Hermitian-adjacency matrices and Hermitian energies of mixed graphs*†

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Abstract

A complex adjacency matrix of a mixed graph is introduced in the present paper, which is an Hermitian matrix and called the Hermitian-adjacency matrix. It incorporates both adjacency matrix of an undirected graph and skew-adjacency matrix of an oriented graph. Some of its properties are studied. Furthermore, properties of its characteristic polynomial are studied. Cospectral problems among mixed graphs, including mixed graphs and their underlying graphs, oriented graphs and their underlying graphs, are studied. We give equivalent conditions for a mixed graph (especially oriented graph) that share the same spectrum with its underlying graph. As a consequence, we reconfirm a conjecture which was proposed by Cui and Hou in [8]. We also show that the spectrum of the Hermitian matrix of a mixed graph is invariant when changing the value of any its cut edge (if any).

Correspondingly, an energy of a mixed graph is introduced and called the Hermitian energy. It incorporates both the energy of an undirected graph and the skew energy of an oriented graph. Some of its bounds are given. Especially, the mixed graphs with optimal upper bound of Hermitian energy are characterized. An infinite family of mixed graphs attaining the maximum Hermitian energy is constructed. Moreover, the Hermitian energy

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of a mixed tree is showed to be equal to the energy of its underlying tree. Finally, the integral formula for Hermitian energy of a mixed graph is given.

Key words: mixed graph; Hermitian-adjacency matrix; Hermitian energy.

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1 Introduction

In this paper we only consider graphs without multiedges and loops. A graph G is denoted by $G = (V, E)$, where V is the vertex set and $E \subseteq V \times V \setminus \{(u, u)|u \in V\}$ is the edge set. A graph $G = (V, E)$ is said to be *mixed* if $(u, v) \in E$ does not always imply $(v, u) \in E$, see for example [23]. In a mixed graph $G = (V, E)$, an edge $(u, v) \in E$ is *undirected* (resp.*directed*) if (v, u) is also in E (resp. (v, u) is not in E) and it is also denoted by $u \leftrightarrow v$ (resp. $u \rightarrow v$). For undirected edge (u, v), it is identical with (v, u) and we just count it one time. For convenience, we denote it by $u \sim v$ not matter it is oriented or not. Hence, in a mixed graph some of edges are oriented, while others are not. For a mixed graph G, the underlying graph G_U of G is a simple undirected graph. Clearly, mixed graphs conclude both possibilities of all edges oriented and all edges undirected as extreme cases. A mixed graph G is called *mixed bipartite* (resp. *mixed tree*) if its underlying graph G_U is a bipartite graph (resp. tree). A subgraph of a mixed graph is called *mixed walk*, *mixed path* or *mixed cycle* if its underlying graph is a walk, path or cycle, respectively. However, the terms of order, size, number of components, degree of a vertex, distance, we mean that they are the same as in their underlying graphs. For undefined terminology and notations we refer the reader to [4].

The Hermitian-adjacency matrix of a mixed graph G of order n is the $n \times n$ matrix $H(G) = (h_{k\ell})$, where $h_{k\ell} = -h_{\ell k} = \mathbf{i}$ if $v_k \rightarrow v_\ell$, where $\mathbf{i}$ is the imaginary number unit and $h_{k\ell} = h_{\ell k} = 1$ if $v_k \leftrightarrow v_\ell$, and $h_{k\ell} = 0$ otherwise. The spectrum $Sp_H(G)$ of G is defined as the spectrum of $H(G)$. It is easy to see that $H(G)$ is an Hermitian matrix, i.e. its conjugation and transposition is itself, that is $H = H^* := \overline{H}^T$. Thus all its eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ are real, and the singular values of $H(G)$ coincide with the absolute values $\{|\lambda_1|, |\lambda_2|, ..., |\lambda_n|\}$ of its eigenvalues. Consequently, the energy of $H(G)$, which is defined as the sum of its singular values [19], is also the sum of the absolute values of its eigenvalues. We call the energy of $H(G)$ as the *Hermitian energy* of the mixed graph G, denoted by $\mathcal{E}_H(G)$. In this paper, we are interested in studying properties of the spectrum and the Hermitian energy of mixed graphs.

2 Basic properties

First we give two examples of mixed graphs and their Hermitian-adjacency matrices and Hermitian energies.

Example 2.1 *Let G be the mixed graph of order 3 on left of Figure 2.1, then its Hermitian-adjacency matrix can be chosen as*

$$H(G) = \begin{pmatrix} 0 & -\mathbf{i} & \mathbf{i} \\ \mathbf{i} & 0 & 1 \\ -\mathbf{i} & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of $H(G)$ is $\lambda^3 - 3\lambda + 2$, the eigenvalues of $H(G)$ are $\{-2, 1, 1\}$, corresponding eigenvectors are $(-\mathbf{i}, -1, 1)^T, (2\mathbf{i}, -1, 1)^T, (0, 1, 1)^T$, and $\mathcal{E}_H(G) = 4$.

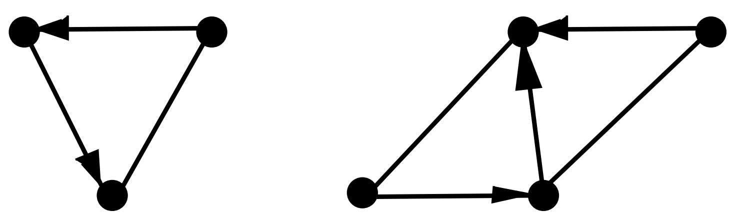

Figure 2.1 Mixed graphs in Examples 2.1 and 2.2

Example 2.2 *Let G be the mixed graph of order 4 on right of Figure 2.1, then its Hermitian-adjacency matrix can be chosen as*

$$H(G) = \begin{pmatrix} 0 & -\mathbf{i} & -\mathbf{i} & 1 \\ \mathbf{i} & 0 & 1 & 0 \\ \mathbf{i} & 1 & 0 & -\mathbf{i} \\ 1 & 0 & \mathbf{i} & 0 \end{pmatrix}.$$

The characteristic polynomial of $H(G)$ is $\lambda^4-5\lambda^2+4$, the eigenvalues of $H(G)$ are $\{-1,1,-2,2\}$, corresponding eigenvectors are $(-\mathbf{i},-2,1,0)^T$, $(1,0,-\mathbf{i},2)^T$, $(-1,0,\mathbf{i},1)^T$, $(-\mathbf{i},1,1,0)^T$, and $\mathcal{E}_H(G)=6$.

The adjacency matrix of an undirected graph G of order n is the $n\times n$ matrix $A(G)=(a_{ij})$, where $a_{ij}=a_{ji}=1$ if $v_i\sim v_j$ and $a_{ij}=0$ otherwise. The spectrum $Sp_A(G)$ of G is defined as the spectrum of $A(G)$. Since $A(G)$ is symmetric matrix, all its eigenvalues, denoted by $\{\mu_1,\mu_2,\ldots,\mu_n\}$, are real.

The energy of graph G (see the survey of Gutman, Li and Zhang [11] and the book of Li, Shi and Gutman [16].) is defined as

$$\mathcal{E}_A(G)=\sum_{\ell=1}^{n}|\mu_\ell|.$$

In theoretical chemistry, the π-electron energy of a conjugated carbon molecule, which usually represented by a simple undirected graph, computed using the Hückel theory, coincides with the energy that defined here. We can see that:

Remark 2.3 *If G is an undirected graph, then $A(G)=H(G)$, $Sp_A(G)=Sp_H(G)$ and $\mathcal{E}_A(G)=\mathcal{E}_H(G)$.*

The skew-adjacency matrix [2] of an oriented graph G of order n is the $n\times n$ matrix $S(G)=(s_{k\ell})$, where $s_{k\ell}=-s_{\ell k}=1$ if $v_k\to v_\ell$, and $s_{k\ell}=0$ otherwise. The spectrum $Sp_S(G)$ of G is defined as the spectrum of $S(G)$. Since $S(G)$ is a skew-symmetric matrix, the eigenvalues of $S(G)$, denoted by $\{s_1,s_2,\ldots,s_n\}$, are purely imaginary numbers. The skew energy of oriented graph G is defined by Adiga et al. in [2] as

$$\mathcal{E}_S(G)=\sum_{\ell=1}^{n}|s_\ell|.$$

For an oriented graph G, we have $H(G)=\mathbf{i}S(G)$. Furthermore, if $X\in\mathbf{C}^n$ is an eigenvector corresponding to an eigenvalue λ of $S(G)$, then $S(G)X=\lambda X$, i.e., $\mathbf{i}S(G)X=\mathbf{i}\lambda X$, then $H(G)X=(\mathbf{i}\lambda)X$. Therefore, $X\in\mathbf{C}^n$ is an eigenvector corresponding to the eigenvalue $(\mathbf{i}\lambda)$ of $H(G)$ and vice versa. Thus, $Sp_H(G)=\mathbf{i}Sp_S(G)$ for any oriented graph G. We then have $\mathcal{E}_S(G)=\sum_{\ell=1}^{n}|\lambda_\ell|=\sum_{\ell=1}^{n}|\mathbf{i}\lambda_\ell|=\mathcal{E}_H(G)$. Therefore,

Remark 2.4 *If G is an oriented graph, then $H(G) = \mathbf{i}S(G), Sp_H(G) = \mathbf{i}Sp_S(G)$ and $\mathcal{E}_H(G) = \mathcal{E}_S(G)$.*

From above, we find that Hermitian-adjacency matrices (resp. Hermitian energies) of mixed graphs incorporate both adjacency matrices (resp. energies) of undirected graphs and skew-adjacency matrices, neglecting the factor $\mathbf{i}$, (resp. skew energies) of oriented graphs. Thus, for convenience, we simply refer the spectrum and the Hermitian energy of $H(G)$ as the spectrum and energy of a mixed graph G, respectively.

