques. This partially answers a guess of Tyomkyn and Uzzell in [Tyomkyn, M., Uzzell, A.J.: A new Turán-type problem on distances of graphs. Graphs Combin. **29**(6), 1927–1942 (2012)].

Keywords: distance, Turán-type problem, forbidden subgraph

1 Introduction

In [10], Tyomkyn and Uzzell introduced a new Turán-type problem on distances of graphs, which is an extension of the problem studied by Bollobás and Tyomkyn in [6], namely, determining the maximum number of paths with length k in a tree T on n vertices.

The problem on counting paths of a given length in a graph G has been studied since 1971, see, e.g., [1, 2, 3, 4, 5, 8, 9] and the references therein. On the other hand, counting paths of length k in trees can be interpreted as counting pairs of vertices at distance k. Tyomkyn and Uzzell asked a natural question as follows.

Question. For a graph G on n vertices, what is the maximum possible number of pairs of vertices at distance k ?

Let $G=(V,E)$ be a connected simple graph. The distance between two vertices u and v in G, denoted by $d_G(u,v)$, is the length of a shortest

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path between u and v in G. Let $N_G(v)$ be the neighborhood of v, and $d_G(v) = |N_G(v)|$ denote the degree of a vertex v. The greatest distance between any two vertices in G is the diameter of G, denoted by $diam(G)$. The set of neighbors of a vertex v in G is denoted by $N(v)$ or $N^1(v)$, and the set of vertices, whose distance is i from v, is denoted by $N^i(v)$, where $i \in \{1, 2, 3, \cdots, diam(G)\}$. Suppose that V' is a nonempty subset of V. The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V' is called the subgraph of G induced by V' and is denoted by $G[V']$; we say that $G[V']$ is an induced subgraph of G. A clique in a graph G is a subset of vertices such that every two vertices in the subset are connected by an edge.

If H is a graph, then we say that G is *H-free* if G does not contain a copy of H as an induced subgraph. A claw, denoted by C, is the complete bipartite graph $K_{1,3}$. Thus, G is said to be *claw-free* if it does not contain any induced subgraph that is isomorphic to C.

A graph G is a *quasi-line graph* if for every vertex $v \in V(G)$, the neighborhood of v can be partitioned into two sets A, B in such a way that both A and B are cliques. Note that there may be edges between A and B. Thus all line graphs are quasi-line graphs, and all quasi-line graphs are claw-free, but if we converse either of the statements, it is not true. For other notation and terminology not defined here, we refer to [7].

For a graph G, a new graph G_k is defined to be the graph with vertex set $V(G)$ and $\{x, y\} \in E(G_k)$ if and only if x and y are at distance k in G. We call G_k the *distance-k graph*. Such vertices x and y are called *k-neighbors*. We call $d_{G_k}(x)$ the *k-degree* of x. We say that a graph G is k-isomorphic to a graph H if G_k is isomorphic to H. Let $\omega(G)$ denote the clique number of a graph G, which is the number of vertices in a maximum clique of G.

It is interesting to maximize the number of edges in G_k over all graphs G on n vertices. In [6], Bollobás and Tyomkyn proved that if G is a tree, then $e(G_k)$ is maximal when G is a t-broom for some t.

Theorem 1.1. *Let $n \geq k$. If G is a tree on n vertices, then $e(G_k)$ is maximal when G is a t-broom. If k is odd, then $t = 2$. If k is even, then t is within 1 of*

$$\frac{1}{4} + \sqrt{\frac{1}{16} + \frac{n-1}{k-2}}.$$

For a general graph G, Tyomkyn and Uzzell [10] gave a conjecture and proved a part of it in the following theorem.

Conjecture 1.1. [10] Let $k \geq 3$ and $t \geq 2$. There is a function $h_2 : N \times N \to N$ such that if $n \geq h_2(k, t)$, then $e(G_k)$ is maximized over all G with $|G| = n$ and $\omega(G_k) \leq t$ when G is k-isomorphic to a t-broom for some t.

