SEMISYMMETRIC GRAPHS OF ORDER $6p^2$ AND PRIME VALENCY

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ABSTRACT. In this paper, we investigate semisymmetric graphs of order $6p^2$ and of prime valency. First, we give a classification of the quasiprimitive permutation group of degree dividing $3p^2$, and then, on the basis of the classification result, we prove that, for primes k and p, a connected graph Γ of order $6p^2$ and valency k is semisymmetric if and only if $k = 3$ and either Γ is the Gray graph, or $p \equiv 1 \,(\mathsf{mod}\ 6)$ and Γ is isomorphic to one known graph.

KEYWORDS. Semisymmetric graph, bi-Cayley graph, normal cover, quasiprimitive permutation group.

1. INTRODUCTION

All graphs considered in this paper are assumed simple, finite and undirected. Let Γ be a graph. We use $V\Gamma$, $E\Gamma$ and $\mathsf{Aut}\Gamma$ to denote its vertex set, edge set and automorphism group, respectively. Each edge $\{u, v\}$ of Γ gives two *arcs* which are the ordered pair (u, v) and (v, u). We denote by $Arc\Gamma$ the set of the arcs of Γ. The action of $\mathsf{Aut}\Gamma$ on vertices induces naturally actions on $E\Gamma$ and $Arc\Gamma$ by $\{u, v\}^g = \{u^g, v^g\}$ and $(u, v)^g = (u^g, v^g)$, respectively. Then the graph Γ is said to be *vertex-transitive*, *edge-transitive*, or *arc-transitive* if $\mathsf{Aut}\Gamma$ acts transitively on $V\Gamma$, $E\Gamma$ or $Arc\Gamma$, respectively. An arc-transitive graph is also said to be *symmetric*. A regular graph Γ is called *semisymmetric* if it is edge-transitive but not vertex-transitive.

The class of semisymmetric graphs was introduced by Folkman [13], who posed a number of problems which spurred the interest in this topic. Afterwards, lots of interesting examples and results were found, see [1, 2, 3, 5, 9, 10, 11, 12, 15, 16, 20, 21, 22, 23, 25] for references.

In [20], a group-theoretic description was given for semisymmetric graphs of prime valency, which says that a semisymmetric graph of prime valency must be one of the seven types. In this paper, we use this result to analyze such graphs of order $6p^2$.

A permutation group is *quasiprimitve* if its non-trivial normal subgroups are all transitive. Our first result gives a complete list of quasiprimitve permutation groups of degree dividing $3p^2$ for a prime p.

Theorem 1.1. *Let G be a quasiprimitve permutation group of degree dividing $3p^2$ for a prime p. Then G is described in Table 4.1, Theorems 4.4, 4.5, 4.6 and 4.9.*

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On the basis of the above classification result, we get a complete classification for semisymmetric graphs of order $6p^2$ and of prime valency.

Theorem 1.2. *Let Γ be a connected graph of order $6p^2$ and valency k, where p and k are primes. Then Γ is semisymmetric if and only if $k = 3$ and either Γ is the Gray graph, or $p \equiv 1 \,(\mathsf{mod}\ 6)$ and Γ is isomorphic the graph Φ defined in Section 2.*

Remarks on Theorem 1.2. A semisymmetric graph must have even order and, by [13], there are no such graphs of orders $2p$ and $2p^2$. Thus a disconnected semisymmetric graph of order $6p^2$ must be a union of p isomorphic semisymmetric graphs of order $6p$. By [11], all semisymmetric graphs of order $6p$ are known. Thus we consider here only the connected graphs.

2. Two examples

In this section, we introduce two examples of graphs involved in Theorem 1.2.

The Gray graph was discovered by Gray in 1932 (unpublished), which was then discovered independently and proved to be semisymmetric by Bouwer [1] in 1968. In the literature, the Gray graph is the first known example of semisymmetric graphs. The Gray graph can be constructed as follows: Taking three copies of the complete bipartite graph $\mathsf{K}_{3,3}$ and, for each given edge, subdividing it in each of the three copies, joining the resulting three vertices to a new vertex. Thus the Gray graph is cubic and has order 54.

The next example was constructed in [20] by using voltage assignment on the arcs of $\mathsf{K}_{3,3}$.

Let p be a prime with $p \equiv 1 \,(\mathsf{mod}\ 6)$. Take $\mu \in \mathbb{Z}_p$ with $\mu^2 + \mu + 1 \equiv 0 \,(\mathsf{mod}\ p)$, and set

$$V = \{(l, i, j) \mid 1 \le l \le 6, i, j \in \mathbb{Z}_p\}.$$

Define a graph Φ on V with edge set

$$\begin{array}{rl} E\Phi = & \{\{(1,i,j),(4,i,j)\} \mid i,j \in \mathbb{Z}_p\} \cup \{\{(1,i,j),(5,i,j)\} \mid i,j \in \mathbb{Z}_p\} \\ & \cup\{\{(1,i,j),(6,i,j)\} \mid i,j \in \mathbb{Z}_p\} \cup \{\{(2,i,j),(4,i,j)\} \mid i,j \in \mathbb{Z}_p\} \\ & \cup\{\{(3,i,j),(4,i,j)\} \mid i,j \in \mathbb{Z}_p\} \\ & \cup\{\{(2,i,j),(5,i{+}1,j)\} \mid i,j \in \mathbb{Z}_p\} \cup \{\{(2,i,j),(6,i,j{-}1)\} \mid i,j \in \mathbb{Z}_p\} \\ & \cup\{\{(3,i,j),(5,i{-}\mu,j)\} \mid i,j \in \mathbb{Z}_p\} \cup \{\{(3,i,j),(6,i,j{+}\mu)\} \mid i,j \in \mathbb{Z}_p\}. \end{array}$$

Then Φ is a well-defined cubic bipartite graph of order $6p^2$.

In [20], it was shown that, up to isomorphism, the above graphs give the complete list of semisymmetric cubic graphs of order $6p^2$.

Theorem 2.1 ([20])**.** *Let Γ be a connected cubic graph of order $6p^2$ for an odd prime p. Then Γ is semisymmetric if and only if either $p = 3$ and Γ is the Gary graph, or $p \equiv 1 \,(\mathsf{mod}\ 6)$ and Γ is isomorphic to the graph Φ defined as above.*

3. Preliminaries

Let Γ be a bipartite graph with bipartition $V\Gamma = U \cup W$. Let X be a *bi-transitive* subgroup of $\mathsf{Aut}\Gamma$, that is, X is transitive on both U and W. Then Γ is called *X-semisymmetric* if it is regular and X acts transitively on $E\Gamma$.

