Rainbow Connection Number and Connectivity

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Abstract

The rainbow connection number, $rc(G)$, of a connected graph G is the minimum number of colors needed to color its edges, so that every pair of vertices is connected by at least one path in which no two edges are colored the same. Our main result is that $rc(G) \leq \lceil \frac{n}{2} \rceil$ for any 2-connected graph with at least three vertices. We conjecture that $rc(G) \leq n/\kappa + C$ for a κ-connected graph G of order n, where C is a constant, and prove the conjecture for certain classes of graphs. We also prove that $rc(G) \leq (2+\varepsilon)n/\kappa + 23/\varepsilon^2$ for any $\varepsilon > 0$.

Keywords: rainbow coloring, rainbow connection number, connectivity, 2-connected graph, ear decomposition, chordal graph, girth

1 Introduction

An *edge coloring* of a graph is a function from its edge set to the set of natural numbers. A path in an edge colored graph with no two edges sharing the same color is called a *rainbow path*. An edge colored graph is said to be *rainbow connected* if every pair of vertices is connected by at least one rainbow path. Such a coloring is called a *rainbow coloring* of the graph. If a rainbow coloring uses k colors, we call it a k-rainbow coloring. The minimum number of colors required to rainbow color a connected graph is called

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its *rainbow connection number*, denoted by $rc(G)$. For example, the rainbow connection number of a complete graph is 1, that of a path is its length, and that of a star is its number of leaves. For a basic introduction to the topic, see Chapter 11 in [7].

The concept of rainbow coloring was introduced by Chartrand, Johns, McKeon and Zhang [6] in 2008. Chakraborty et al. [4] showed that computing the rainbow connection number of a graph is NP-Hard. To rainbow color a graph, it is enough to ensure that every edge of some spanning tree in the graph gets a distinct color. Hence, the order of the graph minus one is an upper bound for the rainbow connection number. Many authors [3, 4, 10] view rainbow connection number as a "quantifiable" way of strengthening the connectivity property of a graph. Hence, tighter upper bounds on the rainbow connection number for a graph with higher connectivity have been a subject of investigation.

The following are the results in this direction reported in literature: Let G be a graph of order n. Caro et al. [3] showed that if G is 2-edge-connected (bridgeless), then $rc(G) \leq 4n/5 - 1$ and if G is 2-connected, then $rc(G) \leq \min\{2n/3, n/2 + O(\sqrt{n})\}$. Li and Shi [11] showed that if G is 3-connected, then $rc(G) \leq 3(n+1)/5$. Krivelevich and Yuster [10] showed that $rc(G) \leq 20n/\delta$, where δ is the minimum degree of G. The result was improved by Chandran et al. [5] where it was shown that $rc(G) \leq 3n/(\delta + 1) + 3$. Hence it follows that $rc(G) \leq 3n/(\lambda+1)+3$ if G is λ-edge-connected and $rc(G) \leq 3n/(\kappa+1)+3$ if G is κ-connected. This is because $\kappa \leq \lambda \leq \delta$.

The main result of our paper is that for any 2-connected graph of order n, $rc(G) \leq \lceil n/2 \rceil$, and the bound is trivially attained by cycles of order at least 4 (Theorem 2.4). This improves the previous best known upper bounds for 2-connected and 3-connected graphs [3, 11] mentioned in last paragraph. We show that the bound of $3n/(\lambda + 1) + 3$ in terms of the edge-connectivity is tight up to additive constants for infinitely many values of λ and n (Example 3.1). We improve the bound for κ-connected graphs to $rc(G) \leq (2 + \epsilon)n/\kappa + 23/\epsilon^2$ for any $\epsilon > 0$ (Theorem 3.6). We conjecture (Conjecture 3.7) that for a κ-connected graph G, $rc(G) \leq n/\kappa + C$ where C is a constant. For $\kappa \geq 3$, we show that the conjecture is true for chordal graphs (Theorem 3.11) and graphs of girth at least 7 (Theorem 3.10). It can be easily shown from existing literature that the conjecture is true for all κ for some other graph classes like AT-free graphs and circular arc graphs too [5]. We remark that an upper bound of $n/\kappa + C$ will be tight up to additive factors.

All graphs considered in this article are finite, simple and undirected. The *length* of a path is its number of edges. If S is a subset of vertices of a graph G, the subgraph of G induced by the vertices in S is denoted by $G[S]$. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The order of G (number of vertices) may be denoted by $|G|$.

2 Result for 2-connected graphs

At first we study the rainbow connection number of 2-connected graphs. As usual, the term κ-vertex-connected will be simply addressed as κ-connected. The following notation and terminology are needed in the sequel.