Thus, the study of Hermitian-adjacency matrices and energies of mixed graphs is meaningful. In fact, we shall see that we can obtain information about the cospectral problem between skew-adjacency matrices of oriented graphs and adjacency matrices of their underlying graphs by using results of Hermitian-adjacency matrices of mixed graphs.

Denote the characteristic polynomial of $H(G)$ of a mixed graph G as:

$$\Phi(G;\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n.$$

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two mixed graphs with disjoint sets of vertices V_1 and V_2, respectively. Then the union G of G_1 and G_2 is defined by $V(G) = V_1 \bigcup V_2$ and $E(G) = E_1 \bigcup E_2$ and denoted by $G = G_1 + G_2$. We immediately have the following result.

Theorem 2.5 *If a mixed graph G is a union of mixed graphs $G_1, G_2, \ldots, G_k$, then*

$$\Phi(G;\lambda) = \prod_{\ell=1}^{k} \Phi(G_\ell;\lambda).$$

Hence the spectrum of G is the union of the spectrums of G_ℓ, $\ell = 1, 2, \ldots, k$.

The *value* of a mixed walk $W = v_1v_2v_3\cdots v_\ell$ is $h(W) = h_{12}h_{23}\cdots h_{(\ell-1)\ell}$. A mixed walk is *positive* or *negative* if $h(W) = 1$ or $h(W) = -1$, respectively. Note that for one direction the value of a mixed walk or a mixed cycle is α, then for the reversed direction its value is $\bar{\alpha}$. Thus, if the value of a mixed cycle is 1 (resp. -1) in a direction, then its value is 1 (resp. -1) for the reversed direction. In these situations, we just termed this mixed cycle as a positive (resp. negative) mixed cycle without mentioning any direction. A graph is *positive* (resp. *negative*) if each its mixed cycle is positive (resp. negative). An *elementary* graph is a mixed graph such that every component is an edge or a mixed cycle, and every its edge-component is defined to be positive. A (real) *spanning elementary subgraph* of a mixed graph G is an elementary subgraph such that it contains all vertices of G and all its mixed cycles are real. The *rank* and the *corank* of a mixed graph G are, respectively,

$$r(G) = n - c; s(G) = m - n + c,$$

where n, m and c are the order, size and number of components of G, respectively.

We can see from definition that we need to check every mixed cycle to make sure whether a mixed graph is positive or not. Our next result shows that we can reduce the number of checks.

Lemma 2.6 *Let G be a mixed graph, then G is positive if and only if every mixed chordless cycle of G is positive.*

Proof. The necessity is obvious. Sufficiency. We shall show every mixed cycle is positive. If the result is not true, then there exists a mixed cycle that is not positive. Then there exists a mixed cycle C in G of least length such that $h(C) \neq 1$. By hypothesis, C contains a chord v_1u_1. Suppose that mixed cycles $C = v_1v_2 \dots v_pu_1u_2 \dots u_qv_1$, $C_1 = v_1v_2 \dots v_pu_1v_1$ and $C_2 = u_1u_2 \dots u_qv_1u_1$ in clock direction. By the choice of C, C_1 and C_2 are chordless and hence positive. Suppose the value of the walk $W_1 = v_1v_2 \dots v_pu_1$ is α. Then $h_{u_1v_1} = \bar{\alpha}$. Thus, the value of the walk $W_2 = u_1u_2 \dots u_qv_1$ is $\bar{\alpha}$. Therefore, $h(C) = \alpha \cdot \bar{\alpha} = 1$, which contradicts to our hypothesis. The result thus follows. ∎

Now we will give two results which similar with those of adjacency matrices [3].

Theorem 2.7 *Let H be the Hermitian-adjacency matrix of a mixed graph G. Then*

$$\det H = \sum_{G'} (-1)^{r(G')+\ell(G')} 2^{s(G')},$$

where the summation is over all real spanning elementary subgraphs G' of G and $\ell(G')$ denotes the number of negative mixed cycles of G'.

Proof. The proof is similar with that of Proposition 7.2 in [3]. Consider a term $\mathrm{sgn}(\pi)h_{1,\pi 1}h_{2,\pi 2} \cdots h_{n,\pi n}$ in the expansion of $\det H$. This term vanishes if, for some $k \in \{1, 2, \cdots, n\}$, $h_{k,\pi k} = 0$; that is, if $v_kv_{\pi k}$ is not an edge of G. In particular, the term vanishes if π fixes any symbol. Thus, if the term corresponding to a permutation π is non-zero, then π can be expressed uniquely as the composition of disjoint cycles of length at least two. Each cycle $(k\ell)$ of length two corresponds to the factors $h_{k\ell}h_{\ell k}$, and signifies $v_k \sim v_\ell$. Each cycle $(pqr\cdots t)$ of length greater than two corresponds to the factors $h_{pq}h_{qr}\cdots h_{tp}$, and signifies a mixed cycle $v_pv_q\cdots v_tv_p$ in G. Consequently, each non-vanishing term in the determinant expansion gives rise to an elementary mixed graph G' of G, with $V(G') = V(G)$. That is, G' is a spanning elementary subgraph of G.

The sign of a permutation π is $(-1)^{N_e}$, where N_e is the number of even cycles (i.e. cycles with even length) in π. If there are c_ℓ of length ℓ, then the equation $\sum \ell c_\ell = n$ show that the number N_o of odd cycles is congruent to n modulo 2. Hence,

$$r(G') = n - (N_o + N_e) \equiv N_e \pmod 2,$$

so the sign of π is equal to $(-1)^{r(G')}$.

Each spanning elementary subgraph G' gives rise to several permutations π for which the corresponding term in the determinant expansion does not vanish. The number of such π arising from a given G' is $2^{s(G')}$, since for each mixed cycle-component in G' there are two ways of choosing the corresponding cycle in π. Furthermore, if for some direction of a permutation π a mixed cycle-component has value $\mathbf{i}$ (or -$\mathbf{i}$), then for the other direction the mixed cycle-component has value $-\mathbf{i}$ (or $\mathbf{i}$) and vice versa. Thus, they cancel each other in the summation. Similarly, if for some direction of a permutation π a mixed cycle-component has value 1 (or -1), then for the other direction the mixed cycle-component has value 1 (or -1) too.

Thus each G' contributes $(-1)^{r(G')+\ell(G')}2^{s(G')}$ to the determinant and the result follows.
∎

We shall now obtain a description of all the coefficients of the characteristic polynomial of a mixed graph G, in terms of some small elementary subgraphs of G. We shall suppose, as before, that

$$\Phi(G;\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n.$$

Theorem 2.8 *The coefficients of the characteristic polynomial of a mixed graph G are given* *by*

$$(-1)^k c_k = \sum_{G'} (-1)^{r(G')+\ell(G')} 2^{s(G')},$$

where the summation is over all real elementary subgraphs G' of G with k vertices and $\ell(G')$ denotes the number of negative mixed cycles of G'.

Proof. The proof is similar with that of Proposition 7.3 in [3]. The number $(-1)^k c_k$ is the sum of all principal minors of $H(G)$ with k rows and columns. Each such minor is the determinant of the Hermitian-adjacency matrix of an induced subgraph of G' with k vertices. Any elementary subgraph with k vertices is contained in precisely one of these induced subgraphs, and so, by applying Theorem 2.7 to each minor, we obtain the required result. ∎

As well as giving explicit expressions for the coefficients of the characteristic polynomial, Theorem 2.8 throws some light on the problem of cospectral mixed graphs. The fact that elementary subgraphs are rather loosely related to the structure of a mixed graph helps to explain why there are many pairs of non-isomorphic mixed graphs having the same spectrum, see examples of undirected graphs [10]. Now we consider the problem of what mixed graphs share the same spectrum with their underlying graphs.

If G is a positive mixed graph, then

$$(-1)^{r(G')+\ell(G')} 2^{s(G')} = (-1)^{r(G')} 2^{s(G')} = (-1)^{r(G'_U)} 2^{s(G'_U)}$$

for every elementary subgraph G' of G. Then $\Phi(G;\lambda) = \Phi(G_U;\lambda)$, i.e.,

Theorem 2.9 *If G is a positive mixed graph, then $Sp_H(G) = Sp_H(G_U)$.*

Note that we will give an equivalent condition for a mixed graph to be positive in Theorem 4.1.

As another application of Theorem 2.8, we investigate a class of mixed graphs which at least half of their coefficients of their characteristic polynomials are zero.

Theorem 2.10 *If G is a mixed graph of order n without real mixed odd cycles, then*

(1) all its coefficients of c_{odd} are equal to zero;

(2) its spectrum is symmetry about zero;

(3) $c_{2k} = (-1)^k b_k$ and $b_k \geq 0$ for all possible integer k.