Suppose that X contains a subgroup G which is regular on both U and W; in this case, Γ is called a *bi-Cayley graph* of G. Take an edge $\{u, w\}$ with $u \in U$ and $w \in W$. Then each vertex in U can be written uniquely as u^g for some $g \in G$, and so does for the vertices in W. Let $\Gamma(u)$ denote the set of neighbors of u in Γ. Set $S = \{s \in G \mid w^s \in \Gamma(u)\}$. Then $1 \in S$ and S is uniquely determined by the choice of $\{u, w\}$, and u^h and w^g are adjacent if and only if $gh^{-1} \in S$. It is well-known that Γ is connected if and only if $\langle S\rangle = G$, and that Γ is vertex transitive if G is abelian, refer to [11]. Let $Y = \mathrm{N}_X(G)$ and $y \in Y_u$. Noting that $w^y \in \Gamma(u) = \{w^s \mid s \in S\}$, there is a unique $t \in S$ with $w^y = w^t$. Then $(u^g)^y = u^{g^y}$ and $(w^g)^y = w^{yg^y} = (w^t)^{g^y}$, so the next lemma holds, see [19].

Lemma 3.1. *For $y \in \mathrm{N}_X(G)$ with $u^y = u$, there are $t \in S$ and $\sigma \in \mathsf{Aut}(G)$ such that $tS^\sigma = S$, $(u^g)^y = u^{g^\sigma}$ and $(w^g)^y = (w^t)^{g^\sigma}$ for all $g \in G$.*

Suppose now that X has a normal subgroup N which is intransitive on one of U and W. Let U_N and W_N denote the sets of N-orbits on U and W, respectively. Define a graph Γ_N on $U_N \cup W_N$ with edge set $\{\{u^N, w^N\} \mid \{u, w\} \in E\Gamma\}$, called the *normal quotient* of Γ with respect to X and N. The graph Γ is called a *normal cover* or an *N-cover* of Γ_N if, for each $\{u, v\} \in E\Gamma$, the induced subgraph $[u^N \cup w^N]$ is a matching in Γ. The next lemma collects several well-known facts about normal quotient.

Lemma 3.2. *Assume that $N \lhd X$ is intransitive on one of U and W.*

1. *X induces a bi-transitive subgroup of $\mathsf{Aut}\Gamma_N$, which is transitive on $E\Gamma_N$ if further X acts transitively on $E\Gamma$.*
2. *If Γ is connected and Γ is a normal cover of Γ_N, then N is semiregular on both U and W.*

For connected semisymmetric graphs of prime valency, [20] gives a group-theoretic description by using normal quotients, which classifies such graphs into seven types as shown below.

Theorem 3.3. *Let Γ be a connected X-semisymmetric graph of prime valency k. Suppose that Γ is not a complete bipartite graph. Then X acts faithfully on both U and W. Moreover, for a minimal normal subgroup $N \cong T^l$ of X, one of the following statements occurs.*

1. *$T = \mathbb{Z}_q$ for a prime q, Γ is a bi-Cayley graph of N, and Γ is a symmetric graph;*
2. *T is non-abelian simple, and Γ is a bi-Cayley graph of N;*
3. *N is non-abelian simple, and Γ is N-semisymmetric;*
4. *$t = 2$, T is non-abelian simple, and Γ is a bi-Cayley graph of T*
5. *Γ is, with respect to N, either a T^{l-1}-cover of a graph described in (3) or a T^{l-2}-cover of a graph described in (4);*

(6) Γ_N *is a k-star, that is, N is transitive on one of U and W and has exactly k orbits on the other one;*
(7) Γ *is a N-cover of a X/N-semisymmetric graph.*

In general, we may reduce a semisymmetric graph by taking normal quotients to an edge transitive bipartite graph which admits a group acting quasiprimitively on at least one of its two parts.

Lemma 3.4 ([14])**.** *Assume that X is faithful on both U and W, and that X is not quasiprimitive on either U or W. Then X has a minimal normal subgroup acting intransitively on both U and W.*

Proof. Since X is not quasiprimitive on U, we may take a minimal normal subgroup N of X which is intransitive on U. If N is intransitive on W, then the result follows. Suppose that N is transitive on W. Take a minimal normal subgroup M of X which is intransitive on W. Then $N \cap M = 1$, and so M centralizes N. Since X is faithful on W, it follows that M is semiregular on W. Thus, $|M|$ is a proper divisor of $|U| = |W|$, hence M is not transitive on U. This completes the proof. ∎

Praeger [26] gave an analogous version of the O'Nan-Scott Theorem for quasiprimitive permutation groups, and classified them into several well-defined types. We just quote here the basic facts for such groups. For a group G, the *socle* $\mathsf{soc}(G)$ is the subgroup generated by all minimal normal subgroup of G.

Theorem 3.5. *Let G be a finite quasiprimitive permutation group on Ω. Then G has at most two minimal normal subgroups, and one of the following statements holds.*

(1) *$G \le \mathrm{AGL}(d,p)$, $|\Omega| = p^d$ and $\mathsf{soc}(G) = \mathbb{Z}_p^d$ is the unique minimal normal subgroup of G, where $d \ge 1$ and p is a prime;*
(2) *$\mathsf{soc}(G) = T^l$ for $l \ge 1$ and a nonabelian simple group T, and either $\mathsf{soc}(G)$ is the unique minimal normal subgroup of G, or $\mathsf{soc}(G) = M \times N$ for two minimal normal subgroups M and N of G with $|M| = |N| = |\Omega|$.*

Note that there is no a non-abelian simple group has order $p^a q^b$, where p and q are two primes. By Theorem 3.5, the next simple result holds.

Lemma 3.6. *Let G be a quasiprimitive permutation group of degree $p^a q^b$, where p and q are primes. Then $\mathsf{soc}(G)$ is the unique minimal normal subgroup of G.*

4. Quasiprimitive Groups with Degree Dividing $3p^2$

In this section we assume that G is a quasiprimitive permutation group on Ω of degree $n = 3p^2$, p^2, $3p$ or p, where p is a prime. For $\alpha \in \Omega$, denote by G_α the stabilizer of α in G.

4.1. **The primitive case.** Assume that G is primitive on Ω, that is, G_α is maximal in G. If $n = p$ then either $\mathbb{Z}_p \le G \le \mathrm{AGL}(1,p)$ or G is 2-transitive, refer to [7, Corollary 3.5B]. Note that all 2-transitive permutation groups are explicitly known, see [4, Chapter 7] for example. Thus all transitive permutation groups of degree p are listed in Table 4.1.

p	11	11	23	p	p	$\frac{q^d-1}{q-1}$
$\mathsf{soc}(G)$	$\mathrm{PSL}(2,11)$	M_{11}	M_{23}	$\mathbb{Z}_p$	A_p	$\mathrm{PSL}(d,q)$
$\mathsf{soc}(G)_\alpha$	A_5	M_{10}	M_{22}	1	A_{p-1}	parabolic

Table 4.1. Transitive groups of prime degree p

For further argument, we need some results on elementary number theory.