Definition 2.1. Let F be a subgraph of a graph G. An *ear* of F in G is a nontrivial path in G whose endpoints are in F but whose internal vertices are not. A nested sequence of graphs is a sequence $(G_0, G_1, \cdots, G_k)$ of graphs such that $G_i \subset G_{i+1}, 0 \leq i < k$. An *ear decomposition* of a 2-connected graph G is a nested sequence $(G_0, G_1, \cdots, G_k)$ of 2-connected subgraphs of G such that: (1) G_0 is a cycle; (2) $G_i = G_{i-1} \bigcup P_i$, where P_i is an ear of G_{i-1} in $G, 1 \leq i \leq k$; (3) $G_k = G$.

The next notions are new ones which will play key roles in our proofs.

Definition 2.2. Let c be a k-rainbow coloring of a connected graph G. If a rainbow path P in G has length k, we call P *a complete rainbow path*; otherwise, it is *an incomplete rainbow path*. A rainbow coloring c of G is *incomplete* if for any vertex $u \in V(G)$, G has at most one vertex v such that all the rainbow paths between u and v are complete; otherwise, it is *complete*.

A complete rainbow path uses all colors of the coloring, while an incomplete rainbow path misses at least one color of the coloring.

For a connected graph G, if a spanning subgraph has an (incomplete) k-rainbow coloring, then G has an (incomplete) k-rainbow coloring. This simple fact will be used in the following proofs.

Lemma 2.1. *Let G be a Hamiltonian graph of order n $(n \geq 3)$. Then G has an incomplete $\lceil \frac{n}{2} \rceil$-rainbow coloring, i.e., $rc(G) \leq \lceil \frac{n}{2} \rceil$.*

Proof. Since G is a Hamiltonian graph, there is a Hamiltonian cycle $C_n = v_1, v_2, \cdots, v_n, v_{n+1}$ $(= v_1)$ in G. Define an edge-coloring c of C_n by $c(v_i v_{i+1}) = i$ for $1 \leq i \leq \lceil \frac{n}{2} \rceil$ and $c(v_i v_{i+1}) = i - \lceil \frac{n}{2} \rceil$ if $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$. It is clear that c is a $\lceil \frac{n}{2} \rceil$-rainbow coloring of C_n, and the shortest path connecting any two vertices in $V(G)$ on C_n is a rainbow path. For any vertex v_i $(1 \leq i \leq n)$, only the antipodal vertex of v_i has no incomplete rainbow path to v_i if n is even. Every pair of vertices in G has an incomplete rainbow path if n is odd. Hence the rainbow coloring c of C_n is incomplete. Since C_n is a spanning subgraph of G, G has an incomplete $\lceil \frac{n}{2} \rceil$-rainbow coloring. □

Let G be a 2-connected non-Hamiltonian graph of order n $(n \geq 4)$. Then G must have an even cycle. In fact, since G is 2-connected, G must have a cycle C. If C is an even cycle, we are done. Otherwise, C is a odd cycle, we then choose an ear P of C such that $V(C) \bigcap V(P) = \{a, b\}$. Since the lengths of the two segments between a, b on C have different parities, P joining with one of the two segments forms an even cycle. Then, starting from an even cycle G_0, there exists a nonincreasing ear decomposition $(G_0, G_1, \cdots, G_t, G_{t+1}, \cdots, G_k)$ of G, such that $G_i = G_{i-1} \bigcup P_i$ $(1 \leq i \leq k)$ and P_i is a longest ear of G_{i-1}, i.e., $\ell(P_1) \geq \ell(P_2) \geq \cdots \geq \ell(P_k)$. Suppose that $V(P_i) \bigcap V(G_{i-1}) = \{a_i, b_i\}$ $(1 \leq i \leq k)$. We call the distinct vertices a_i, b_i $(1 \leq i \leq k)$ the *endpoints* of the ear P_i. Without loss of generality, suppose that $\ell(P_t) \geq 2$ and $\ell(P_{t+1}) = \cdots = \ell(P_k) = 1$. So G_t is a

2-connected spanning subgraph of G. Since G is a non-Hamiltonian graph, we have $t \geq 1$. Denote the order of G_i $(0 \leq i \leq k)$ by n_i. All these notations will be used in the sequel.

Lemma 2.2. *Let G be a 2-connected non-Hamiltonian graph of order n $(n \geq 4)$. If G has at most one ear with length 2 in the nonincreasing ear decomposition, then G has a incomplete $\lceil \frac{n}{2} \rceil$-rainbow coloring, i.e., $rc(G) \leq \lceil \frac{n}{2} \rceil$.*

Proof. Since G_t $(t \geq 1)$ in the nonincreasing ear decomposition is a 2-connected spanning subgraph of G, it only needs to show that G_t has an incomplete $\lceil \frac{n}{2} \rceil$-rainbow coloring. We will apply induction on t.