Proof. If G is a mixed graph of order n without real mixed odd cycles, then there exists no any real elementary subgraphs with odd number of vertices in G, which means that all its coefficients of c_{odd} are equal to zero by Theorem 2.8. It follows that the characteristic polynomial of G has the form

$$\Phi(G;\lambda) = \lambda^n + c_2\lambda^{n-2} + c_4\lambda^{n-4} + \ldots = \lambda^\sigma p(\lambda^2),$$

where $\sigma = 0$ or 1, and p is a polynomial function. Thus the spectrum of G is symmetry about zero. Hence if λ_ℓ is an eigenvalue of $H(G)$ with multiplicity k, so is $-\lambda_\ell$. It follows that the characteristic polynomial is a power of λ times a product of terms of the form $\lambda^2 - \lambda_\ell^2$. Assume $\Phi(G;\lambda) = \lambda^a \prod(\lambda^2 - \lambda_\ell^2) := \lambda^a p'(\lambda^2)$, where a is the largest possible integer. Hence the roots of the polynomial $p'(\lambda)$ are real and positive, then its coefficients alternate in sign. ∎

Remark 2.11 *The converse of Theorem 2.10 is not true, see the mixed graph G in Example 2.2 for example. We can see that G satisfies conditions (1),(2) and (3), but it has two real mixed odd cycles.*

Since a mixed bipartite graph contains not mixed odd cycles and hence not real mixed odd cycles, we immediately have:

Corollary 2.12 *Let G be a mixed bipartite graph, then (1) all its coefficients of c_{odd} are equal to zero; (2) its the spectrum is symmetry about zero; (3) $c_{2k} = (-1)^k b_k$ and $b_k \geq 0$ for all possible integer k.*

A *generalized orientation* ϕ of an undirected graph G is to specify an orientation (or direction) according to ϕ to each edge in a subset S of $E(G)$. It is called *orientation* of G if $S = E(G)$ and the resulting graph is called *oriented graph.* It is called *trivial* if $S = \emptyset$. Thus a mixed graph can be viewed as a resulting graph of a generalized orientation ϕ of its underlying graph. For an oriented graph, its mixed odd cycle is never real. Thus, the class of oriented graphs is another special class of mixed graphs containing not real mixed odd cycles. Therefore,

Corollary 2.13 *If G is an oriented graph, then (1) all its coefficients of c_{odd} are equal to zero; (2) its spectrum is symmetry about zero; (3) $c_{2k} = (-1)^k b_k$ and $b_k \geq 0$ for all possible integer k.*

The next result gives a characterization of an undirected graph without even cycles.

Theorem 2.14 *Let G be an undirected graph, then G has no even cycles if and only if $Sp_H(G^{\phi_1}) = Sp_H(G^{\phi_2})$ for any generalized orientations ϕ_1 and ϕ_2 of G that G^{ϕ_1} and G^{ϕ_2} contain not real mixed odd cycles.*

Proof. The proof is similar to the proof of Theorem 4.2 in [7]. Necessity. If G has no even cycles and ϕ_1 and ϕ_2 are two arbitrary generalized orientations of G that G^{ϕ_1} and G^{ϕ_2} contain not real mixed odd cycles. Then each real elementary subgraph with even number of vertices in G^{ϕ_j} $(j = 1, 2)$ is comprised by matching, i.e. disjoint edges. Thus for any possible odd number $2k + 1$, we have $c_{2k+1}(G^{\phi_1}) = c_{2k+1}(G^{\phi_2}) = 0$ and for any possible even number $2k$, we have

$$\begin{aligned} c_{2k}(G^{\phi_1}) &= \sum_{G'_1} (-1)^{r(G'_1)+\ell(G'_1)} 2^{s(G'_1)} = \sum_{G'_1} (-1)^{r(G'_1)} = \sum_{G'_2} (-1)^{r(G'_2)} \\ &= \sum_{G'_2} (-1)^{r(G'_2)+\ell(G'_2)} 2^{s(G'_2)} = c_{2k}(G^{\phi_2}), \end{aligned}$$

where G'_1 and G'_2 are k matchings of G^{ϕ_1} and G^{ϕ_2}, respectively. Thus, $\Phi(G^{\phi_1}; \lambda) = \Phi(G^{\phi_2}; \lambda)$ and hence $Sp_H(G^{\phi_1}) = Sp_H(G^{\phi_2})$.

Sufficiency. Note that G^{ϕ} contains no real mixed odd cycles for any orientation ϕ. We will proof by contradiction if there is an undirected graph G with finite even cycles such that $Sp_H(G^{\phi_1}) = Sp_H(G^{\phi_2})$ for any orientations ϕ_1 and ϕ_2.

Let C be an even cycle of least length, say 2ℓ in G. By Theorem 2.8, the first 2ℓ coefficients of the characteristic polynomial of $H(G^{\phi})$ for an orientation ϕ are

$$c_k = m_k(G^{\phi}) \quad \text{when} \quad k < 2\ell \quad \text{and} \quad c_{2\ell} = m_{2\ell}(G^{\phi}) - 2 \sum_{\ell(C')=2\ell} h(C'), \tag{2.1}$$

where $m_k(G)$ is the number of matchings in G covering k vertices and the sum is taken over all mixed cycles C' in G^{ϕ} of length 2ℓ. Let $e \in C$ in G^{ϕ} and $n_{\alpha}(e)$ be the number of mixed $2\ell-$cycles in G^{ϕ} that contain e and have values α in one direction, where $\alpha \in \{\pm 1\}$. Suppose that $n_+(e) \neq n_-(e)$. If the direction of the arc e is reversed, then in (2.1) the contribution from the matchings will be unaffected as will that from the mixed 2ℓ-cycles not containing e. But the contribution from the mixed $2\ell-$cycles that contain e equals $-2(n_+(e) - n_-(e))$ and will be negated. Consequently, $c_{2\ell}$ will change. Thus G will have an orientation ϕ' such that $Sp_H(G^{\phi'}) \neq Sp_H(G^{\phi})$ and the sufficiency will have been proved.

Suppose then that $n_+(e) = n_-(e)$ for all edges e in all orientations G^ϕ of G. We shall see that this leads to a contradiction. For $t \in \{1, \ldots, 2\ell\}$, let $n_+(e_1, \ldots, e_t)$ be the number of positive $2\ell-$cycles in G^ϕ and contain all of $e_1, \ldots, e_t$. Define $n_-(e_1, \ldots, e_t)$ analogously.

We claim that for each $t \in \{1, \ldots, 2\ell\}$, $n_+(e_1, \ldots, e_t) = n_-(e_1, \ldots, e_t)$ for all orientations G^ϕ and all edges $e_1, \ldots, e_t$. We proceed by induction on t. The case $t = 1$ is assumed. Suppose that the claim holds for each $t < 2\ell$ and let G^ϕ be an orientation of G. For edges $e_1, e_2, \ldots, e_t, e_{t+1}$ in G, let $n_+(e_1, \ldots, e_t, \overline{e}_{t+1})$ denotes the number of positive $2\ell-$cycles that contain edges $e_1, \ldots, e_t$, but not edge e_{t+1}. Define $n_-(e_1, \ldots, e_t, \overline{e}_{t+1})$ analogously. Then

$$n_+(e_1, \ldots, e_t) = n_+(e_1, \ldots, e_t, e_{t+1}) + n_+(e_1, \ldots, e_t, \overline{e}_{t+1}),$$
$$n_-(e_1, \ldots, e_t) = n_-(e_1, \ldots, e_t, e_{t+1}) + n_-(e_1, \ldots, e_t, \overline{e}_{t+1}),$$

and $n_+(e_1, \ldots, e_t) = n_-(e_1, \ldots, e_t)$ by assumption. Now we consider the orientation $G^{\phi'}$ obtained from G^ϕ by reversing the orientation of e_{t+1}. Then

$$n'_+(e_1, \ldots, e_t) = n_-(e_1, \ldots, e_t, e_{t+1}) + n_+(e_1, \ldots, e_t, \overline{e}_{t+1}),$$
$$n'_-(e_1, \ldots, e_t) = n_+(e_1, \ldots, e_t, e_{t+1}) + n_-(e_1, \ldots, e_t, \overline{e}_{t+1}),$$

and $n'_+(e_1, \ldots, e_t) = n'_-(e_1, \ldots, e_t)$ by assumption. Consequently,

$$\begin{aligned} & n_+(e_1, \ldots, e_t, \overline{e}_{t+1}) - n_-(e_1, \ldots, e_t, \overline{e}_{t+1}) \\ & = n_-(e_1, \ldots, e_t, e_{t+1}) - n_+(e_1, \ldots, e_t, e_{t+1}) \\ & = n_+(e_1, \ldots, e_t, e_{t+1}) - n_-(e_1, \ldots, e_t, e_{t+1}). \end{aligned}$$

Lines 1 and 3 above are equal and sum to zero. Thus $n_-(e_1, \ldots, e_t, e_{t+1}) = n_+(e_1, \ldots, e_t, e_{t+1})$, as desired. Hence the claim holds. We then have $n_+(e_1, \ldots, e_{2\ell}) = n_-(e_1, \ldots, e_{2\ell})$ for any orientation G^ϕ, where $e_1, \ldots, e_{2\ell}$ are edges of a mixed $2\ell-$cycle of G^ϕ. This is a contradiction, since a mixed even cycle in an oriented graph is either positive or negative and hence one member of the equality is 0, while the other is 1. Thus we complete the proof. ∎

Note that any orientation ϕ of an undirected graph G contains not real mixed odd cycles. We immediately arrive at the result of Cavers et al.[7]:

Corollary 2.15 *[7] An undirected graph G has no even cycles if and only if all of its oriented graphs are all cospectral.*

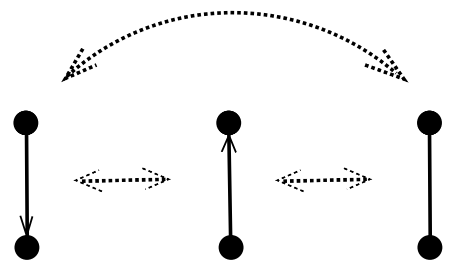

Figure 2.2 Changing the value of an edge

For a mixed graph G with cut edges, we see that cut edges are not contained in any mixed cycles of G. The next result shows that the spectrum of G is invariant when changing the value of any cut edges (if any), for example reversing the cut arc's orientation or unorienting it or orienting an undirected cut edge, see Figure 2.2.