Theorem 1.2. *[10] There is a constant k_0 and a function $n_0 : N \to N$ such that for all $k \geq k_0$, all $n \geq n_0(k)$ and all graphs G of order n with no three vertices pairwise at distance k,*

$$e(G_k) \leq (n-k+1)^2/4.$$

Moreover, if the equality holds, then G is k-isomorphic to the double broom.

For the detailed proof and some terminology, we refer to [10]. Actually, Tyomkyn and Uzzell tried to do better about the bound, but there is no good way. For the case of $k = 2$, they believe that for $n \geq 5$, a triangle-free G_2 can have no more than $(n-1)^2/4+1$ edges. They mentioned that this is clearly true for $n = 5$ and a computer search verifies that it also holds for $6 \leq n \leq 11$. However, they could not prove it in general. In this paper, we will give an partial answer to their guess. Our main result is as follows:

Theorem 1.3. *Let G be a graph on n vertices, which has no three vertices pairwise at distance 2. If there exists an vertex $v \in V(G)$, whose neighbors are covered by at most two cliques, then G has at most $(n-1)^2/4+1$ pairs of vertices at distance 2.*

From Theorem 1.3, we can get the following corollary.

Corollary 1.1. *Let G be a quasi-line graph on n vertices, which has no three vertices pairwise at distance 2. Then G has at most $(n-1)^2/4+1$ pairs of vertices at distance 2.*

2 Preliminaries

In this section, we will figure out the structure of a graph G with the property that G_2 is triangle-free.

Let C_k be the cycle with k vertices $v_1, v_2, \cdots, v_k$ such that $v_1v_k, v_iv_{i+1} \in E(C_k)$ for $i = 1, 2, \cdots, k-1$. First of all, we define two graphs $C_6^{'}$ and $C_6^{''}$ which can be obtained from C_6 as follows: $C_6^{'} = C_6 + v_1v_3$, $C_6^{''} = C_6 + v_1v_3 + v_3v_5$.

Lemma 2.1. *If G_2 is triangle-free, then G is claw-free, C_6-free, $C_6^{'}$-free and $C_6^{''}$-free.*

Proof. By contradiction. Suppose that G has a claw C as an induced subgraph. Let $V(C) = \{v, u_1, u_2, u_3\}$ and v is adjacent to u_i, $i = 1, 2, 3$. Then the three vertices u_1, u_2 and u_3 form a triangle in G_2, a contradiction.

Suppose that G is not C_6-free. Let $C_6 = v_1v_2 \ldots v_6v_1$. Then v_1v_3, v_3v_5, $v_1v_5 \in e(G_2)$, which implies that the three vertices v_1, v_3 and v_5 form a triangle in G_2, a contradiction.

Similarly, G is $C_6^{'}$-free and $C_6^{''}$-free. □

The *independence number* $\alpha(G)$ of a graph G is the cardinality of a maximum independent set of G. Recall that an independent set of G is a subset of vertices in G such that no two of them are connected by an edge of G. For a claw-free graph, there is a well-known result [9] as follows.

Lemma 2.2. *Let G be a claw-free graph with independence number at least three, then every vertex v satisfies exactly one of the following:*
(1) $N_G(v)$ is covered by two cliques,
(2) $N_G(v)$ contains an induced C_5.

From Lemma 2.2, we know that for a claw-free graph G, the subgraph induced by the neighborhood of a vertex $v \in G$ is covered by at most two cliques, or contains an included C_5. In the following, we give an observation about the subgraph induced by $N_G^2(v)$.

Observation 1. Let G be a graph with diameter two and $v \in V(G)$. If G_2 is triangle-free, then $G[N_G^2(v)]$ is a clique.

Let $G^{(d)}$ denote a graph G with diameter d. Let $v_0v_1 \dots v_d$ be a spindle of $G^{(d)}$ and $V_d = N_{G^{(d)}}(v_{d-1}) \backslash v_{d-2}$. We define an operation as follows: Delete all the edges between v_{d-1} and V_d, and then join v_{d-2} with all the vertices of V_d. We call this operation *"move V_d to v_{d-2}"*.