First, consider the case that $t = 1$. Let G be a 2-connected non-Hamiltonian graph with $t = 1$ in the nonincreasing ear decomposition. The spanning subgraph $G_1 = G_0 \bigcup P_1$ of G consists of an even cycle G_0 and an ear P_1 of G_0. Without loss of generality, suppose that $G_0 = v_1, v_2, \cdots, v_{2k}, v_{2k+1} (= v_1)$ where $k \geq 2$. We color the edges of G_0 with k colors. Define the edge-coloring c_0 of G_0 by $c_0(v_i v_{i+1}) = i$ for $1 \leq i \leq k$ and $c_0(v_i v_{i+1}) = i - k$ if $k+1 \leq i \leq 2k$. From the proof of Lemma 2.1, the coloring c_0 is an incomplete k-rainbow coloring of G_0. Now consider G_1 according to the parity of $\ell(P_1)$. If $\ell(P_1)$ is even, then n_1 is odd and color the edges of P_1 with $\frac{\ell(P_1)}{2}$ new colors. In the first $\frac{\ell(P_1)}{2}$ edges of P_1 the colors are all distinct, and the same ordering of colors is repeated in the last $\frac{\ell(P_1)}{2}$ edges of P_1. It is easy to verify that the obtained coloring c_1 of G_1 is an incomplete $\lceil \frac{n_1}{2} \rceil$-rainbow coloring such that for any pair of vertices in G, there exists an incomplete rainbow path connecting them. If $\ell(P_1)$ is odd, then n_1 is even and color the edges of P_1 with $\frac{\ell(P_1)-1}{2}$ new colors. The middle edge of P_1 receives any color that already appeared in G_0. The first $\frac{\ell(P_1)-1}{2}$ edges of P_1 all receive distinct new colors and in the last $\frac{\ell(P_1)-1}{2}$ edges of P_1 this coloring is repeated in the same order. It is easy to verify that the obtained coloring c_1 of G_1 is an incomplete $\lceil \frac{n}{2} \rceil$-rainbow coloring.

Let G be a 2-connected non-Hamiltonian graph with $t \geq 2$ in the nonincreasing ear decomposition. Assume that the subgraph G_i $(1 \leq i \leq t-1)$ has an incomplete $\lceil \frac{n_i}{2} \rceil$-rainbow coloring c_i such that when n_i is odd, any pair of vertices have an incomplete rainbow path. We distinguish the following three cases.

Case 1. $\ell(P_t)$ (≥ 3) is odd.

Suppose that $P_t = v_0(= a_t), v_1, \cdots, v_r, v_{r+1}, \cdots, v_{2r}, v_{2r+1}(= b_t)$, where $r \geq 1$. We color the edges of P_t with r new colors to obtain an incomplete coloring c_t of G_t. Define an edge-coloring of P_t by $c(v_{i-1}v_i) = x_i$ $(1 \leq i \leq r)$, $c(v_r v_{r+1}) = x$ and $c(v_{i-1}v_i) = x_{i-r-1}$ $(r+2 \leq i \leq 2r+1)$, where $x_1, x_2, \cdots, x_r$ are new colors and x is a color appearing in G_{t-1}. It is easy to check that the obtained coloring c_t of G_t is a $\lceil \frac{n}{2} \rceil$-rainbow coloring.

Now we show that c_t is incomplete such that when n_t is odd, any pair of vertices have an incomplete rainbow path. For any pair of vertices $u, v \in V(G_{t-1}) \times V(G_{t-1})$, the rainbow path P from u to v in G_{t-1} is incomplete in G_t, because the new colors $x_1, x_2, \cdots, x_r$ $(r \geq 1)$ do not appear in P. For any pair of vertices $u, v \in V(P_t) \times V(P_t)$, if there exists a rainbow path P from u to v on P_t, then P is incomplete in G_t, since some color in G_{t-1} does not appear in P; if not, there exists an incomplete rainbow path P from u to v through some vertices of G_{t-1} such that at least one new color does not appear in P. For any pair of vertices $u, v \in V(G_{t-1}) \times (V(P_t)\backslash\{v_r, v_{r+1}\})$, there exists an incomplete rainbow path from u to v in which at least one new color does not appear. If there exists a vertex all of whose rainbow paths to a_t (resp. b_t) in G_{t-1} are complete, we denote the vertex by a'_t (resp. b'_t). For vertex v_r (resp. v_{r+1}), only the vertex a'_t (resp. b'_t) possibly has no incomplete rainbow path to v_r (resp. v_{r+1}) in G_t. So there possibly exist two pairs of vertices a'_t, v_r and b'_t, v_{r+1} which have no incomplete rainbow path. Since a'_t, b'_t are distinct in G_{t-1}, the rainbow coloring c_t is incomplete. If n is odd, then n_{t-1} is odd. By induction, a'_t, b'_t do not exist when n_{t-1} is odd. Hence every pair of vertices have a incomplete rainbow path.

Case 2. $\ell(P_t)$ (≥ 2) is even and n_{t-1} is even.