Theorem 2.16 *Let G be an undirected graph with cut edges, then $Sp_H(G^{\phi_1}) = Sp_H(G^{\phi_2})$ for any generalized orientations ϕ_1 and ϕ_2 of G that differ only on some cut edges of G.*

Proof. Let S be the cut edges that differ in ϕ_1 and ϕ_2. If a real elementary subgraph G' of G^{ϕ_1} contains not edges from S, then G' is also a real elementary subgraph of G^{ϕ_2}. Obviously, we have $(-1)^{r(G')+\ell(G')}2^{s(G')} = (-1)^{r(G')+\ell(G')}2^{s(G')}$. If a real elementary subgraph G' of G^{ϕ_1} contains some edges S' from S, then correspondingly there is a real elementary subgraph G'' of G^{ϕ_2} that satisfies $G'_U \cong G''_U$ and differs with G' only on S' and vice versa. We have $(-1)^{r(G')+\ell(G')}2^{s(G')} = (-1)^{r(G'')+\ell(G'')}2^{s(G'')}$. Thus,

$$c_k(G^{\phi_1}) - c_k(G^{\phi_2}) = \sum_{G'_1}(-1)^{r(G'_1)+\ell(G'_1)}2^{s(G'_1)} - \sum_{G'_2}(-1)^{r(G'_2)+\ell(G'_2)}2^{s(G'_2)} = 0,$$

for any integer k and $Sp_H(G^{\phi_1}) = Sp_H(G^{\phi_2})$. ∎

Thus for any mixed graph with cut edges G_1, we can changing the value of some cut edges such that the resulting mixed graph G_2 is not isomorphic to G_1 and $Sp_H(G_1) = Sp_H(G_2)$.

From Theorem 2.8, we also see that for a mixed graph real mixed cycles play important role in determining coefficients of its characteristic polynomial. If a mixed graph contains no any real mixed cycles, then its coefficients of its characteristic polynomial are determining solely by their matchings.

Corollary 2.17 *Let G be a mixed graph that contains no any real mixed cycles, then $c_i = 0$ for i odd; and $(-1)^r c_{2r}$ is the number of ways of choosing r disjoint edges in G.*

Since a mixed forest contains no real mixed cycles, we have:

Corollary 2.18 *Let T be a mixed forest, then $c_i = 0$ for i odd; and $(-1)^r c_{2r}$ is the number of ways of choosing r disjoint edges in T.*

Example 2.19 *Let $\overrightarrow{S_n}$ be a mixed tree of order n such that its underlying graph is a star, then $(-1)^r c_{2r} = 0$ for $r \geq 2$ and $-c_2 = n-1$. Then $\Phi(\overrightarrow{S_n}; \lambda) = \lambda^n - (n-1)\lambda^{n-2} = \lambda^{n-2}(\lambda - \sqrt{n-1})(\lambda + \sqrt{n-1})$.*

Example 2.20 *Let $\overrightarrow{P_n}$ be a mixed tree of order n such that its underlying graph is a path. In fact, since $(-1)^r c_{2r}$ is the number of ways of choosing r disjoint edges in T. Then $(-1)^r c_{2r} = \binom{n-r}{r}$ and $\Phi(\overrightarrow{P}_n; \lambda) = \sum\limits_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \binom{n-r}{r} \lambda^{n-2r}$.*

Since any elementary subgraph of a mixed forest is a union of disjoint edges and then any minor with same order has identical value, we have:

Corollary 2.21 *For any undirected forest T, we have $Sp_H(T) = Sp_H(T^{\phi})$ for any not trivial generalized orientation ϕ of T.*

Thus, for any pairs of non-isomorphic mixed forests T, T' that share the same underlying graph, we have $Sp_H(T) = Sp_H(T')$. We have the following result:

Corollary 2.22 *For any mixed forest T with $n (n \geq 2)$ vertices, there exists an non-isomorphic mixed forest T' such that $Sp_H(T) = Sp_H(T')$.*

Note that any edge of a forest is also its cut edge, the Corollary 2.22 can be seen as a corollary of Theorem 2.16. This result and Theorem 2.16 generalized that of Schwenk [20] that if we select a tree T with n vertices, all such trees being equally likely, then the probability that T belongs to a cospectral pair tends to 1 as n tends to infinity.

3 Bounds of Hermitian Energy

In this section we will give some bounds to Hermitian energy of a mixed graph. First, we need the following result.

Theorem 3.1 *If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of an n-vertex mixed graph G with size m, then*
(1) $\sum\limits_{k=1}^{n} \lambda_k^2 = 2m$;
(2) for $1 \le i \le n, |\lambda_i| \le \Delta$, the maximum degree of the underlying graph.

Proof. (1) We have

$$\sum_{k=1}^{n} \lambda_k^2 = \text{trace of } H^2 = \sum_{k=1}^{n}\sum_{\ell=1}^{n} h_{k\ell}h_{\ell k} = \sum_{k=1}^{n}\sum_{\ell=1}^{n} h_{k\ell}h_{k\ell}^* = \sum_{k=1}^{n}\sum_{\ell=1}^{n} |h_{k\ell}|^2 = 2m.$$

(2) is a direct consequence of the Gershgorin Circle theorem. ∎

An upper bound for $\mathcal{E}_H(G)$, similar to McClelland's inequality for energy of an undirected graph, is given in the next theorem.

Theorem 3.2 *For an n-vertex mixed graph G, $\sqrt{2m + n(n-1)p^{2/n}} \le \mathcal{E}_H(G) \le \sqrt{2mn} \le n\sqrt{\Delta}$, where $p = |\det H(G)|$.*

Proof. The proof is similar with that in [2]. We have

$$[\mathcal{E}_H(G)]^2 = \left[\sum_{k=1}^{n} |\lambda_k|\right]^2 = \sum_{k=1}^{n} \lambda_k^2 + \sum_{k \ne \ell} |\lambda_k||\lambda_\ell| \ge 2m + n(n-1)p^{2/n},$$

the last inequality being a consequence of Theorem 3.1 and the AM-GM inequality. On the other hand, Schwarz's inequality applied to the Euclidean vectors $(|\lambda_1|, \ldots, |\lambda_n|)$ and $(1, \ldots, 1)$, yields

$$\mathcal{E}_H(G) = \sum_{k=1}^{n} |\lambda_k| \le \sqrt{\sum_{k=1}^{n} \lambda_k^2}\sqrt{n} = \sqrt{2mn} \le \sqrt{(n\Delta)n} = n\sqrt{\Delta}. \tag{3.2}$$

∎

Corollary 3.3 *For a mixed graph G and its Hermitian-adjacency matrix H, the following conditions are equivalent:*

(1) $\mathcal{E}_H(G) = n\sqrt{\Delta}$*;*

(2) $H^2 = \Delta \mathbf{I}_n$*;*

(3) $(H')^2 = \mathbf{I}_n$*, where* $H' = \frac{1}{\sqrt{\Delta}} H$*.*

Proof. The proof is similar with that in [2]. Equality holds in (3.2) if and only if the Schwarz's inequality becomes equality and $2m = n\Delta$, if and only if, there exists a constant α such that $|\lambda_i|^2 = \alpha$ for all i and G is a $\Delta-$regular mixed graph, if and only if, $H^*H = \alpha \mathbf{I}_n$ and $\alpha = \Delta$, i.e., $H^2 = \Delta \mathbf{I}_n$. I.e., $(H')^2 = \mathbf{I}_n$, and $(H')^* = \frac{1}{\sqrt{\Delta}} H^* = \frac{1}{\sqrt{\Delta}} H = H'$. ∎

It follows from the proof above that if $\mathcal{E}_H(G) = n\sqrt{\Delta}$, then G must be a $\Delta-$regular mixed graph. So each column and row of $H(G)$ contains exactly Δ nonzero entries from $\mathbf{T} = \{1, \pm \mathbf{i}\}$ and inner products $H(u,:) \cdot H(v,:) = 0, H(:,u) \cdot H(:,v) = 0$ for different vertices u and v, where $H(u,:)$ and $H(:,u)$ represent row vector and column vector corresponding to vertex u in $H(G)$, respectively.

We want to characterize all mixed graphs satisfying these conditions. First, we have

Lemma 3.4 *For a mixed graph G with $H^2(G) = kI$ $(k \geq 2)$, then $g(G_U) \leq 4$, where $g(G_U)$ denotes the girth of G_U.*

Proof. For otherwise, if $g(G_U) \geq 5$ (or $g(G_U)$ is infinity). Let u and v be a pair of vertices with distance two in G_U and $w \in N(u) \cap N(v)$. Then $H(u,:) \cdot H(v,:) = h_{u,w} h_{w,v} \neq 0$ which contradicts to (2) of Corollary 3.3. ∎

In fact, we can say more about it.