Lemma 4.1. *There are no primes r and p such that $r^e+1=p^2$ unless $p=2$ or 3.*

Proof. Suppose that p is an odd prime and $p^2=r^e+1$ for some prime r. Then $r=2$ as p is odd. Then $(p+1)(p-1)=p^2-1=2^e$. So $p+1=2^k$ and $p-1=2^l$ with $l+k=e$ and $2^k-2^l=2$, which implies that $l=1$ and $k=2$, hence $p=3$ and $r=2$. ∎

Now we list a well-known result in number theory. For integers $a>0$ and $n>0$, a prime divisor of a^n-1 is called *primitive* with respect to a and n if it does not divide a^i-1 for any $0<i<n$.

Theorem 4.2 (Zsigmondy). *For two integers a, $n\geq 2$, if a^n-1 has no primitive prime divisors, then either $(a,n)=(2,6)$, or $n=2$ and $a+1$ is a power of 2.*

Lemma 4.3. *Let q be a notrivial power of some prime and $d\geq 2$. If $\frac{q^d-1}{q-1}\in\{p,3p,p^2,3p^2\}$ for an odd prime p, then p is primitive with respect to q and d.*

Proof. The result is trivial for $\frac{q^d-1}{q-1}=p$. If $p=3$, then $q=8$ and $d=2$, and the result holds. If $(q,d)=(2,6)$, then $\frac{q^d-1}{q-1}\notin\{p,3p,p^2,3p^2\}$ for any prime p. If $d=2$, then $q+1=\frac{q^d-1}{q-1}\in\{3p,3p^2\}$ by Lemma 4.1, and the result also holds by noting that p is not a divisor of $q-1$. Thus we assume that $p>3$, $d>2$, $\frac{q^d-1}{q-1}\in\{3p,p^2,3p^2\}$ and there is a primitive prime divisor r with respect to q and d. If $\frac{q^d-1}{q-1}=p^2$, then $r=p$. Assume that $\frac{q^d-1}{q-1}\in\{3p,3p^2\}$. Then 3 is a divisor of $q^{d-1}+\cdots+q+1$, yielding $q\equiv 1\ (\mathsf{mod}\ 3)$, so 3 is not primitive. Thus $r=p$. This completes the proof. ∎

We are ready to classify the primitive permutation groups of degree $3p$, p^2 and $3p^2$.

Theorem 4.4. *Let G be a primitive permutation group of degree $3p$, where p is prime. Then either $G\leq\mathbb{Z}_3^2{:}\mathrm{GL}(2,3)\cong\mathbb{Z}_3^2{:}2\mathrm{S}_4$, or G is one of the groups listed in Table 4.2.*

Proof. By [7, Appendix B], we can determine G when $p=2$ or 3. If $p=2$ then G has degree 6 and $\mathsf{soc}(G)=\mathrm{A}_6$ or A_5. If $p=3$ then G has degree 9, and so either G is of affine type, or $\mathsf{soc}(G)=\mathrm{A}_9$ or $\mathrm{PSL}(2,8)$.

Assume next that $p\geq 5$. If G is not 2-transitive then, by [27], G is one of the groups listed in lines 2-5 of Table 4.2.

Assume that G is 2-transitive. By checking the degrees of 2-transitive permutation groups, we conclude that $(3p,\mathsf{soc}(G))$ is one of $(3p,\mathrm{A}_{3p})$, $(15,\mathrm{A}_7)$ and $(\frac{q^d-1}{q-1},\mathrm{PSL}(d,q))$, where $q=r^e$ for a prime r. We assume that the third pair occurs in the following.

Degree	G	Action or Remark
6	$\mathrm{A}_5, \mathrm{S}_5$	cosets of D_{10} in A_5
15	$\mathrm{A}_6, \mathrm{S}_6$	2-subsets
21	$\mathrm{A}_7, \mathrm{S}_7$	2-subsets
21	$\mathrm{PSL}(3,2).2$	point-line incendent pairs
57	$\mathrm{PSL}(2,19)$	cosets of A_5 (two actions)
15	A_7	cosets of $\mathrm{PSL}(2,7)$ (two actions)
$3p$	$\mathrm{A}_{3p}, \mathrm{S}_{3p}$	
15	$\mathrm{PSL}(4,2)$	points, hyperplanes
2^e+1	$\mathrm{PSL}(2,2^e), \mathrm{P\Gamma L}(2,2^e)$	ponits; e odd prime
q^2+q+1	$\mathrm{PSL}(3,q).O$	points, hyperplanes; $q \equiv 1 \,(\mathsf{mod}\ 3)$

TABLE 4.2. Primitive permutation groups of degree $3p$.

Suppose that $d \geq 4$ is even. Then $3p = \frac{q^d-1}{q-1} = (q^{\frac{d}{2}-1} + \cdots + q + 1)(q^{\frac{d}{2}} + 1)$, yielding $3 = q^{\frac{d}{2}-1} + \cdots + q + 1$ and $p = q^{\frac{d}{2}} + 1$. The first equation implies that $q = 2$ and $d = 4$. Hence $p = 5$ and $\mathsf{soc}(G) = \mathrm{PSL}(4,2) \cong \mathrm{A}_8$. Checking the maximal subgroups of A_8 and S_8 in the Atlas [6], we know that $G = \mathrm{PSL}(4,2)$ and $G_\alpha \cong \mathbb{Z}_2^3{:}\mathrm{PSL}(3,2)$.

If $d = 2$, then $3p = q + 1 = r^e + 1$, so $r = 2$ as $3p$ is odd, which implies that e is an odd prime.

Finally, let d be odd. Note that $3p = \frac{q^d-1}{q-1} = q^{d-1} + \cdots + q + 1$. It follows that $q \equiv 1 \,(\mathsf{mod}\ 3)$ and 3 is a divisor of d. If d is not a prime, then $\frac{q^d-1}{q-1} = \frac{q^s-1}{q-1}((q^s)^{t-1} + \cdots + q^s + 1)$ for odd integers $1 < s \leq t$ with $d = st$, yielding that $3 = \frac{q^s-1}{q-1}$, which is impossible. Thus d is a prime, so $d = 3$. Then $3p = q^2 + q + 1$ with $q \equiv 1 \,(\mathsf{mod}\ 3)$. ∎

The next result gives a classification of primitive permutation groups of degree p^2.