In this case, n is odd. Suppose that $P_t = v_0(= a_t), v_1, \cdots, v_r, v_{r+1}, \cdots, v_{2r-1}, v_{2r}(= b_t)$, where $r \geq 1$. Define an edge-coloring of P_t by $c(v_{i-1}v_i) = x_i$ for $1 \leq i \leq r$ and $c(v_{i-1}v_i) = x_{i-r}$ for $r+1 \leq i \leq 2r$. It is clear that the obtained coloring c_t of G_t is a $\lceil \frac{n}{2} \rceil$-rainbow coloring.

Now we prove that c_t is incomplete such that when n_t is odd, any pair of vertices have an incomplete rainbow path. For any pair of vertices in $V(G_{t-1}) \times V(G_{t-1})$ or $V(P_t) \times V(P_t)$, there is an incomplete rainbow path connecting them in G_t, similar to the Case 1. For any pair of vertices $u \in V(G_{t-1}), v \in V(P_t)$ $(v \neq v_r)$, there is an incomplete rainbow path P from u to v such that at least one new color does not appear in P. For any vertex $u \in V(G_{t-1})$, since the coloring c_{t-1} is incomplete, u has an incomplete rainbow path P' in G_{t-1} to one of a_t, b_t (say a_t). Then P' joining with $a_t P_t v_r$ is an incomplete rainbow path from u to v_r in G_t. Therefore, the rainbow coloring c_t of G_t is incomplete such that any pair of vertices has an incomplete rainbow path.

Case 3. $\ell(P_t)$ (≥ 2) is even and n_{t-1} is odd.

In this case, n is even. We consider the following three subcases.

Subcase 3.1. $[V(P_t) \bigcap V(P_{t-1})] \backslash V(G_{t-2}) = \emptyset$.

If $\ell(P_{t-1})$ is odd, let $G'_{t-1} = G_{t-2} \bigcup P_t$ and $G_t = G'_{t-1} \bigcup P_{t-1}$. By induction, G'_{t-1} has an incomplete $\lceil \frac{n'_{t-1}}{2} \rceil$-rainbow coloring ($n'_{t-1}$ is the order of G'_{t-1}). Similar to Case 1, we can obtain an incomplete $\lceil \frac{n}{2} \rceil$-rainbow coloring of G_t from G'_{t-1} .

Suppose that $\ell(P_{t-1})$ is even. By induction, G_{t-2} has an incomplete $\lceil \frac{n_{t-2}}{2} \rceil$-rainbow coloring c_{t-2}. Suppose that $P_{t-1} = v_0(= a_{t-1}), v_1, \cdots, v_r, v_{r+1}, \cdots, v_{2r-1}, v_{2r}(= b_{t-1})$ and $P_t = v'_0(= a_t), v'_1, \cdots, v'_s, v'_{s+1}, \cdots, v'_{2s-1}, v'_{2s}(= b_t)$, where $r \geq 2, s \geq 1$. Since c_{t-2} is incomplete and a_i, b_i $(1 \leq i \leq k)$ are two distinct vertices, then a_{t-1} has an incomplete rainbow path P' to one of a_t, b_t (say a_t) and b_{t-1} has an incomplete rainbow path P'' to the other vertex. Suppose that x is the color in G_{t-2} that does not appear in P'. Now color the edges of P_{t-1}, P_t with $r+s-1$ new colors and the color x. Define an edge-coloring of P_{t-1} by $c(v_{i-1}v_i) = x_i$ $(1 \leq i \leq r)$ and $c(v_{i-1}v_i) = x_{i-r}$ $(r+1 \leq i \leq 2r)$, where $x_1, x_2, \cdots, x_r$ are new colors. Define an edge-coloring of P_t by $c(v'_{i-1}v'_i) = y_i$ $(1 \leq i \leq s-1), c(v'_{s-1}v'_s) = x, c(v'_s v'_{s+1}) = x_1$ and $c(v'_{i-1}v'_i) = y_{i-s-1}$ $(s+2 \leq i \leq 2s)$, where $y_1, y_2, \cdots, y_{s-1}$ are new colors.

Similar to Case 2, the obtained coloring c_{t-1} of G_{t-1} is an incomplete $\lceil \frac{n_{t-1}}{2} \rceil$-rainbow coloring such that every pair of vertices have an incomplete rainbow path. It is obvious that G_t is rainbow connected. The path $(v'_s P_t a_t)P'(a_{t-1}P_{t-1}v_r)$ is a rainbow path from v'_s to v_r which is possibly complete. For any other pair of vertices in G_t, there is an incomplete rainbow path connecting them. Hence the rainbow coloring c_t of G_t is incomplete.

Subcase 3.2. $[V(P_t)\bigcap V(P_{t-1})]\backslash V(G_{t-2}) = \{b_t\}$.