Lemma 3.5 *Let G be a connected $k-$regular mixed graph with order n $(n \geq 3)$, then $\mathcal{E}_H(G) = n\sqrt{k}$ if and only if for any pair of vertices u and v with distance not more than two in G_U such that $N(u) \cap N(v) \neq \emptyset$, there are edge-disjoint mixed $4-$cycles $uxvy$ of the following three types; for any edge $u \sim v$ such that $N(u) \cap N(v) = \emptyset$, there are edge-disjoint mixed $4-$cycles $uvxy$ of the following three types.*

Proof. By Corollary 3.3, we have $H^2(G) = kI$. For different vertices u and v,

$$H(u,:) \cdot H(v,:) = \sum_{w \in N(u) \cap N(v)} h_{uw} h_{vw} = 0. \tag{3.3}$$

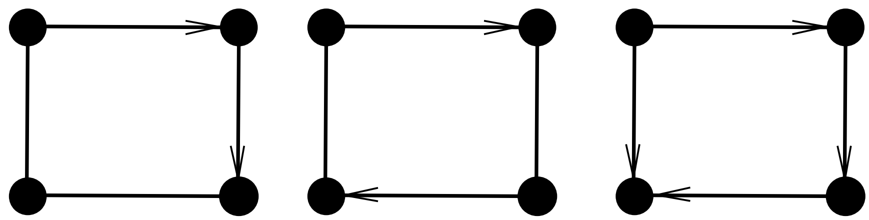

Figure 3.1 Three types of mixed $4-$cycles

If $d(u,v) \geq 3$, then $H(u,:) \cdot H(v,:) = 0$ holds obvious. If $d(u,v) = 2$, then $N(u) \cap N(v) \neq \emptyset$. Because of $h_{ux}h_{vx} \in \{\pm 1, \pm \mathbf{i}\}$ for a vertex $x \in N(u) \cap N(v)$ and (3.3), we have another vertex $y \in N(u) \cap N(v)$ such that $h_{ux}h_{xv} + h_{uy}h_{yv} = 0$. Thus if $h_{ux}h_{xv} = p$, then $h_{uy}h_{yv} = -p$. By checking all possible combinations, we find that the mixed $4-$cycle $uxvy$ are one of three forms in Figure 3.1. If $d(u,v) = 1$ and $N(u) \cap N(v) \neq \emptyset$. We will have a pair of vertices $x, y \in N(u) \cap N(v)$ such that $uxvy$ is one of the three mixed $4-$cycles in Figure 3.1. If $d(u,v) = 1$ and $N(u) \cap N(v) = \emptyset$. We have $d(x,u) = 2$ for any $x \in N(v) \backslash \{u\}$. There exists a vertex $y \in (N(x) \cap N(u)) \backslash \{v\}$ such that $uvxy$ is one of the three mixed $4-$cycles in Figure 3.1. The converse is easily checked. ∎

From the proof of Lemma 3.5, we can see that $|N(u) \cap N(v)|$ is even for pair of vertices u and v with distance no more than two. Therefore, Lemma 3.5 generalizes a result of Adiga et al [2].

We want to determine which $k-$regular mixed graphs on n vertices with $\mathcal{E}_H(G) = n\sqrt{k}$. To avoid triviality, we assume $k \neq 0$. It follows that n must be even since $\sum\limits_{p=1}^{n} \lambda_p = 0$ and $\lambda_p^2 = k$ for $p = 1, 2, \ldots, n$. It is not hard to see that $1-$regular mixed graph on n vertices with $\mathcal{E}_H(G) = n\sqrt{k}$ if and only if it is a union of $\frac{n}{2}$ edges with n even. If $k = 2$, then the $2-$regular graph must be a union of mixed cycles. Lemma 3.5 shows that all $2-$regular mixed graph on n vertices with $\mathcal{E}_H(G) = n\sqrt{2}$ if and only if it is a union of $\frac{n}{4}$ mixed $4-$cycles from the three types of mixed $4-$cycles in Lemma 3.5. For $k = 3$, the case of oriented graphs is solved by Gong and Xu in [13]. And the underlying component of the oriented graphs of order n with $\mathcal{E}_H(G) = n\sqrt{3}$ is either K_4 the complete graph on 4 vertices or Q_3 the hypercube. For $k = 4$, Chen et al. [9] and Gong et al. [14] respectively constructed infinitely many connected oriented graphs of order n with $\mathcal{E}_H(G) = 2n$. For large k, the problem seems rather difficult because it is closely related to the famous Hardamard Matrix Conjecture in combinatorial design when we restrict it to oriented graphs [18]. We thus would like to propose the following problem:

Problem 3.6 *Determine all the $k-$regular mixed graphs G on n vertices with $\mathcal{E}_H(G) = n\sqrt{k}$ for each k, $3 \leq k \leq n$.*

or

Problem 3.7 *Determine all the matrices H of order n for all integer k ($3 \leq k \leq n$) such that:*
(1) $H^* = H$, i.e. H is Hermitian;
(2) $H^2 = \mathbf{I}_n$, i.e. H is unitary;
(3) each row of H contains exactly k nonzero entries of $\mathbf{T}' = \{\frac{1}{k}, \pm\frac{\mathbf{i}}{k}\}$.

In order to construct infinite many mixed graphs with maximum Hermitian energies, we consider two kinds of products of mixed graphs. Let G_1 be an undirected graph and G_2 a mixed graph. The Kronecker product $G_1 \times G_2$ of G_1 and G_2 is a mixed graph with vertex set $V(G_1) \times V(G_2)$ and there is an undirected edge (resp. arc) from (u_1, v_1) to (u_2, v_2) if and only if $u_1 \sim u_2$ in G_1 and (v_1, v_2) is an edge (resp. arc) in G_2. And the Kronecker product $G_2 \times G_1$ is defined analogously. Since $H(G_1 \times G_2) = H(G_1) \otimes H(G_2)$, where the symbol $\otimes$ stands for the Kronecker product of two matrices (see [6]).

Remark 3.8 *Let G_1 be an undirected graph (resp. mixed graph) of order p and $Sp_H(G_1) = \{\mu_1, \mu_2, \ldots, \mu_p\}$. Let G_2 be a mixed graph (resp. undirected graph) of order q and $Sp_H(G_2) = \{\lambda_1, \lambda_2, \ldots, \lambda_q\}$. Then the eigenvalues of the mixed graph $G_1 \times G_2$ are $\mu_k\lambda_\ell$ for all possible $k \in \{1, 2, \ldots, p\}$ and $\ell \in \{1, 2, \ldots, q\}$.*

Example 3.9 *Let G_1 be an undirected graph K_2 and G_2 an $\ell-$regular mixed graph of order q with $\mathcal{E}_H(G_2) = q\sqrt{\ell}$. Then the mixed graph $G_1 \times G_2$ has the maximum Hermitian energy $\mathcal{E}_H(G_1 \times G_2) = 2q\sqrt{\ell}$.*

Let G_1 and G_2 be mixed graphs. The Cartesian product $G_1 \Box G_2$ of G_1 and G_2 is a mixed graph with vertex set $V(G_1) \times V(G_2)$ and there is an undirected edge (resp. arc) from (u_1, v_1)

to (u_2, v_2) if and only if $u_1 = u_2$ and (v_1, v_2) is an edge (resp. arc) in G_2, or $v_1 = v_2$ and (u_1, u_2) is an undirected edge (resp. arc) in G_1.

If G_2 is a bipartite oriented graph with bipartition X and Y, we modify the orientation of $G_1 \Box G_2$ with the following method. If there is an arc from (u, v_1) to (u, v_2) in $G_1 \Box G_2$ and $u \in Y$, then we reverse the direction of the arc. All the other arcs or edges keep unchanged. We denote this new orientation of $G_1 \Box G_2$ by $(G_1 \Box G_2)^*$.

Theorem 3.10 *Let G_1 be an oriented bipartite graph of order p and $Sp_H(G_1) = \{\mu_1, \mu_2, \ldots, \mu_p\}$. Let G_2 be a mixed graph of order q and $Sp_H(G_2) = \{\lambda_1, \lambda_2, \ldots, \lambda_q\}$. Then the eigenvalues of the mixed graph $G_1 \Box G_2$ are $\pm\sqrt{\mu_k^2 + \lambda_\ell^2}$, $k = 1, 2, \ldots, p$ and $\ell = 1, 2, \ldots, q$.*

Proof. The proof is similar with that in [1]. Let G_1 be an oriented bipartite graph with bipartition X and Y, where $|X| = x$ and $|Y| = p - x$. With suitable labeling of the vertices of $G_1 \Box G_2$, the Hermitian-adjacency matrix $H = H((G_1 \Box G_2)^*)$ can be chosen as follows:

$$H = \mathbf{I}'_p \otimes H(G_2) + H(G_1) \otimes \mathbf{I}_q,$$

where $\mathbf{I}'_p$ is a diagonal matrix with first x diagonal elements equal to 1 and last $p-x$ diagonal elements equal to -1; $H(G_1)$ it the partition matrix $\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$, where B is an $x \times x$ matrix.

Now we want to determine the singular values of H, we have

$$\begin{aligned} H^*H = H^2 &= (\mathbf{I}'_p \otimes H(G_2) + H(G_1) \otimes \mathbf{I}_q)(\mathbf{I}'_p \otimes H(G_2) + H(G_1) \otimes \mathbf{I}_q) \\ &= \mathbf{I}_p \otimes H^2(G_2) + H^2(G_1) \otimes \mathbf{I}_q + \mathbf{I}'_p H(G_1) \otimes H(G_2) + H(G_1)\mathbf{I}'_p \otimes H(G_2) \\ &= \mathbf{I}_p \otimes H^2(G_2) + H^2(G_1) \otimes \mathbf{I}_q + (\mathbf{I}'_p H(G_1) + H(G_1)\mathbf{I}'_p) \otimes H(G_2) \\ &= \mathbf{I}_p \otimes H^2(G_2) + H^2(G_1) \otimes \mathbf{I}_q \end{aligned}$$

because $\mathbf{I}'_p H(G_1) + H(G_1)\mathbf{I}'_p = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & -B \\ B^* & 0 \end{pmatrix} = \mathbf{0}$. Therefore, the eigenvalues of H^*H are $\mu^2 + \lambda^2$, where $\mu \in Sp_H(G_1)$ and $\lambda \in Sp_H(G_2)$. Thus the result follows. ∎

As an application of Theorem 3.10, we can construct infinite mixed graphs with maximum Hermitian energy. First, we need the following result.