Theorem 4.5. *Let G be a primitive permutation group of degree p^2. Then one of the following holds.*

(1) $G \leq \mathrm{AGL}(2,p)$;
(2) $G = \mathrm{A}_{p^2}$ *or* S_{p^2};
(3) *either* $\mathsf{soc}(G) = \mathrm{PSL}(2,8)$ *and* $p = 3$, *or* $\mathsf{soc}(G) = \mathrm{PSL}(d,q)$, d *is an odd prime and* $p^2 = \frac{q^d-1}{q-1}$;
(4) $T^2 \lhd G \leq H \wr \mathrm{S}_2$, *where H is a transitive permutation group of degree p with* $\mathsf{soc}(H) = T$ *is listed in Table* 4.1.

Proof. If G is affine, then (1) holds. Thus we assume that $\mathsf{soc}(G)$ is nonabelian. Assume that G is 2-transitive. Then, by [8], either $\mathsf{soc}(G) = A_{p^2}$ and (2) occurs, or $\mathsf{soc}(G) = \mathrm{PSL}(d,q)$ with $p^2 = \frac{q^d-1}{q-1}$. If the latter case occurs, then $p^2 = \frac{q^d-1}{q-1}$ yields that d is a prime and, by Lemma 4.1, $p = 3$ and $q = 8$ while $d = 2$, hence (3) holds. Assume that G is not 2-transitive. Then G has a Sylow p-subgroup isomorphic to $\mathbb{Z}_p^2$, and G has a normal subgroup of index 2 which is the direct product of two intransitive groups, refer to [28, Theorems 25.2 and 27.2]. In particular, $\mathsf{soc}(G)$ is not simple. Thus (4) follows from [18]. ∎

Finally, we classify the primitive permutation groups of degree $3p^2$.

Theorem 4.6. *Let G be a primitive permutation group of degree $3p^2$ for some prime p. Then eithher $G \leq \mathrm{AGL}(3,3)$ or G is one of the groups listed in Table 4.3.*

$3p^2$	G	2-trans. ?	Condition
12	$\mathrm{PSL}(2,11), \mathrm{PGL}(2,11)$	Yes	
	M_{11}	Yes	
	M_{12}	Yes	
27	$\mathrm{PSU}(4,2), \mathrm{PSU}(4,2).2$	No	
$3p^2$	$\mathrm{A}_{3p^2}, \mathrm{S}_{3p^2}$	Yes	
q^2+q+1	$\mathrm{PSL}(3,q).O$	Yes	$q \equiv 1 \,(\mathsf{mod}\ 3)$

Table 4.3. Primitive permutation groups of degree $3p^2$.

Proof. By [7, Appendix B], we can determine G when $p = 2$ or 3. If $p = 2$ then G has degree 12, and G is one of A_{12} or S_{12}, $\mathrm{PSL}(2,11)$, $\mathrm{PGL}(2,11)$, M_{11} and M_{12}. If $p = 3$, then G has degree 27, and so either G is of affine type, or G is one of A_{27}, S_{27}, $\mathrm{PSU}(4,2)$, and $\mathrm{PSU}(4,2).2$.

Assume next that $p \geq 5$. In particular, $3p^2$ is odd and not a power of an integer less than $3p^2$. By [18], $\mathsf{soc}(G)$ is a nonabelian simple group. Since G is primitive, T is transitive on Ω; in particular, T has a subgroup of index $3p^2$. We next prove the result by checking all possible candidates for $L := \mathsf{soc}(G)$ one by one. Let $\alpha \in \Omega$. Then $3p^2 = |\Omega| = |L : L_\alpha|$.

Case 1. Assume that L is an alternating group A_c. Then $3p^2 = |L : L_\alpha| = \binom{c}{k}$ for $1 \leq k < \frac{c}{2}$. Thus p^2 divides $c!$, yielding $c \geq 2p$. If $k = 2$, then $3p^2 = \frac{c(c-1)}{2}$, yielding p^2 divides c or $c-1$, hence $3p^2 \geq \frac{p^2(p^2-1)}{2}$, which is impossible. If $k \geq 3$, then $3p^2 \geq \binom{c}{3} = \frac{c(c-1)(c-2)}{6}$ as $k < \frac{c}{2}$, so

$$3p^2 \geq \frac{2p(2p-1)(2p-2)}{6} > \frac{2p(2p-\frac{p}{2})(2p-\frac{p}{2})}{6} = \frac{3p^3}{4} > 3p^2,$$

a contradiction. Then $k = 1$, $c = 3p^2$ and $L = \mathrm{A}_{3p^2}$.

Case 2. Suppose that L is a sporadic simple group. Then, inspecting the orders of sporadic simple groups, we get $p \leq 13$ as p^2 is a divisor of $|L|$. Then $3p^2 < 1000$. Thus G is a primitive permutation group of degree less than 1000. Checking the tables in [7, Appendix B], we conclude that there is no such a primitive permutation group, a contradiction.

Case 3. Let $L = L(q)$ be a simple group of Lie type over $GF(q)$. The estimation on the prime divisors of $|L : L_\alpha|$ excludes most of the candidates for L. These computations are straightforward, but quite tedious. We give details only in the cases $L = \mathrm{PSL}(d,q)$ and $L = \mathrm{PSU}(d,q)$.

Subcase 3.1. Let $L = \mathrm{PSL}(d,q)$, $q = r^e$ for a prime r. Then L_α is a parabolic subgroup of L and

(1) $|L:L_\alpha| = \frac{(q^d-1)(q^{d-1}-1)\cdots(q^{d-m+1}-1)}{(q^m-1)(q^{m-1}-1)\cdots(q-1)}$ for $m \le \frac{d}{2}$ and G acts on m or $(d-m)$-dimensional subspaces; or

(1') $|L:L_\alpha| = \frac{(q^d-1)(q^{d-1}-1)\cdots(q^{d-2m+1}-1)}{[(q^m-1)(q^{m-1}-1)\cdots(q-1)]^2}$ for $1 \le m < \frac{d}{2}$; or

(1") $|L:L_\alpha| = \frac{q^{md-m^2}(q^d-1)(q^{d-1}-1)\cdots(q^{d-m+1}-1)}{(q^m-1)(q^{m-1}-1)\cdots(q-1)}$ for $1 \le m < \frac{d}{2}$.