If $\ell(P_{t-1})$ is odd, suppose that $P_{t-1} = v_0(= a_{t-1}), v_1, \cdots, v_r, v_{r+1}, \cdots, v_{2r}, v_{2r+1}(= b_{t-1})$. Since P_{t-1} is a longest ear of G_{t-2} and $b_t \in V(P_{t-1}) \setminus V(G_{t-2})$, we have $r \geq 2$. Define an edge-coloring of P_{t-1} by $c(v_{i-1}v_i) = x_i$ $(1 \leq i \leq r)$, $c(v_r v_{r+1}) = x$ and $c(v_{i-1}v_i) = x_{i-r-1}$ $(r+2 \leq i \leq 2r+1)$, where $x_1, x_2, \cdots, x_r$ are new colors and x is a color appearing in G_{t-2}. Similar to Case 1, the obtained coloring c_{t-1} of G_{t-1} is an incomplete $\lceil \frac{n_{t-1}}{2} \rceil$-rainbow coloring such that every pair of vertices have an incomplete rainbow path. If $\ell(P_{t-1})$ is even, suppose that $P_{t-1} = v_0(= a_{t-1}), v_1, \cdots, v_r, v_{r+1}, \cdots, v_{2r-1}, v_{2r}(= b_{t-1})$, where $r \geq 2$. Define an edge-coloring of P_{t-1} by $c(v_{i-1}v_i) = x_i$ $(1 \leq i \leq r)$, and $c(v_{i-1}v_i) = x_{i-r}$ $(r + 1 \leq i \leq 2r)$, where $x_1, x_2, \cdots, x_r$ are new colors. Similar to Case 2, the obtained coloring c_{t-1} of G_{t-1} is an incomplete $\lceil \frac{n_{t-1}}{2} \rceil$-rainbow coloring such that every pair of vertices have an incomplete rainbow path.

Without loss of generality, assume that b_t belongs to the first half of P_{t-1} and that $P_t = v'_0(= a_t), v'_1, \cdots, v'_s, v'_{s+1}, \cdots, v'_{2s-1}, v'_{2s}(= b_t)$, where $s \geq 1$. We color the edges of P_t with $s - 1$ new colors. Define an edge-coloring of P_t by $c(v'_{i-1}v'_i) = y_i$ $(1 \leq i \leq s - 1)$, $c(v'_{s-1}v'_s) = x_1, c(v'_s v'_{s+1}) = y$ and $c(v'_{i-1}v'_i) = y_{i-s-1}$ $(s+2 \leq i \leq 2s)$, where $y_1, y_2, \cdots, y_{s-1}$ are new colors, y is a color in G_{t-2} and $x \neq y$. It is easy to verify that the obtained coloring c_t of G_t is a $\lceil \frac{n}{2} \rceil$-rainbow coloring.

For any pair of vertices $v' \in V(P_t)(v' \neq v'_s)$ and $v \in V(G_{t-1})$, there exists an incomplete rainbow path P connecting them, since the path from v' to the nearest endpoint of P_t joining with the incomplete rainbow path from the endpoint to v in $V(G_{t-1})$ is an incomplete rainbow path from v' to v in G_t. For v'_s, there is an incomplete rainbow path from v'_s to any vertex in $V(G_{t-2})\bigcup V(b_{t-1}P_{t-1}v_{r+2})$ through edge $e = v'_{s-1}v'_s$, and an incomplete rainbow path from v'_s to any vertex in $V(a_{t-1}P_{t-1}v_{r+1})$ through edge $e = v'_s v'_{s+1}$. For any pair of vertices in $V(P_t) \times V(P_t)$, there is an incomplete rainbow path connecting them. Hence the rainbow coloring c_t is incomplete.

Subcase 3.3. $[V(P_t)\bigcap V(P_{t-1})]\backslash V(G_{t-2}) = \{a_t, b_t\}$.

We can prove this subcase in a way similar to Subcase 3.2. Without loss of generality, we can assume that $a_t = v_p(1 \leq p \leq r - 1)$ and $b_t = v_q(q \geq p + 2)$. Color all the edges of P_{t-1} and P_t as in Subcase 3.2 but only the edge $e = v'_{s-1}v'_s$ which is colored by x_{p+1} in P_{t-1} instead. The obtained coloring c_t of G_t is an incomplete $\lceil \frac{n}{2} \rceil$-rainbow coloring. □

Lemma 2.3. *Let G be a 2-connected non-Hamiltonian graph of order n $(n \geq 4)$. If G has at least 2 ears of length 2 in the nonincreaing ear decomposition, then $rc(G) \leq \lceil \frac{n}{2} \rceil$.*