Theorem 3.11 *Let G_1 be an oriented $k-$regular bipartite graph of order p with $\mathcal{E}_H(G_1) = p\sqrt{k}$ and G_2 an $\ell-$regular mixed graph of order q with $\mathcal{E}_H(G_2) = q\sqrt{\ell}$. Then the mixed graph $(G_1 \Box G_2)^*$ has the maximum Hermitian energy $\mathcal{E}_H((G_1 \Box G_2)^*) = pq\sqrt{k+\ell}$.*

Proof. First we have $H^2(G_1) = k\mathbf{I}_p$ and $H^2(G_2) = \ell\mathbf{I}_q$. Then the Hermitian eigenvalues of G_1 and G_2 are all $\pm\sqrt{k}$ and $\pm\sqrt{\ell}$ respectively. By Theorem 3.10, the Hermitian eigenvalues of $(G_1 \Box G_2)^*$ are $\pm\sqrt{k+\ell}$. The result follows. ∎

Note that if we let G_2 to be an $\ell-$regular oriented graph of order q and maximum energy $H^2(G_2) = \ell\mathbf{I}_q$ in Theorem 3.11, we arrive at a result of Anuradha et al [1]. And if we let G_1 be an oriented P_2 in Theorem 3.11, we arrive is exactly a result of Cui and Hou [8].

In the following example, we present a family of mixed graphs with maximum Hermitian energy.

Example 3.12 *Let G_1 be a mixed tree of order two and $\overrightarrow{K_2}$ an oriented path of order two. For each $k \geq 2$, set $G_k = (G_{k-1} \Box \overrightarrow{K_2})^*$. Then G_k is a mixed $k-$regular graphs with order 2^k and Hermitian energy $2^k\sqrt{k}$. Note that G_k is a mixed hypercube Q_k of dimension k. The result generalizes a result of Tian [22] that there exists a k-regular graph with $n = 2^k$ vertices (i.e., Q_k) having an orientation with maximum skew energy.*

We now give bounds of Hermitian energy of a mixed graph with respect to its size.

Theorem 3.13 *For a mixed graph G with m edges, $2\sqrt{m} \leq \mathcal{E}_H(G) \leq 2m$.*

Proof. We have three relations for the eigenvalues:

$$\sum_{k=1}^{n} \lambda_k = 0$$

$$\sum_{k=1}^{n} \lambda_k^2 = 2m$$

$$\sum_{k<\ell} \lambda_k \lambda_\ell = -m$$

From the definition of the Hermitian energy of a mixed graph, we have

$$\begin{aligned}
\mathcal{E}_H^2(G) &= \sum_{k=1}^{n} \lambda_k^2 + 2\sum_{k<\ell} |\lambda_k \lambda_\ell| \\
&= 2m + 2\sum_{k<\ell} |\lambda_k \lambda_\ell| \\
&\geq 2m + 2|\sum_{k<\ell} \lambda_k \lambda_\ell| \\
&= 4m.
\end{aligned}$$

Combining the facts that $n \leq 2m$ and $\mathcal{E}_H(G) \leq \sqrt{2mn}$, we can obtain the upper bound. ∎

Bapat and Pati [5] showed that energy of an undirected graph is never an odd integer. Adiga et al. [2] also showed that skew energy of an oriented graph is never an odd integer. Along this line, we have:

Theorem 3.14 *The Hermitian energy of a mixed graph, if it is a rational number, must be an even positive integer.*

Proof. The proof is similar with that in [2]. Let $\{\lambda_1, \dots, \lambda_n\}$ be the eigenvalues of $H(G)$. Then

$$\text{trace of } H(G) = \sum_{k=1}^{n} \lambda_k = 0.$$

W.O.L.G. let $\lambda_1, \dots, \lambda_k$ be positive and the rest of the λ_ℓ's non-positive. Then

$$\mathcal{E}_H(G) = \sum_{\ell=1}^{n} |\lambda_\ell| = 2(\lambda_1 + \cdots + \lambda_k).$$

Since $\lambda_1, \dots, \lambda_k$ are algebraic integers, so is their sum. Hence $(\lambda_1 + \cdots + \lambda_k)$ must be a rational integer if $\mathcal{E}_H(G)$ is rational. ∎

Similar to a result in [2], we have a result that each even positive integer is the Hermitian energy of some mixed graph. From Example 2.19, we see that $\mathcal{E}_H(\overrightarrow{S}_{n+1}) = 2\sqrt{n}$. For each even positive integer $2r$, we take $n = r^2$. Then $\mathcal{E}_H(\overrightarrow{S}_{n+1}) = 2r$. Thus, we have the following result.

Theorem 3.15 *Each even positive integer $2r$ is the Hermitian energy of a mixed star.*

4 Mixed graphs that share the same spectrum with their underlying graphs

In this section, we want to characterize which mixed graphs that share the same spectrum with their underlying graphs. First we need some concepts. A switching function is any function $\theta : V \to \mathbf{T}$, where $\mathbf{T} = \{1, \pm\mathbf{i}\}$. Switching a mixed graph G to a mixed graph G' means that there is a diagonal matrix $D(\theta) := \mathrm{diag}(\theta(v_k) : v_k \in V)$ such that $H(G) = D(\theta)^{-1}H(G')D(\theta)$. Equivalently, $h_{k\ell} = \theta(v_k)^{-1}h'_{k\ell}\theta(v_\ell)$ for any k, ℓ, where $h_{k\ell}$ and $h'_{k\ell}$ are the $(k,\ell)-$elements in $H(G)$ and $H(G')$, respectively. We say G_1 and G_2 are switching equivalent, denoted by $G_1 \sim G_2$, when there exists a switching function, such that $H(G_2) = D(\theta)^{-1}H(G_1)D(\theta)$. Note that the definition of switching equivalence between oriented graphs is given in [8]. Switching equivalence forms an equivalence relation on Hermitian-adjacency matrices for a fixed underlying graph. It is straightforward to see that if two mixed graphs G_1 and G_2 are switching equivalent, then $Sp_H(G_1) = Sp_H(G_2)$.

Now we consider the problem of determining what kind of mixed graphs having the same spectrum with their underlying graphs.

Theorem 4.1 *Let G be a mixed graph with Hermitian-adjacency matrix $H(G)$, then the following are equivalent:*
(1) G is positive.
(2) $G \sim G_U$.

Proof. We only need to show the equivalence for connected mixed graphs. Let G be a connected mixed graph of order n and size m. If G is positive. We shall prove (2) by induction. If $m = n - 1$, then G is a mixed tree and we define a function $\theta : V \to \mathbf{T}$ by the following procedure. First, pick an arbitrary vertex v and set $\theta(v) = 1$, then we expand the definition of θ through adjacency relation by setting the value of $\theta(w)$ such that $\theta(u)^{-1}h_{uw}\theta(w) = 1$, i.e. $\theta(w) = \theta(u)h_{uw}^{-1}$, for defined vertex u and its undefined neighbor w. The process end when all vertex are defined. We can see that θ is a switching function such that $H(G_U) = D(\theta)^*H(G)D(\theta)$, that is $H(G) \sim H(G_U)$. Now we assume the result (2) hold for any connected mixed graph of order n and size $m(\geq n-1)$. Let G be a connected mixed graph of order n and size $m+1$. Then G contains at least a mixed cycle. Let $C : v_1v_2 \cdots v_kv_1$

be any mixed cycle of G. Consider the mixed connected graph $G-v_1v_2$ with order n and size m. For convenience, denote by G' the mixed graph $G-v_1v_2$ and G'_U the underlying graph of $G-v_1v_2$, respectively. By induction, we have a switching function $\theta : V(G') \to \mathbf{T}$ such that

$$H(G'_U) = D(\theta)^* H(G') D(\theta).$$

In the following, we want to show that θ is also a switching function from $H(G)$ to $H(G_U)$. To see this, consider

$$\begin{aligned}
&D(\theta)^* H(G) D(\theta) \\
&= D(\theta)^* \Big[H(G') + h_{v_1v_2} E_{12} + h_{v_2v_1} E_{21} \Big] D(\theta) \\
&= D(\theta)^* H(G') D(\theta) + D(\theta)^* \Big[h_{v_1v_2} E_{12} + h_{v_2v_1} E_{21} \Big] D(\theta) \\
&= H(G'_U) + E_{12} + E_{21} \\
&= H(G_U)
\end{aligned}$$

where E_{12} (resp. E_{21}) is a $n \times n$ matrix with all elements equal to zero except the $(1,2)-$entry (resp. $(2,1)-$entry), which is equal to one. The third equality holds because G is positive and so is the mixed cycle C. Thus,

$$h_{v_1v_2} h_{v_2v_3} \cdots h_{v_kv_1} = 1,$$

hence

$$\begin{aligned}
&\theta(v_1)^* \Big[h_{v_1v_2} h_{v_2v_3} \cdots h_{v_kv_1} \Big] \theta(v_1) = \theta(v_1)^* \theta(v_1) = 1, \\
&\Big[\theta(v_1)^* h_{v_1v_2} \theta(v_2) \Big] \Big[\theta(v_2)^* h_{v_2v_3} \theta(v_3) \Big] \cdots \Big[\theta(v_k)^* h_{v_kv_1} \theta(v_1) \Big] = 1.
\end{aligned}$$

Since

$$H(G'_U) = D(\theta)^* H(G') D(\theta).$$

we have $\theta(v_p)^* h_{v_pv_{p+1}} \theta(v_{p+1}) = 1$ for $p = 2, 3, \ldots, k (\bmod\ k)$ and then $\theta(v_1)^* h_{v_1v_2} \theta(v_2) = 1$ and $\theta(v_2)^* h_{v_2v_1} \theta(v_1) = 1$ holds analogously. Therefore, (2) holds for mixed graph G of order n and size $m+1$. We proved (2) by induction.