First, (1") can be easily excluded. Observe that r divides $|L:L_\alpha|$, so r is odd. If $r=3$ then $m=1$ and $d=2$, a contradiction. Thus $r=p$, yielding $md-m^2=2$, so $d=3$ and $m=1$, hence $3p^2 = q^2\frac{q^d-1}{q-1}$, which is impossible.

Suppose that $q=2$. Assume that (1') occurs. If $m \ge 2$ then, by Theorem 4.2, $|L:L_\alpha|$ has at least three distinct prime divisors. Thus $m=1$ and $3p^2 = |L:L_\alpha| = (2^d-1)(2^{d-1}-1)$ with $d \ge 3$, which is impossible. Thus, (1) occurs. If $d-m \ge 4$ and $m \ge 4$ then, by Theorem 4.2, $|L:L_\alpha|$ has at least three distinct prime divisors. Thus either $d-m<4$ or $m<4$. So $|L:L_\alpha|$ is one of 2^d-1 for $d-m=1$ or $m=1$, $\frac{(2^d-1)(2^{d-1}-1)}{3}$ for $d-m=2$ or $m=2$, $\frac{(2^d-1)(2^{d-1}-1)(2^{d-2}-1)}{21}$ for $d-m=3$ or $m=3$. The latter two cases all imply $9 = 2^{d-1}-1$, a contradiction. For the first case, setting $p=2k+1$ with $k \ge 2$, then $3k(k+1)+1 = 2^{d-2}$, which is not true.

Assume next that $q \ne 2$. By Theorem 4.2, $|L:L_\alpha|$ has at least three distinct prime divisors if $m \ge 3$ for (1) or $m \ge 2$ for (1'), which is not the case. For (1), if $m=2$ then $|L:L_\alpha| = \frac{(q^d-1)(q^{d-1}-1)}{(q^2-1)(q-1)}$, which can not have the form $3p^2$ as $q>2$ and $d \ge 2m = 4$. For (1'), if $m=1$ then $|L:L_\alpha| = \frac{(q^d-1)(q^{d-1}-1)}{(q-1)^2}$, which can not have the form $3p^2$ as $q>2$ and $d \ge 3$. Thus we get $3p^2 = |L:L_\alpha| = \frac{q^d-1}{q-1}$ and G acts on 1 or $(d-1)$-dim. subspaces. Further, $3p^2 = \frac{q^d-1}{q-1}$ yields that d is a prime. Suppose that $d=2$. Then $3p^2 = q+1$, yielding $q=2^e$ for odd $e>6$ as $p \ge 5$ and 3 divides $q+1$. Hence $p^2 = \sum_{i=0}^{e-1}(-2)^i$, so $p^2-1 \equiv 2 \pmod{4}$, a contradiction. Thus d is an odd prime. Since 3 divides $\frac{q^d-1}{q-1}$, we conclude that $d=3$ and $q \equiv 1 \pmod{3}$.

Subcase 3.2. Let $L = \mathrm{PSU}(d,q)$. Then L_α must be the stabilizer of some m-dim. subspaces; otherwise, $|L:L_\alpha|$ is divided by q^{md-m^2} with $m < \frac{d}{2}$, yielding $m=1$, $d=3$, $q=r=p$, hence $|L:L_\alpha| = 3p^2 l$ for some $l>3$, a contradiction. Thus

$$|L:L_\alpha| = \frac{\prod_{i=d-2m+1}^{d}(q^i-(-1)^i)}{\prod_{i=1}^{m}(q^{2i}-1)} = 3p^2$$

for $1 \le m \le [\frac{d}{2}]$. Note that a primitive prime divisor of $q^{2i}-1$ must divide q^i+1. If $m=1$, then $|L:L_\alpha| = \frac{(q^d-(-1)^d)(q^{d-1}-(-1)^{d-1})}{q^2-1}$, yielding $3 = \frac{q^d-(-1)^d}{q^2-1}$ or $\frac{q^{d-1}-(-1)^{d-1}}{q^2-1}$, hence $d=3$ and $p^2 = q^3+1$, so $q=2$; however $\mathrm{PSU}(3,2)$ is not simple, a contradiction. By Theorem 4.2, $|L:L_\alpha| = 3p^2$ yields that either $m=3$ and $d=6$, or $m=2$ and $d=4$. For the first case, $|L:L_\alpha| = (q+1)(q^3+1)(q^5+1)$ which has not the form of $3p^2$. Thus $m=2$, $d=4$ and $|L:L_\alpha| = (q^3+1)(q+1)$, it follows that $p=3$, contradicts the assumption that $p>3$. ∎

4.2. **The quasiprimitive case.** In this part, we assume that G is a quasiprimitive permutation group on Ω of degree $n = 3p^2$, p^2 or $3p$.

Let $\Delta \neq \Omega$ be a block of G, that is, $\Delta \cap \Delta^g = \emptyset$ or Δ for all $g \in G$. Set $\mathcal{B} = \{\Delta^g \mid g \in G\}$. Let $G^{\mathcal{B}}$ be the permutation group induced by G on $\mathcal{B}$. Then $G^{\mathcal{B}}$ has degree $\frac{3p^2}{|\Delta|}$. Since G is quasiprimitive on Ω, we know that G acts faithfully on $\mathcal{B}$, that is, $G^{\mathcal{B}} \cong G$. Assume that $G^{\mathcal{B}}$ is primitive on $\mathcal{B}$. Note that the order of a primitive permutation group of degree p is not divisible by p^2, yielding the non-existence of subgroups of index dividing by p^2. It follows that $(n, |\Delta|)$ is one of $(3p, 1)$, $(3p, 3)$ with $p \neq 2$, $(p^2, 1)$, $(3p^2, 1)$, $(3p^2, 3)$ with $p \neq 2$, and $(3p^2, p)$. In particular, we get the following lemma.

Lemma 4.7. *Every quasiprimitive permutation group of degree* p^2 *is also primitive.*

Further, it is easily shown the following lemma holds.

Lemma 4.8. *Every affine quasiprimitive permutation group is also primitive.*

The next result classifies quasiprimitve permutation groups of degree dividing $3p^2$.