Proof. We only need to prove that there exists a rainbow coloring c_t of the spanning subgraph G_t in the nonincreasing ear decomposition that uses at most $\lceil \frac{n}{2} \rceil$ colors. If G has 2 or 3 ears of length 2 in the nonincreaing ear decomposition, then G_{t-2} has at most one ear of length 2 and $\ell(P_{t-1}) = \ell(P_t) = 2$. From Lemmas 2.1 and 2.2, G_{t-2}

has an incomplete $\lceil \frac{n_{t-2}}{2} \rceil$-rainbow coloring c_{t-2}. Assume that $P_{t-1} = a_{t-1}, v, b_{t-1}$ and $P_t = a_t, v', b_t$. Since P_{t-1} is a longest ear of G_{t-2}, we have $a_t, b_t \in V(G_{t-2})$. Since the coloring c_{t-2} is incomplete, a_{t-1} has an incomplete rainbow path P to one of a_t, b_t (say a_t) such that the color x in G_{t-2} does not appear in P. Define an edge-coloring of P_{t-1} and P_t by $c(a_{t-1}v) = c(b_{t-1}v) = c(b_t v') = x_1$ and $c(a_t v') = x$, where x_1 is a new color. It is clear that $va_{t-1}Pa_tv'$ is a rainbow path from v to v', and the obtained coloring of G_t is a $\lceil \frac{n}{2} \rceil$-rainbow coloring.

Now consider the case that G has at least 4 ears of length 2 in the nonincreaing ear decomposition. Suppose that $\ell(P_{t'-1}) \geq 3$ and $\ell(P_{t'}) = \ell(P_{t'+1}) = \cdots = \ell(P_t) = 2$. Since $P_i (1 \leq i \leq k)$ is a longest ear of G_{i-1}, we have that $a_{t'}, b_{t'}, \cdots, a_t, b_t \in V(G_{t'-1})$, i.e., $V(G_{t'-1})$ is a connected 1-step dominating set. From Lemmas 2.1 and 2.2, there exists a $\lceil \frac{n_{t'-1}}{2} \rceil$-rainbow coloring $c_{t'-1}$ of $G_{t'-1}$. Color one edge of $P_i (t' \leq i \leq t)$ with x_1 and the other with x_2, where x_1, x_2 are two new colors. It is obvious that G_t is rainbow connected. Since G has at least 4 ears of length 2, the obtained rainbow coloring of G_t uses at most $\lceil \frac{n}{2} \rceil$ colors. □

From the above three lemmas and the fact that $rc(C_n) = \lceil \frac{n}{2} \rceil$ $(n \geq 4)$, we can derive our following main result.

Theorem 2.4. *Let G be a 2-connected graph of order n $(n \geq 3)$. Then $rc(G) \leq \lceil \frac{n}{2} \rceil$, and the upper bound is tight for $n \geq 4$.*

Since for any two distinct vertices in a κ-connected graph G of order n, there exist at least κ internal disjoint paths connecting them, the diameter of G is no more than $\lfloor \frac{n}{\kappa} \rfloor$. One could think of generalizing Theorem 2.4 to the case of higher connectivity. Therefore we conjecture that for any κ-connected graph G, $rc(G) \leq \lceil \frac{n}{\kappa} \rceil$.

3 Results for graphs with higher connectivity

Now we will deal with graphs with higher (edge-) connectivity. At first, we settle the question of a tight upper bound for rainbow connection number in terms of edge-connectivity by showing that the bound of $3n/(\lambda + 1) + 3$, which directly follows from the minimum degree bound of $3n/(\delta + 1) + 3$ [5], is tight up to additive factors. We show the tightness by constructing a family of λ-edge-connected graphs for infinitely many values of λ and order n with diameter $d = \frac{3n}{\lambda+1} - 3$. Since diameter is a lower bound on the rainbow connection number, the construction suffices for our purpose.

Example 3.1 (Construction of a λ-edge-connected graph G on n vertices with diameter $d = \frac{3n}{\lambda+1} - 3$)**.** *Let $d \geq 1$ be a natural number, and λ a natural number such that $\lambda + 1$ is a multiple of 3 and $\lambda \geq 8$. Set $k := \frac{\lambda+1}{3}$, and set $V(G) = V_0 \uplus V_1 \uplus \cdots \uplus V_d$, where $|V_i|$ is $2k$ for $i = 0$ and $i = d$, and k for $1 \leq i < d$. Two distinct vertices $u \in V_i$ and $v \in V_j$ are adjacent in G if and only if $|i - j| \leq 1$. It is easy to see that the diameter of G is d, $n = |V(G)| = k(d + 3)$ and hence $d = \frac{n}{k} - 3 = \frac{3n}{\lambda+1} - 3$. By considering all pairs of vertices, it can be seen that G is λ-edge-connected.*

Next, we try to obtain an upper bound on the rainbow connection number for κ-connected graphs that is tighter than the $3n/(\kappa+1)$ bound implied by the degree bound in [5]. We will show that for any κ-connected graph G, $rc(G) \le (2+\epsilon)n/\kappa + 23/\epsilon^2$ for any $\epsilon > 0$. The following notation and terminology are needed.

Definition 3.1. Given a graph G, a set $D \subseteq V(G)$ is called an *ℓ-step dominating set* of G, if every vertex in G is at a distance at most ℓ from D. Furthermore, if $G[D]$ is connected, then D is called a *connected ℓ-step dominating set* of G.