Lemma 2.3. *If G be a graph with diameter $d \geq 2$, then $e(G_2^{(d)}) \leq e(G_2^{(d-1)})$, where $G^{(d-1)}$ is obtained by applying the above operation on $G^{(d)}$.*

Proof. Let $v_0v_1 \dots v_d$ be a spindle of $G^{(d)}$. After applying the above operation on $G^{(d)}$, the only change on the number of vertex pairs $\{u, v\}$ such that $d_{G^{(d)}}(u, v) = 2$ is brought by the movement of V_d. Thus, to prove $e(G_2^{(d)}) \leq e(G_2^{(d-1)})$, it suffices to show that for every spindle of $G^{(d)}$, the number of vertex pairs such that the distance between them is two is not decreasing after using the operation "move V_d to v_{d-2}". So we only need to show that for some spindle, after using the above operation, $|\{u|d_{G^{(d)}}(u, v_d) = 2\}| \leq |\{u'|d_{G'}(u', v_d) = 2\}|$, where G' is obtained by moving V_d to v_{d-2} in $G^{(d)}$. Since $|\{u|d_{G^{(d)}}(u, v_d) = 2\}| = |N_{G^{(d)}}(v_{d-1}) \backslash v_d|$ and $|\{u'|d_{G'}(u', v_d) = 2\}| = |N_{G^{(d)}}(v_{d-2}) \backslash v_{d-2} \cup N_{G^{(d)}}(v_{d-1}) \backslash v_d \backslash v_{d-2}|$, we can get that $|\{u|d_{G^{(d)}}(u, v_d) = 2\}| \leq |\{u'|d_{G'}(u', v_d) = 2\}|$. Therefore, we have $e(G_2^{(d)}) \leq e(G_2^{(d-1)})$. $\square$

Corollary 2.1. *If G is a graph with diameter $d \geq 2$, then $e(G_2^{(d)}) \leq e(G_2^{(2)})$.*

Proof. By Lemma 2.3, we know that $e(G_2^{(d)}) \leq e(G_2^{(d-1)}) \leq \cdots \leq e(G_2^{(2)})$. $\square$

By Corollary 2.1, we know that to maximize $|e(G_2^{(d)})|$ $(d \geq 2)$, it suffices to get the maximum value of $|e(G_2^{(2)})|$. Thus, in the following part of the paper, we focus on the graph $G^{(2)}$. For convenience, we write G instead of $G^{(2)}$.

3 Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. For convenience, we use $\{x, y\}$ or vertex-pair to stand for the vertex-pair such that $d_G(x, y) = 2$. Actually, Theorem 1.3 can be stated as the following theorem.

Theorem 3.1. *Let G be a graph with $|V(G)| \geq 5$. If there is a vertex $v \in V(G)$ whose neighborhood is covered by at most two cliques, then a triangle-free G_2 can have no more than $(n-1)^2/4 + 1$ edges.*

Proof. Since G_2 is triangle-free, Lemma 2.1 implies that G is claw-free. Let $v \in V(G)$, whose neighbors is covered by at most two cliques. Then the proof will be given by the following cases.

Case 1. $N_G(v)$ is covered by only one clique.

Let $V_1 = N_G(v)$ and $V_2 = N_G^2(v)$. By Observation 1, $G[N_G^2(v)]$ is a clique. Suppose $V_1 = V_{11} \cup V_{12}$ and $V_2 = V_{21} \cup B \cup V_{22}$, where a vertex of V_{21} is only adjacent to vertices of V_{11} but not any vertex of V_{12}, a vertex of V_{22} is only adjacent to vertices of V_{12} but not any vertex of V_{11}, and a vertex of B is adjacent to vertices of both V_{11} and V_{12}. Without loss of generality, we suppose that $|V_{11}| \geq |V_{12}|$.