Theorem 4.9. *Let G be a quasiprimitve group on Ω of degree n dividing $3p^2$. Let $N = \mathsf{soc}(G)$. If G is imprimitive, then one of the following occurs.*

(1) $n = 12$ *or* 15, $N = \mathrm{A}_5$ *and* $N_\alpha \cong \mathbb{Z}_5$ *or* $\mathbb{Z}_2^2$ *respectively;*
(2) $n = 21$, $N = \mathrm{PSL}(3, 2)$ *and* $N_\alpha = \mathrm{D}_8$*;*
(3) $n = 39$, $N = \mathrm{PSL}(3, 3)$ *and* $N_\alpha = \mathbb{Z}_3^2{:}2\mathrm{D}_8$*;*
(4) $n = 3p = 3(1 + 2^{2^s})$, $N = \mathrm{PSL}(2, 2^{2^s})$ *for integer* $s \geq 1$, *and* $N_\alpha = \mathbb{Z}_2^{2^s}{:}\mathbb{Z}_{\frac{2^{2^s}-1}{3}}$*;*
(5) $n = 3\frac{q^d-1}{q-1} = 3p$ *or* $3p^2$, $N = \mathrm{PSL}(d, q)$ *with odd prime* d *and* $q \equiv 1 \ (\mathsf{mod}\ 3(q-1, d))$, *and* $N_\alpha = [q^{d-1}].\mathbb{Z}_{(q-1,d-1)}.\mathrm{PSL}(d-1, q).\mathbb{Z}_{\frac{q-1}{3(q-1,d)}}$*;*
(6) $n = 75$, $N = \mathrm{A}_5 \times \mathrm{A}_5$, $N_\alpha = \mathbb{Z}_2^4{:}\mathbb{Z}_3$*;*
(7) $n = 3(1 + 2^{2^s})^2$, $N = \mathrm{PSL}(2, 2^{2^s}) \times \mathrm{PSL}(2, 2^{2^s})$ *and* $N_\alpha = (\mathbb{Z}_2^{2^s}{:}\mathbb{Z}_{\frac{2^{2^s}-1}{3}} \times \mathbb{Z}_2^{2^s}{:}\mathbb{Z}_{\frac{2^{2^s}-1}{3}}){:}\mathbb{Z}_3$*;*
(8) $n = 3(\frac{q^d-1}{q-1})^2$, $N = \mathrm{PSL}(d, q) \times \mathrm{PSL}(d, q)$ *with odd prime* d *and* $q \equiv 1 \ (\mathsf{mod}\ 3(q-1, d))$, *and* $N_\alpha = (M_1 \times M_2).\mathbb{Z}_3$ *with* $M_1 \cong M_2 \cong [q^{d-1}].\mathbb{Z}_{(q-1,d-1)}.\mathrm{PSL}(d-1, q).\mathbb{Z}_{\frac{q-1}{3(q-1,d)}}$.

Proof. Assume that G is not primitive on Ω. Let $\Delta \neq \Omega$ be a block of G such that $G^{\mathcal{B}}$ is a primitive permutation group on $\mathcal{B} = \{\Delta^g \mid g \in G\}$. Then $(n, |\Delta|) = (3p, 3)$, $(3p^2, 3)$ or $(3p^2, p)$, and so $G^{\mathcal{B}}$ is of degree p, p^2 or $3p$, respectively. Moreover, by Lemma 4.8, both G and $G^{\mathcal{B}}$ are not of affine type. Then $\mathsf{soc}(G^{\mathcal{B}})$ is listed in Tables 4.1, 4.2 or Theorem 4.5 (2)-(4). Let N be socle of G. Then $N = \mathsf{soc}(G^{\mathcal{B}})$ as $G \cong G^{\mathcal{B}}$, and N is transitive on both Ω and $\mathcal{B}$. Take $\alpha \in \Delta$. Then $|\Delta| = |N_\Delta : N_\alpha|$, that is, N_Δ has a subgroup N_α of index $|\Delta|$.

Assume that $p = 2$. Then $(n, |\Delta|) = (12, 2)$ and $G^{\mathcal{B}}$ is of degree 6. Thus $N = \mathsf{soc}(G^{\mathcal{B}}) = \mathrm{A}_6$ or A_5. Note that A_6 has no subgroups of index 12. Thus, $N = \mathrm{A}_5$ and $N_\alpha \cong \mathbb{Z}_5$. Therefore, we assume next that p is odd.

Suppose that $(n, |\Delta|) = (3p^2, p)$. Then $G^{\mathcal{B}}$ is a primitive permutation group of degree $3p$ listed in Table 4.2. Note that $|G|$ is divisible by p^2. By Lemma 4.3 and checking orders of candidates for N, the only possibility is that $N = \mathrm{A}_{3p}$. But, in this case, $N_\Delta = \mathrm{A}_{3p-1}$ has no subgroups of index p.

Case 1. Assume that $(n, |\Delta|) = (3p, 3)$. Then N is a nonabelian simple group listed in Table 4.1. Checking those groups and their stabilizers, either $N = \mathrm{A}_5$, $p = 5$ and $N_\alpha \cong \mathbb{Z}_2^2$, or $N = \mathrm{PSL}(d, q)$ and $p = \frac{q^d-1}{q-1}$ for a prime d.

Assume that $N = \mathrm{PSL}(d, q)$. Then

$$N_\Delta = [q^{d-1}].\mathbb{Z}_{(q-1,d-1)}.\mathrm{PSL}(d-1, q).\mathbb{Z}_{\frac{q-1}{(q-1,d)}}.$$

Note that N_α is a subgroup of N_Δ of index 3. If $\mathrm{PSL}(d-1, q)$ is nonabelian simple, then $q \equiv 1 \,(\mathsf{mod}\ 3(q-1, d))$ and

$$N_\alpha = [q^{d-1}].\mathbb{Z}_{(q-1,d-1)}.\mathrm{PSL}(d-1, q).\mathbb{Z}_{\frac{q-1}{3(q-1,d)}}.$$

If $d = 3$ and $q \in \{2, 3\}$, then $N = \mathrm{PSL}(3, 2)$ and $N_\alpha = \mathrm{D}_8$, or $N = \mathrm{PSL}(3, 3)$ and $N_\alpha = \mathbb{Z}_3^2{:}2\mathrm{D}_8$. Let $d = 2$. Then $q > 3$ and, since $q + 1 = p$ is prime, we know $q = 2^{2^s}$ for some integer $s \geq 1$. So $N_\Delta = \mathbb{Z}_2^{2^s}{:}\mathbb{Z}_{2^{2^s}-1}$. Thus $q \equiv 1 \,(\mathsf{mod}\ 3)$ and $N_\alpha = \mathbb{Z}_2^{2^s}{:}\mathbb{Z}_{\frac{2^{2^s}-1}{3}}$. Thus one of parts (1)-(5) occurs.

Case 2. Let $(n, |\Delta|) = (3p^2, 3)$. Then either $G^{\mathcal{B}}$ is 2-transitive and $N = \mathrm{A}_{p^2}$ or $\mathrm{PSL}(d, q)$, or $N = T^2$ with T nonabelian simple and listed in Table 4.1. Noting that A_{p^2-1} is simple, we know it has no subgroups of index 3. Thus $N \neq \mathrm{A}_{p^2}$.