From [1] we have the following lemma.

Lemma 3.2 ([1]). *If G is a bridgeless graph, then for every connected ℓ-step dominating set D^ℓ of G, $\ell \ge 1$, there exists a connected $(\ell-1)$-step dominating set $D^{\ell-1} \supset D^\ell$ such that*

$$rc(G[D^{\ell-1}]) \le rc(G[D^\ell]) + 2\ell + 1.$$

The following three lemmas are used to prove our theorem.

Lemma 3.3. *If G is a bridgeless graph, and D^ℓ is a connected ℓ-step dominating set of G, then*

$$rc(G) \le rc(G[D^\ell]) + \ell(\ell+2) \le |D^\ell| - 1 + \ell(\ell+2).$$

Proof. Note that the only 0-step dominating set in G is $V(G)$. Hence the first inequality follows from repeated application of Lemma 3.2. The second inequality follows since $rc(G[D^\ell]) \le |D^\ell| - 1$. □

Lemma 3.4. *Every κ-connected ($\kappa \ge 1$) graph G of order n has a connected 2ℓ-step dominating set of size at most $\left(\frac{2\ell+1}{\kappa\ell+1}\right)n$ for every natural number $\ell \ge 0$.*

Proof. If $k \le 2$, the bound is trivial for any $\ell \ge 0$ since we can take $V(G)$ as the dominating set. Similarly, if r is the radius of G, for $\ell \ge r/2$ we can take any central vertex of G as the 2ℓ-step dominating set. Hence we assume $\kappa > 2$ and $\ell < r/2$.

The following procedure is used to construct a 2ℓ-step dominating set D. Let $N^i(S) := \{x : d_G(x,S) = i\}$ and $\overline{N^i}(S) := \{x : d_G(x,S) \le i\}$ for any $S \subset V(G)$. $N^i(s) = N^i(\{s\})$ and $\overline{N^i}(s) = \overline{N^i}(\{s\})$ for any $s \in V(G)$.

$D = \{u\}$, for some $u \in V(G)$.
While $N^{2\ell+1}(D) \ne \emptyset$,
{
Pick any $v \in N^{2\ell+1}(D)$. Let $(v, x_{2\ell}, x_{2\ell-1}, \ldots, x_0)$, $x_0 \in D$ be a shortest v–D path.
D *is*$D \cup \{x_1, x_2, \ldots, x_{2\ell}, v\}$.
}

Clearly, D remains connected after every iteration of the procedure. Since the procedure ends only when $N^{2\ell+1}(D) = \emptyset$, the final D is a 2ℓ-step dominating set. Let t be the number of iterations executed by the procedure. From Menger Theorem (see [2] for example) we know that when the procedure starts $|\overline{N^\ell}(D)| = |\overline{N^\ell}(u)| \geq \kappa\ell + 1$. This is because $\ell < r$ and $|N^i(u)| \geq \kappa$ for $1 \leq i < r$ since G is κ-connected. Note that $v \in N^{2\ell+1}(D)$ ensures that $\overline{N^\ell}(v) \cap \overline{N^\ell}(D) = \emptyset$, and $|\overline{N^\ell}(v)| \geq \kappa\ell+1$ due to κ-connectivity of G. Hence the addition of v to D increases $|\overline{N^\ell}(D)|$ by at least $\kappa\ell+1$ in every iteration. Therefore, when the procedure ends, $(\kappa\ell + 1)(t + 1) \leq |\overline{N^\ell}(D)| \leq n$. Since D starts as a singleton set and each iteration adds $2\ell+1$ more vertices, $|D| = (2\ell+1)t+1 \leq \frac{(2\ell+1)n}{\kappa\ell+1} - 2\ell \leq \left(\frac{2\ell+1}{\kappa\ell+1}\right)n$. □

Lemma 3.5. *If G is a κ-connected ($\kappa \geq 1$) graph of order n, then for every natural number $\ell \geq 0$,*

$$rc(G) \leq \left(\frac{2\ell+1}{\kappa\ell+1}\right)n + 2\ell(2\ell+2) - 1.$$

Proof. The case $\kappa = 1$ is trivial. Hence we assume $\kappa \geq 2$ and therefore G is bridgeless. Since G is κ-connected, by Lemma 3.4, for every $\ell \geq 0$ we have a 2ℓ-step dominating set D of size at most $\left(\frac{2\ell+1}{\kappa\ell+1}\right)n$. Now an application of Lemma 3.3 gives the result. □

Theorem 3.6. *For every $\kappa \geq 1$, if G is a κ-connected graph of order n, then for every $\epsilon \in (0, 1)$,*

$$rc(G) \leq \left(\frac{2+\epsilon}{\kappa}\right)n + \frac{23}{\epsilon^2}.$$

Proof. Given an $\epsilon \in (0, 1)$, choose $\ell = \lceil \frac{1}{\epsilon} \rceil$. Then the result follows from Lemma 3.5. Note that $2\ell(2\ell + 2) - 1 \leq 23/\epsilon^2$. □

The above bound may not be tight, and we are tempted to believe that the following conjecture might be true.