By considering whether $V_{12} = \emptyset$ or not, we discuss as follows.

Subcase 1.1. $V_{12} = \emptyset$.

Then, $V_{22} = \emptyset$ and $G[V_1, V_2]$ is a complete bipartite graph. Hence, $e(G_2) = |V_2| \leq n - 2 \leq (n-1)^2/4 + 1$.

Subcase 1.2. $V_{12} \neq \emptyset$.

In this subcase, we will give the proof in detail as follows.

Subsubcase 1.2.1. $V_{22} \neq \emptyset$.

Let $d = |V_{22}|$. We can define a new graph G' as follows: $V(G') = V(G)$, $V_1' = N_{G'}(v) = V_{11}' \cup V_{12}'$ and $V_2' = N_{G'}^2(v) = V_{21}' \cup V_{22}'$, where both $G'[V_1']$ and $G'[V_2']$ are cliques, both $G'[V_{11}', V_{21}']$ and $G'[V_{12}', V_{22}']$ are complete bipartite graphs, and $|V_{11}'| = 1$, $|V_{22}'| = 1$.

Suppose $d = 1$. In the following, we show that G' can be obtained from G by applying the corresponding operations mentioned in the following paper. Moreover, we show that such operations ensure that the number of vertex-pairs remains the same or increases.

Firstly, delete the edges between V_{12} and B, and it is obvious that the number of $\{u_1, u_2\}$'s is not decreasing, where $u_1, u_2 \in V(G)$. Then $V_{21}' = V_{21} \cup B$ and $V_{22}' = V_{22}$.

Secondly, suppose $u \in V_{11}$ such that u is adjacent to all the vertices in V'_{21}. Delete the edges between $V_{11} \setminus \{u\}$ and V'_{21}, meanwhile, connect all the vertices in $V_{11} \setminus u$ and all the vertices in V'_{12}. Then $V'_{11} = \{u\}$ and $V'_{12} = \{V_{11} \setminus \{u\}\} \cup V_{12}$. Therefore, we get the graph G' (see Figure 1). Let $a = |V_{11}|,\ b = |V_{12}|,\ c = |V_{21}| + |B|$. Since $e(G_2) = a + bc + c + 1$ and $e(G'_2) = (a + c - 1)b + (a + c + 1)$, then $e(G'_2) - e(G_2) = (a - 1)b \geq 0$, that is, after applying the above operation, we ensure that the number of $\{u, v\}$'s is not decreasing.

For the graph G', we have $e(G'_2) = xy + x + 2$ where $x = |V'_{12}|,\ y = |V'_{21}|$, such that $x + y = n - 3$ and $x \geq 1,\ y \geq 1$. Thus, $e(G'_2) \leq (n-2)^2/4 + 2 < (n-1)^2/4 + 1$, where $n \geq 5$.

Now suppose $d \geq 2$. Let $u_1 \in V_{21}$ and $u_2 \in V_{12}$. Delete the edges between $V_{21} \setminus \{u_1\} \cup B$ and V_{22} and the edges between $V_{12} \setminus \{u_2\}$ and V_{11}; meanwhile, move the vertices of $V_{21} \setminus \{u_1\} \cup B$ to V_{11} (the new vertex set obtained is denoted by V'_{12}), and the vertices in $V_{12} \setminus \{u_2\}$ with V_{22} (the new vertex set obtained is denoted by V'_{21}), therefore, we get the graph G' (see Figure 1), where $V'_{11} = \{u_1\}$ and $V'_{22} = \{u_2\}$. Let $a = |V_{11}|,\ b = |V_{12}|,\ c = |V_{21}| + |B|$. Since $e(G_2) = ad + bc + c + d$ and $e(G'_2) = (a + c - 1)(b + d - 1) + (a + c - 1 + 1) + 1$, then $e(G'_2) - e(G_2) = a(b - 1) + (c - 1)(d - 2) \geq 0$, that is, using the above operation we ensure that the number of $\{w_1, w_2\}$'s is not decreasing, where $w_1, w_2 \in V(G')$.