Assume first $N = \mathrm{PSL}(d, q)$. Then $p^2 = \frac{q^d-1}{q-1}$, yielding d a prime. A similar argument as in above case implies that

$$N_\alpha = [q^{d-1}].\mathbb{Z}_{(q-1,d-1)}.\mathrm{PSL}(d-1, q).\mathbb{Z}_{\frac{q-1}{3(q-1,d)}} \text{ with } q \equiv 1 \,(\mathsf{mod}\ 3(q-1, d));$$

or $d = 2$ with $q > 3$; or $d = 3$ with $q \in \{2, 3\}$. The last case implies that $\frac{q^d-1}{q-1}$ is a prime, which is impossible. The second case implies that $p^2 = 1 + q$, so $q = 8$ and $p = 3$ by Lemma 4.1. However, $\mathrm{PSL}(2, 8)$ has no a subgroup of index 27, a contradiction. Thus G is described as in part (5) of the theorem.

Assume that $N = T^2$ with T listed in Table 4.1. By Theorem 4.5, $N \lhd G \leq H \wr \mathrm{S}_2$ and G has product action. Then, writing $N = T_1 \times T_2$ and choosing a suitable 'point' Δ in $\mathcal{B}$, we may assume that $N_\Delta = H_1 \times H_2$ with $|T_1 : H_1| = |T_2 : H_2| = p$. Further, we may take $g \in G_\alpha$ such that $\alpha \in \Delta$, $T_1^g = T_2$, $T_2^g = T_1$, $H_1^g = H_2$ and $H_2^g = H_1$. Then $H_1, H_2 \not\leq N_\alpha$, so $H_1 N_\alpha = H_2 N_\alpha = N_\Delta$. Thus $3 = |N_\alpha H_i : N_\alpha| = |H_i : (H_i \cap N_\alpha)|$, where $i = 1, 2$. Thus both H_1 and H_2 have subgroups of index 3.

By the above argument, we know that one of the following holds:

$N = \mathrm{A}_5 \times \mathrm{A}_5$, $p = 5$ and $N_\alpha \cap H_i \cong \mathbb{Z}_2^2$;
$N = \mathrm{PSL}(3, 2) \times \mathrm{PSL}(3, 2)$ and $N_\alpha \cap H_i = \mathrm{D}_8$;
$N = \mathrm{PSL}(3, 3) \times \mathrm{PSL}(3, 3)$ and $N_\alpha \cap H_i = \mathbb{Z}_3^2{:}2\mathrm{D}_8$;
$N = \mathrm{PSL}(2, 2^{2^s}) \times \mathrm{PSL}(2, 2^{2^s})$ and $N_\alpha \cap H_i = \mathbb{Z}_2^{2^s}{:}\mathbb{Z}_{\frac{2^{2^s}-1}{3}}$;
$N = \mathrm{PSL}(d, q) \times \mathrm{PSL}(d, q)$ and $N_\alpha \cap H_i = [q^{d-1}].\mathbb{Z}_{(q-1,d-1)}.\mathrm{PSL}(d-1, q).\mathbb{Z}_{\frac{q-1}{3(q-1,d)}}$ with odd prime d and $q \equiv 1 \,(\mathsf{mod}\ 3(q-1, d))$.

Noting that $(H_1 \cap N_\alpha) \times (H_2 \cap N_\alpha) \lhd N_\alpha$, we can easily exclude $N = \mathrm{PSL}(3,2) \times \mathrm{PSL}(3,2)$ or $\mathrm{PSL}(3,3) \times \mathrm{PSL}(3,3)$. Thus one of parts (6)-(8) of the theorem occurs. ∎

Finally, Theorem 1.1 follows from the arguments in the above two subsections.

5. Proof of Theorem 1.2

Let Γ be a connected bipartite regular graph of order $6p^2$ with bipartition $V\Gamma = U \cup W$. Assume that Γ is G-semisymmetric and of prime valency k, where $G \leq \mathsf{Aut}\Gamma$ and p is a prime. Thus $|U| = |W| = 3p^2$.

Clearly, a cycle is symmetric. By [16], all semisymmetric graphs of order 24 have valency 4 or 6. So we assume further both p and k are odd primes.

Note that $3p^2$ is not a prime as p is odd. Then Γ is not a complete bipartite graph. By Theorem 3.3, G acts faithfully on both U and W. Denote by G^U and G^W the permutation groups induced by G on U and W, respectively. Then $G^U \cong G^W \cong G$.

Lemma 5.1. *Suppose that G is quasiprimitive on one of U and W. Then Γ is a bi-Cayley graph of $\mathbb{Z}_3^3$.*

Proof. Without loss of generality, we assume that G^U is a quasiprimitive permutation group. Let $N = \mathsf{soc}(G)$. Then, by Lemma 3.6 and Theorems 4.6 and 4.9, N is the unique minimal normal subgroup of G and one of the following three cases occurs: $N \cong \mathbb{Z}_3^3$, or N is given in Theorem 4.9 (6)-(8), or N is nonabelain simple.

Assume that $N \cong \mathbb{Z}_3^3$. Then, by Theorem 3.3, either Γ is a bi-Cayley graph of $\mathbb{Z}_3^3$ or Γ_N is a k-star. Suppose that Γ_N is a k-star. Then $k = 3$, that is, Γ is a cubic G-semisymmetric graph of order 54. However, by [20] or [22], G has a normal subgroup of order 9 acting semiregularly on both U and W, which is a contradiction. Thus Γ is a bi-Cayley graph of $\mathbb{Z}_3^3$.

Thus we shall suppose that N is insoluble and deduce a contradiction.

Case 1. Suppose that G^U is described as in Theorem 4.9 (6)-(8). Then $N = T^2$ and $T = \mathrm{A}_5$, $\mathrm{PSL}(2, 2^{2^s})$ or $\mathrm{PSL}(d,q)$ with $q \equiv 1\,(\mathsf{mod}\ 3)$. Then, by Theorem 3.3, we conclude that Γ_N is a k-star, that is, N has k-orbits on W; in particular, k is a divisor of $3p^2$.

If $k = 3$ then, by [25], G has a semiregular normal subgroup M such that G/M is either soluble or almost simple, which is not the case. Thus $k = p$. Then each N-orbit on W has size $3p$.