If $G \sim G_U$, then there is a switching function $\theta : V \to \mathbf{T}$ such that $H(G_U) = D(\theta)^* H(G) D(\theta)$. Thus,

$$\theta(v_p)^* \cdot h_{v_pv_q} \cdot \theta(v_q) = 1,$$

for any pair of adjacent vertices v_p and v_q. For any mixed cycle $C: v_1v_2\cdots v_kv_1$ of G, we have

$$\Big[\theta(v_1)^*h_{v_1v_2}\theta(v_2)\Big]\Big[\theta(v_2)^*h_{v_2v_3}\theta(v_3)\Big]\cdots\Big[\theta(v_k)^*h_{v_kv_1}\theta(v_1)\Big]=1,$$
$$\theta(v_1)^*\Big[h_{v_1v_2}h_{v_2v_3}\cdots h_{v_kv_1}\Big]\theta(v_1)=\theta(v_1)^*\theta(v_1)=1.$$

Therefore,

$$h_{v_1v_2}h_{v_2v_3}\cdots h_{v_kv_1}=1,$$

i.e., the mixed cycle C is positive. Thus G is positive. ∎

Since a mixed forest contains no mixed cycle and then is positive, we obtain Corollary 2.21 again from this theorem.

The next result shows that Cartesian product of positive mixed graphs is still positive.

Theorem 4.2 *Let G' and G'' are positive mixed graphs. Then $G'\Box G''$ is a positive mixed graph too.*

Proof. Let $C=(u_1,v_1)(u_2,v_2)\dots(u_k,v_k)(u_1,v_1)$ be a mixed chordless cycle in $G'\Box G''$. If $u_1=u_2=\cdots=u_k$ or $v_1=v_2=\cdots=v_k$, then C is obviously positive since both G' and G'' are positive. For otherwise, then k is even. And if there is an edge $(u_p,v_p)(u_{p+1},v_{p+1})$ belongs to C such that $u_p=u_{p+1}$ (or $v_p=v_{p+1}$) for one direction, then correspondingly there is an edge $(u_q,v_q)(u_{q+1},v_{q+1})$ belongs to C such that $u_q=u_{q+1}, v_q=v_{p+1}, v_{q+1}=v_p$ (or $v_p=v_{p+1}, u_q=u_{p+1}, u_{q+1}=u_p$), all subscripts are modulo of k, for this direction by the definition of Cartesian product and the fact that C is a mixed chordless cycle. Then $h_{(u_p,v_p)(u_{p+1},v_{p+1})}\cdot h_{(u_q,v_q)(u_{q+1},v_{q+1})}=1$ and hence $h(C)=1$. Therefore, $G'\Box G''$ is a positive mixed graph. ∎

5 Oriented Graphs

We shall focus on the oriented graphs in this section. We will reconfirm some results with respect to oriented graphs. Especially, we will reconfirm a conjecture given by Cui and Hou in [8].

For an oriented graph G with odd mixed cycles (i.e. oriented odd cycle), we can see that none value of its oriented odd cycle equal to one, therefore G is not positive and then $Sp_H(G) \neq Sp_H(G_U)$ by Theorem 4.1. Thus the underlying graph of a positive oriented graph shall be bipartite.

Let G^γ be an oriented graph with G as its underlying graph. An oriented even cycle C of G^γ is said to be evenly or oddly oriented if the number of arcs of C in each direction is even or odd [1]. An oriented even cycle C_{2k} of length $2k$ in G^γ is said have a *parity-linked orientation* if it is evenly oriented whenever k is even and oddly oriented whenever k is odd. If every oriented even cycle in G^γ has a parity-linked orientation, then the orientation γ is defined to be a parity-linked orientation of G and G^γ is called as a *parity-linked oriented graph.* Note that the parity-linked orientation is termed as uniform orientation in [8]. Assume there are ℓ arcs in one direction in a parity-linked oriented even cycle C_{2k}, then $h(C_{2k}) = x^\ell(-x)^{2k-\ell} = x^{2k}(-1)^{2k-\ell} = 1$ for $x \in \{\pm\mathbf{i}\}$. Thus, a parity-linked oriented even cycle C is actually positive. Conversely, if an oriented cycle C is positive in an oriented graph, we can see that C must be a parity-linked oriented even cycle. Therefore, the notion of a parity-linked oriented even cycle is equivalent to that of a positive oriented cycle in an oriented graph. And we call a parity-linked oriented graph as a positive oriented graph.

Let $G = G(X, Y)$ be a bipartite graph with bipartition (X, Y). The canonical orientation of G is that orientation which orients all the edges from one partite set to the other. It is immaterial if it is from X to Y or from Y to X. Since for any oriented even cycle $C_{2k} : v_1v_2v_3 \dots v_{2k}v_1$, we have $h(C_{2k}) = h_{v_1v_2}h_{v_2v_3}\dots h_{v_{2k-1}v_{2k}}h_{v_{2k}v_1} = \mathbf{i}^k(-\mathbf{i})^k = 1$, which means that the oriented graph is positive. Thus, we obtained a result of Shader and So [21] from Theorem 4.1 that

Theorem 5.1 *For the canonical orientation σ of $G = G(X, Y)$, $Sp_H(G^\sigma) = Sp_H(G)$.*

From this point onward, σ stands for the canonical orientation with respect to a bipartite graph G with a fixed bipartition (X, Y).

Let U be any proper subset of $V(G)$ of an oriented graph G^{γ_1} and $\overline{U} = V(G)\backslash U$ its complement. Reversing the orientations of all the oriented edges between U and $\overline{U}$ results in another oriented graph G^{γ_2}. This process is called the *switch* of G^{γ_1} with respect to U. The oriented graph got by two successive switches with respect to U_1 and U_2 is just the oriented graph obtained from G^{γ_1} by the switch with respect to the set $U_1 \bigtriangleup U_2$, the symmetric difference of U_1 and U_2. The next lemma shows that if an oriented graph G_1 is obtained from an oriented graph G_2 by a switch then $G_1 \sim G_2$ and vice versa.

Lemma 5.2 *Let G_1 and G_2 be two connected oriented graphs, then the following are equivalent:*
(1) $G_1 \sim G_2$.
(2) G_2 is obtained from G_1 by a switch.

Proof. If G_2 is obtained from G_1 by a switch with respect to a subset of $V(G_1)$, say U. Let $H(G_1)$ be the Hermitian-adjacency matrix of G_1 with respect to a labeling of its vertex set. If the cardinality of U is zero, then $H(G_2) = \mathbf{I}_n^* H(G_1)\mathbf{I}_n$ and the result follows. If the cardinality of U is one, say $U = \{v_1\}$. Suppose the orientations of all the arcs incident at vertex v_1 of G_1 are reversed. Let the resulting oriented graph be G_2. Then $H(G_2) = P_1^* H(G_1)P_1$ where P_1 is the diagonal matrix obtained from $\mathbf{i}\mathbf{I}_n$ by changing the first diagonal entry to $-\mathbf{i}$. Similarly, if the cardinality of U is k, say $U = \{v_1, \ldots, v_k\}$. Suppose P_U is the diagonal matrix obtained from $\mathbf{i}\mathbf{I}_n$ by changing the first k diagonal entries to $-\mathbf{i}$, then by replacing P_1 with P_U in the above proof the result of (1) follows.

Now assume (1) is true. If $G_1 \cong G_2$, the result is obvious. Now we assume $G_1 \ncong G_2$. Then there is a switching function $\theta : V \to \mathbf{T}$, where $\mathbf{T} = \{1, \pm\mathbf{i}\}$ and a diagonal matrix $D(\theta) := \mathrm{diag}(\theta(v_k) : v_k \in V)$ such that $H(G_2) = D(\theta)^{-1}H(G_1)D(\theta)$. I.e., $h_{k\ell}(G_2) = \theta(v_k)^{-1}h'_{k\ell}(G_1)\theta(v_\ell)$ for any k, ℓ. We have $h_{k\ell}(G_p) \in \{\pm\mathbf{i}\}$ for $p = 1, 2$, therefore either both $\theta(v_k)$ and $\theta(v_\ell)$ are equal to one or $\theta(v_k)$ and $\theta(v_\ell)$ are belong to $\{\pm\mathbf{i}\}$ for any k, ℓ. But the first case would come to the conclusion that $G_1 \cong G_2$, which is impossible. If the latter case holds. If $\theta(v_k) = \mathbf{i}$ for all k or $\theta(v_k) = -\mathbf{i}$ for all k, then $G_1 \cong G_2$, which is impossible too. Let $U = \{v|\theta(v) = -\mathbf{i}\}$, then from the proof of first part, we can see that G_2 can be obtained from G_1 by a switch with respect to U. ∎

Theorem 5.3 *Suppose γ is an orientation of a bipartite graph $G = G(X, Y)$. Then the following are equivalent:*
(1) $Sp_H(G^\gamma) = Sp_H(G)$;
(2) G^γ is positive;
(3) Every mixed chordless cycle of G^γ is positive;

(4) $G^\gamma \sim G^\sigma$, where σ is the canonical orientation of G;

(5) G^γ can be obtained from G^σ by a switch.