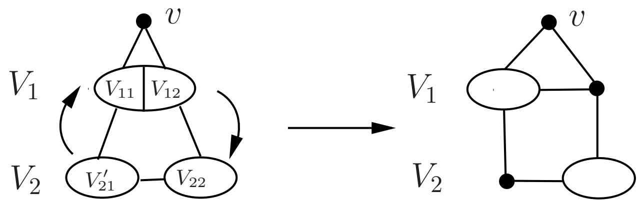

Figure 1: A new graph G' for $d \geq 2$

For the graph G', we have $e(G'_2) = xy + x + 2$, where $x = |V'_{12}|,\ y = |V'_{21}|$ such that $x + y = n - 3$ and $x \geq 1,\ y \geq 1$. Thus, $e(G'_2) \leq (n-2)^2/4 + 2 < (n-1)^2/4 + 1$, where $n \geq 5$.

Subsubcase 1.2.2. $V_{22} = \emptyset$.

For this case, delete the edges between V_{11} and B, then it returns to **Subsubcase 1.2.1**.

By combining all the situations in **Subcase 1.1**, we get that $e(G) \leq (n-2)^2/4 + 2 < (n-1)^2/4 + 1$, where $n \geq 5$.

Case 2. $G[N_G(v)]$ is covered by two cliques.

In this case, $G[N_G(v)]$ is covered by two cliques, denoted by V_1 and U_1. According to the condition whether $e(U_1, V_1) = 0$, we prove it by the following subcases.

Subcase 2.1. $e(U_1, V_1) = 0$.

Let $V_2 = N_G^2(v)$. V_2 can be divided into three parts A, B and C, where a vertex of A is only adjacent to vertices of V_1, a vertex of B is adjacent to vertices of both V_1 and U_1, and a vertex of C is only adjacent to vertices of U_1. Now we give the following claim.

Claim 1. $G[V_1, A]$ and $G[U_1, C]$ are both complete bipartite graphs.

Proof. If $G[V_1, A]$ is not a complete bipartite graph, then there are vertices $u_1 \in A$, $u_2 \in V_1$, $w \in U_1$ such that $d_G(u_1, u_2) = d_G(u_1, w) = d_G(u_2, w) = 2$. Thus, the three vertices u_1, u_2, w form a triangle in G_2, a contradiction.

Similarly, $G[U_1, C]$ is also a complete bipartite graph. □

Now we divide vertex set B into three parts B_1, B_2 and B_3, where a vertex of B_1 is only adjacent to vertices of V_1, a vertex of B_2 is adjacent to vertices of both V_1 and U_1, and a vertex of B_3 is only adjacent to vertices of U_1.

Claim 2. Both $G[V_1, B_1 \cup B_2]$ and $G[U_1, B_2 \cup B_3]$ are complete bipartite graphs.

Proof. To prove Claim 2, it suffices to show that every vertex $u \in B$ is adjacent to all the vertices of V_1 or U_1 , that is, at least one of $G[V_1, u]$ and $G[U_1, u]$ is a complete bipartite graph. Suppose that there is a vertex $u \in B$ such that neither $G[V_1, u]$ nor $G[U_1, u]$ is a complete bipartite graph. Then there are vertices $w_1 \in V_1$, $w_2 \in U_1$ such that $d_G(u, w_1) = d_G(u, w_2) = d_G(w_1, w_2) = 2$. Thus, $uw_1, w_1w_2, w_2u \in e(G_2)$, contradicting to the fact that G_2 is triangle-free. □

In the following, we apply some operations on G to maximize $|e(G_2)|$.

We define a new graph G' as follows. G' with $V_1' = V_1$, $U_1' = U_1$, $V_2' = A' \cup C'$ ($A' = A \cup B_1 \cup B_2$ and $C' = B_3 \cup C$), where $G[V_1', A']$, $G[V_2', C']$ and $G[A', C']$ are complete bipartite graphs, $G[V_1']$, $G[V_2']$, $G[A']$ and $G[B']$ are complete graphs.