Conjecture 3.7. *For every $\kappa \geq 1$, if G is a κ-connected graph of order n, then $rc(G) \leq n/\kappa + C$, where C is a constant.*

Now we show some cases in which Conjecture 3.7 is true, namely high girth graphs (where the *girth* of a graph is the size of a shortest cycle in the graph, denoted by $girth(G)$), chordal graphs (where a graph is called *chordal* if it contains no induced cycles of length greater than 3). We mainly consider κ-connected graphs with $\kappa \geq 3$, since for 2-connected graphs we have shown in above section that the conjecture is true, and the upper bound is tight.

Lemma 3.8. *Every connected graph G of order n, minimum degree $\delta \geq 3$ and girth at least $2g + 1$ has a connected $2g$-step dominating set of size at most $\left(\frac{2g+1}{C_{\delta,g}}\right)n - 2g$, where $C_{\delta,g} = \frac{\delta(\delta-1)^g-2}{\delta-2}$.*

Proof. Note that $\delta \geq 3$ ensures that G is not a tree and hence the girth is finite, that is, $1 \leq g < \infty$. Now observe that for any vertex $v \in V(G)$, $|\overline{N^g}(\{v\})| \geq 1 + \delta + \delta(\delta - 1) + \ldots + \delta(\delta - 1)^{g-1}$. The summation on the right hand side is equal to $C_{\delta,g}$, for $\delta \geq 3$. Now the proof follows the same steps as in that of Lemma 3.4 after setting $\ell = g$. Hence we omit the details. $\square$

Lemma 3.9. *Let G be a connected graph of minimum degree δ. Then,*

1. *if $\delta \geq 3$ and $girth(G) \geq 7$, then $rc(G) < n/\delta + 41$ and*
2. *if $\delta \geq 5$ and $girth(G) \geq 5$, then $rc(G) < n/\delta + 19$.*

Proof. For the first result, from substituting the mentioned values of minimum degree and girth in Lemma 3.8 and then by applying Lemma 3.3 we get that $rc(G) \leq \frac{7}{C_{\delta,3}}n - 6 - 1 + 6 \times 8 = \frac{7}{C_{\delta,3}}n + 41 = \frac{n}{\delta} \cdot \frac{7\delta(\delta-2)}{\delta(\delta-1)^3-2} + 41 \leq \frac{n}{\delta} \cdot \frac{21}{22} + 41 < \frac{n}{\delta} + 41$, in which we used the monotonicity of the function $f(\delta) = \frac{7\delta(\delta-2)}{\delta(\delta-1)^3-2}$ for the second inequality "$\leq$". The proof for the second result is similar. $\square$

Since vertex connectivity of a graph is a lower bound for minimum degree, the following results is immediate.

Theorem 3.10. *Let G be a κ-connected graph. Then,*

1. *if $\kappa \geq 3$ and $girth(G) \geq 7$, then $rc(G) < n/\kappa + 41$ and*
2. *if $\kappa \geq 5$ and $girth(G) \geq 5$, then $rc(G) < n/\kappa + 19$.*

Theorem 3.11. *For every κ-connected chordal graph G of order n,*

$$rc(G) \leq \frac{n}{\kappa} + 3.$$

Proof. The case of $\kappa = 1$ is trivial since rainbow coloring a spanning tree of G suffices. Hence let us assume $\kappa \geq 2$ and hence G is bridgeless. We claim that G has a 1-step connected dominating set D which can be rainbow colored using $\frac{|D|}{\kappa}$ colors. Then by Lemma 3.3, $rc(G) - 3 \leq rc(D) \leq \frac{|D|}{\kappa} \leq \frac{n}{\kappa}$. Hence it remains to prove the above claim. Consider a maximal connected set $D \subset V(G)$ that can be rainbow colored using $\frac{|D|}{\kappa}$ colors. Such a set exists since any singleton set of vertices can be rainbow colored using $0 < \frac{1}{\kappa}$ colors. Suppose for contradiction that D is not a 1-step dominating set. Then $N_G(D)$ is a vertex separator and hence contains a minimal separator $S \subset N_G(D)$. Since G is κ-connected, $|S| \geq \kappa$, and since G is chordal, S induces a clique [8]. Giving a single new color to every D–S and S–S edge extends the rainbow coloring of $G[D]$ to $G[D \cup S]$. Thus $D \cup S$ is a connected set containing D which can be colored using $rc(G[D]) + 1 \leq \frac{|D|}{\kappa} + 1 \leq \frac{|D \cup S|}{\kappa}$ colors, contradicting the maximality of D. So D is a 1-step connected dominating set and thus the result follows. $\square$

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