Proof. By Theorem 4.1 and Theorem 2.6 we have the equivalences of (2),(3) and (4). By Lemma 5.2, we have the equivalence of (4) and (5). We now only need to show the equivalence of (1) and (2). By Theorem 5.1, we see that G^σ is positive. W.O.L.G. assume the canonical orientation σ of G is the orientation which orients all the edges from X to Y. Let a diagonal matrix $D(\theta) := \mathrm{diag}(\theta(v_k)|\theta(v_k) = 1 \text{ if } v_k \in X; \theta(v_k) = \mathbf{i} \text{ if } v_k \in Y)$. Then $H(G^\sigma) = D(\theta)^{-1}H(G)D(\theta)$, i.e. $G \sim G^\sigma$. Thus if $G^\gamma \sim G^\sigma$, then $G^\gamma \sim G$. By Theorem 2.9, we can see that (2) implies (1). Now assume (1) holds. If (2) is not true. Then there exists a bipartite graph G and an orientation γ of G such that $Sp_H(G^\gamma) = Sp_H(G)$ and G^γ is not positive. Thus there exists a negative mixed cycle in G^γ. Since G is bipartite, let C^γ is a negative mixed cycle with least length $2k$ in G^γ. By Theorem 2.8,

$$c_{2k}(G^\gamma) - c_{2k}(G) = c\Big((-1)^{(2k-1)+1} - (-1)^{(2k-1)+0}\Big) \cdot 2^1 = 4c \neq 0,$$

where c is the number of negative mixed cycles with length $2k$ in G^γ. Then $Sp_H(G^\gamma) \neq Sp_H(G)$, which contradicts to our assumption. Therefore, (1) implies (2). ∎

Note that Cui and Hou in [8] first showed the equivalence of (1) and (2) for oriented graph. They conjectured that (1) and (5) are equivalent. Anuradha et al. [1] first gave an affirmative proof to the conjecture. They also showed that (3) implies (1).

6 Integral representation for Hermitian energy

We now present an integral representation for the Hermitian energy of a mixed graph which enables one to compute the Hermitian energy and compare the Hermitian energy between two mixed graphs without actually computing the eigenvalues. This generalizes the cases of energy of undirected graph [12] and skew energy of oriented graph [2].

Let G be a mixed graph and $\Phi(G;\lambda)$ the characteristic polynomial of $H(G)$, and let

$\lambda_1, \lambda_2, \ldots, \lambda_n$ be its zeros. Then

$$\Phi(G;\lambda) = \prod_{k=1}^{n}(\lambda - \lambda_k) \quad \text{and} \quad \Phi'(G;\lambda) = \sum_{k=1}^{n}\prod_{\substack{\ell=1\\ \ell\neq k}}^{n}(\lambda - \lambda_\ell),$$

from which follows

$$\mathbf{i}\lambda\frac{\Phi'(G;\mathbf{i}\lambda)}{\Phi(G;\mathbf{i}\lambda)} = \mathbf{i}\lambda\sum_{k=1}^{n}\frac{1}{\mathbf{i}\lambda - \lambda_k}.$$

We have

$$\begin{aligned}
|\lambda_k| = |\lambda_k| + \mathbf{i}0 &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\lambda_k^2}{\lambda_k^2+\lambda^2}d\lambda + \mathbf{i}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\lambda_k\lambda}{\lambda_k^2+\lambda^2}d\lambda\\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\lambda_k^2+\mathbf{i}\lambda_k\lambda}{\lambda_k^2+\lambda^2}d\lambda = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(1 - \frac{\mathbf{i}\lambda}{\mathbf{i}\lambda - \lambda_k}\right)d\lambda.
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{E}_H(G) = \sum_{k=1}^{n}|\lambda_k| &= \frac{1}{\pi}\int_{-\infty}^{\infty}\sum_{k=1}^{n}\left(1 - \frac{\mathbf{i}\lambda}{\mathbf{i}\lambda - \lambda_k}\right)d\lambda\\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(n - \mathbf{i}\lambda\frac{\Phi'(G;\mathbf{i}\lambda)}{\Phi(G;\mathbf{i}\lambda)}\right)d\lambda.
\end{aligned}$$

Example 6.1 *Let G be the mixed graph in Example 2.2. The characteristic polynomial of $H(G)$ is $\lambda^4 - 5\lambda^2 + 4$, hence*

$$\begin{aligned}
\mathcal{E}_H(G) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(4 - \frac{4\lambda^4 + 10\lambda^2}{\lambda^4 + 5\lambda^2 + 4}\right)d\lambda\\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{10\lambda^2 + 16}{\lambda^4 + 5\lambda^2 + 4}d\lambda\\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{8}{\lambda^2+4} + \frac{2}{\lambda^2+1}\right)d\lambda\\
&= 6,
\end{aligned}$$

as noted already.

Here we need a result of Mateljević et al. [17]:

Theorem 6.2 *[17] Let ϕ be a real polynomial of degree n with leading coefficient 1, let $\lambda_k, 1 \leq k \leq n$, be its zeros. Then*

$$\sum_{k=1}^{n}|Re\ \lambda_k| = \frac{1}{\pi}\int_{-\infty}^{\infty}\ln|\lambda^n\Phi(G;\frac{\mathbf{i}}{\lambda})|\frac{d\lambda}{\lambda^2} = \frac{2}{\pi}\int_{0}^{\infty}\ln|\lambda^n\Phi(G;\frac{\mathbf{i}}{\lambda})|\frac{d\lambda}{\lambda^2}.$$

Thus

$$\mathcal{E}_H(G) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\lambda^2}\ln|\lambda^n\Phi(G;\frac{\mathbf{i}}{\lambda})|d\lambda = \frac{2}{\pi}\int_{0}^{\infty}\frac{1}{\lambda^2}\ln|\lambda^n\Phi(G;\frac{\mathbf{i}}{\lambda})|d\lambda \tag{6.4}$$

If G is a mixed graph without real mixed odd cycles (or a mixed bipartite graph or an oriented graph) of order n, then by Theorem 2.10 (or Corollary 2.12 and Corollary 2.13), $\Phi(G;\lambda) = \sum_{k\geq 0}(-1)^k b_k(G)\lambda^{n-2k}$ and $b_k \geq 0$ for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$. Then the integral formula (6.4) is simplified as:

$$\mathcal{E}_H(G) = \frac{2}{\pi}\int_{0}^{\infty}\frac{1}{\lambda^2}\ln\left[1+\sum_{k\geq 1} b_k(G)\lambda^{2k}\right]d\lambda \tag{6.5}$$

Furthermore, if G is a mixed tree (or more generally, mixed forest), then

$$\Phi(G;\lambda) = \sum_{k\geq 0}(-1)^k m_k(G)\lambda^{n-2k}$$

and

$$\mathcal{E}_H(G) = \frac{2}{\pi}\int_{0}^{\infty}\frac{1}{\lambda^2}\ln\left[1+\sum_{k\geq 1} m_k(G)\lambda^{2k}\right]d\lambda$$

Consider now Eq. (6.5) and let G_1 and G_2 be two mixed graphs of order n without real mixed odd cycles. If inequalities

$$b_k(G_1) \leq b_k(G_2) \tag{6.6}$$

are satisfied for all values of k, then from Eq. (6.5), it follows that $\mathcal{E}_H(G_1) \leq \mathcal{E}_H(G_2)$. In addition, if at least one of these inequalities is strict, then $\mathcal{E}_H(G_1) < \mathcal{E}_H(G_2)$. Bearing this in mind we define a partial order $\prec$ and write $G_1 \preceq G_2$ or $G_2 \succeq G_1$ if the conditions (6.6) are obeyed for all k. Moreover, if at least one of the inequalities in (6.6) is strict, then we write $G_1 \prec G_2$ or $G_2 \succ G_1$. Then we have:

$$\begin{aligned} G_1 \preceq G_2 &\implies \mathcal{E}_H(G_1) \leq \mathcal{E}_H(G_2) \\ G_1 \prec G_2 &\implies \mathcal{E}_H(G_1) < \mathcal{E}_H(G_2) \end{aligned}$$

Therefore, if G_1 and G_2 are two mixed graphs without real mixed odd cycles, then

$$G_1 \preceq G_2 \iff (\forall k)\ b_k(G_1) \leq b_k(G_2).$$

Furthermore, if T_1 and T_2 are mixed trees of order n (or more generally, mixed forests), then

$$T_1 \preceq T_2 \iff (\forall k)\ m_k(T_1) \leq m_k(T_2).$$

Practically many results on undirected graphs or oriented graphs that are extremal with regard to energy or skew energy were obtained by establishing the existence of the relation $\preceq$ between the elements of some class of undirected graphs and oriented graphs (see surveys [11] and [15] and the references cited therein). Analogously, many results on mixed graphs that are extremal regard to Hermitian energy can be obtained too. For example, we have the following generalizations (corresponding results for energies of trees can be found in [11]):

Corollary 6.3 *For a mixed tree $\overrightarrow{T}$ of order n, we have*

$$\mathcal{E}_H(\overrightarrow{S_n}) \leq \mathcal{E}_H(\overrightarrow{T}) \leq \mathcal{E}_H(\overrightarrow{P_n}),$$

where $\overrightarrow{P_n}$ and $\overrightarrow{S_n}$ are mixed path and mixed star of order n, respectively, and the left equality holds if and only if $\overrightarrow{T}$ is a mixed star and the right equality holds if and only if $\overrightarrow{T}$ is a mixed path.

Corollary 6.4 *For a mixed tree $\overrightarrow{T}$ of order n with diameter at least d, we have*

$$\mathcal{E}_H(\overrightarrow{T}) \geq \mathcal{E}_H(\overrightarrow{B}_{n,d}),$$

where $\overrightarrow{B}_{n,d}$ is a mixed comet of order n such that its underlying graph a comet $B_{n,d}$ that is obtained from the path P_d by attaching $n-d$ pendent edges to one end vertex of P_d, and the equality holds if and only if $\overrightarrow{T}_U \cong B_{n,d}$.

Corollary 6.5 *For a mixed tree $\overrightarrow{T}$ of order n with ℓ leaves, we have*

$$\mathcal{E}_H(\overrightarrow{T}) \geq \mathcal{E}_H(\overrightarrow{B}_{n,n-\ell+1}),$$

where the equality holds if and only if $\overrightarrow{T}_U \cong B_{n,n-\ell+1}$.

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