Form G to G', we perform the following operations: Delete the edges between B_1 and U_1, remove the edges between B_2 and U_1, and the edges between B_3 and V_1. It is obvious that the number of vertex-pairs at distance two is not decreasing.

Now we construct a new graph G'' (see Figure 2) which is obtained from G', that is, $V(G'') = V(G')$, by moving the vertex in $A' \setminus \{u\}$ to V_1', and the vertices in $C' \setminus \{w\}$ to U_1'. The new vertex sets are denoted by V_1'' and U_1'', where $u \in A'$ and $w \in B'$.

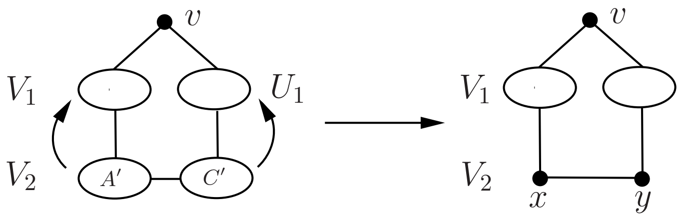

Figure 2: A new graph G''

Let $a = |V_1'|$, $b = |V_2'|$, $c = |A'|$, $d = |B'|$. Since $e(G_2') = ab+ad+bc+c+d$ and $e(G_2'') = (a+c-1)(b+d-1) + (a+c-1) + (b+d-1) + 2$, then $e(G_2'') - e(G_2') = (c-1)(d-1) \geq 0$, that is, after using the move operation the number of vertex-pairs whose distance is two is not decreasing.

Now, for the graph G'', let $x = |V_1''|$ and $y = |U_1''|$. Then $e(G_2'') = xy + x + y + 2$, where $x + y = n - 3$ and $x, y \geq 1$. By some calculations, we get that $e(G_2'') \leq (n-1)^2/4 + 1$. And the equality holds if and only if n is odd and $x = y = (n-1)/2$, where $n \geq 5$.

Combining all the situations in **Case 1**, $e(G_2) \leq (n-1)^2/4 + 1$ follows from the condition that $N_G(v)$ is covered by at least two cliques.

Subcase 2.2. $e(U_1, V_1) \neq 0$.

In this subcase, if there is not a subset B of V_2 such that the vertices in B can form the vertex-pairs with some vertices of $V_{11} \subseteq V_1$ and meanwhile with some vertices of $U_{11} \subseteq U_1$, then we delete the edges between U_1 and V_1. Now it returns to **Subcase 2.1**.

If there is a subset B of V_2 such that the vertices in B can form the vertex-pairs both with some vertices of $V_{11} \subseteq V_1$ and with some vertices of $U_{11} \subseteq U_1$, then divide B into two parts B_1 and B_2, delete the edges between V_1 and U_1, move B_1 to V_1, and move B_2 to U_1. Now we want to prove that there always exist such B_1 and B_2 to ensure that the number of the vertex-pairs is not decreasing. Let $a = |V_{11}|$, $b = |U_{11}|$, $c = |B_1|$, $d = |B_2|$. By only considering the vertex-pairs of those sets, the change of the number is at least $(a+c)(b+d) - (c+d)(a+b) = (a-d)(b-c)$, which is no less than 0 (by some knowledge of the inequality, no matter how much $(a+c)$ and $(c+d)$ are, we can find some number to make it right). Now this subcase returns to **Subcase 2.1**.

Combining all the above cases, we complete the proof of Theorem 3.1. $\square$

From the proof of Theorem 3.1, we can easily get Corollary 1.1. By

Lemmas 2.1 and 2.2, if we can prove the following statement: All the claw-free graphs with diameter two, which has no three vertices pairwise at distance 2, has at most $(n-1)^2/4+1$ pairs of vertices at distance 2, then we can confirm the guess of Tyomkyn and Uzzell.

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