Let B be an N-orbit on W and consider the action of N on B. Writing $N = T_1 \times T_2$. Then T_1 and T_2 is the only minimal normal subgroups of N. If both T_1 and T_2 are transitive on B, then both of them are regular on B, so $|B| = |T|$, which is impossible. Thus we may assume that T_1 is intransitive on B. Then T_1 acts trivially on B; otherwise one of T_1 and T_2 will have a transitive permutation representation of degree 3, which is impossible as T is nonabelian simple. Let $w \in B$ and $u \in \Gamma(w)$. Then $u^{T_1} \subseteq \Gamma(w)$, and so $k \geq |u^{T_1}| = |T_1 : (T_1)_u|$. Since N is a minimal normal subgroup of G, there is some $g \in G$ such that $T_1^g = T_2$. Thus $(T_2)_u = (T_1^g)_u =$

$T_1^g \cap G_u = (T_1 \cap G_u^{g^{-1}})^g = (T_1 \cap G_{u^{g^{-1}}})^g = ((T_1)_{u^{g^{-1}}})^g$. Since N is transitive on U and T_1 is normal in N, we know that all T_1-orbits on U have the same size. It follows that $|(T_1)_{u^{g^{-1}}}| = |(T_1)_u|$. Thus $|(T_2)_u| = |(T_1)_u|$. Noting $(T_1)_u \times (T_2)_u \le N_u$, we know that $3p^2 = |N : N_u|$ is a divisor of $|N : ((T_1)_u \times (T_2)_u| = |T_1 : (T_1)_u|^2$; in particular, $|T_1 : (T_1)_u| > p$. Thus $k \ge |u^{T_1}| = |T_1 : (T_1)_u| > p$, a contradiction.

Case 2. Suppose that N is nonabelian simple, that is, G is almost simple.

Assume first that G^U is primitive. Then G is known by Theorem 4.6. Suppose that $\mathsf{soc}(G) = \mathrm{PSU}(4,2)$ acting on 27 points. Then G is also primitive on W as G has no a subgroup of index properly dividing 27 by the Atlas [6]. Moreover, all subgroups of index 27 in G are all conjugate. Thus the actions of G on U and W are permutationally equivalent, which implies that Γ has valency the size of a suborbit of G^U. Then Γ has valency 16 or 10, a contradiction.

Suppose that G^U is 2-transitive. By [17, Theorem 5.2.2], G has no a permutation representation of degree properly dividing $3p^2$. It follows that G^W is also 2-transitive. Thus either Γ is the complete bipartite graph with a matching deleted, or Γ is the point-line incidence graph of the projective plane $\mathrm{PG}(2,q)$. So Γ has valency $k = 3p^2 - 1$ or $q+1$. Since k is an odd prime, $k = q+1$ and Γ is the point-line incidence graph of the projective plane $\mathrm{PG}(2,q)$ with $q \equiv 1 \pmod 3$. So $q+1$ is an odd prime, yielding $q = 4$ and $3p^2 = q^2+q+1 = 21$, a contradiction.

Note that the above argument is also available when we consider the action of G on W. Thus, assume now that G^U and G^W are both imprimitive. Since G^U is quasprimitive, $N = \mathrm{PSL}(d,q)$ and $N_u = [q^{d-1}].\mathbb{Z}_{(q-1,d-1)}.\mathrm{PSL}(d-1,q).\mathbb{Z}_{\frac{q-1}{3(q-1,d)}}$ by Theorem 4.9, where $u \in U$, $p^2 = \frac{q^d-1}{q-1}$, $q \equiv 1 \pmod{3(q-1,d)}$ and d is an odd prime.

Suppose that N is not transitive on W. Note that all N-orbits on W have the same size dividing $3p^2$. Since $\mathrm{PSL}(d,q)$ has no a permutation representation of degree less than $p^2 = \frac{q^d-1}{q-1}$ (see [17, Table 5.2.A], for example), we conclude that N has exactly three orbits on W. Thus Γ has valency 3, which is impossible by [25]. Therefore, N is also transitive on W. Then G^W is also quasiprimitive, and Γ is N-semisymmetric.

By Theorem 4.9 again, $N_w \cong N_u$ for $w \in W$. Then it is easily shown that the induced permutation group $N_u^{\Gamma(u)}$ by N_u on $\Gamma(u)$ is $\mathrm{PSL}(d-1,q).O$, where O is cyclic and of order dividing $\frac{q-1}{3(q-1,d)}$. Since Γ has prime valency $k = |\Gamma(u)|$ and Γ is N-semisymmetric, $N_u^{\Gamma(u)}$ is a transitive permutation group of prime degree k. Checking the groups given in Table 4.1 and noting that $\mathrm{A}_5 \cong \mathrm{PSL}(2,4)$, we know that $(\mathsf{soc}(N_u^{\Gamma(u)}), k) = (\mathrm{PSL}(2,11), 11)$ or $(\mathrm{PSL}(d-1,q), \frac{q^{d-1}-1}{q-1})$. The former case implies that $d = 3$, $q = 11$ and $p^2 = \frac{q^d-1}{q-1} = 133$, a contradiction. Assume that $k = \frac{q^{d-1}-1}{q-1}$. Then $d-1$ is a prime as k is a prime. Since d is a prime, $d = 3$. Thus $k = q+1$ is an odd prime, it follows that $q = 2^{2^s}$ for some $s \ge 0$. Since $q \equiv 1 \pmod{3(q-1,d)}$, we know that $s > 1$ and $q-1$ is divisible by 9, which is impossible. ∎

Lemma 5.2. *Either Γ is a bi-Cayley graph of an abelian group, or G has a normal subgroup of order p^2 which is semiregular on both U and W.*

Proof. Assume that Γ is not a bi-Cayley graph of $\mathbb{Z}_3^3$. Then, by Lemma 5.1, G is not quasiprimitive on either U or W. By Lemma 3.4, there is a minimal normal subgroup of G, says N, which is intransitive on both U and W. Then, by Theorem 3.3 and Lemma 3.2, we know that N is semiregular on both U and W, Γ is a normal cover of $\Sigma := \Gamma_N$ and Σ is G/N-semisymmetric by identifying G/N with a subgroup of $\mathsf{Aut}\Sigma$. In particular, Γ_N has valency k. Denote by V_N the vertex set of Σ, and by U_N and W_N the two bipartition subsets of Σ corresponding to U and W, respectively.

Note that $|N|$ is a divisor of $3p^2$ as N is semiregular on U. Then $|N| = 3$, p, or p^2. If $|N| = p^2$ then the lemma holds. Thus in the following we assume that $|N|$ is a prime. Then Σ is not a complete bipartite graph as it has prime valency. Thus G/N is faithful on both U_N and W_N by Theorem 3.3.

Case 1. Assume that $p = 3$. In this case G/N is a transitive permutation group of degree 9 on both U_N and W_N.

Suppose first that G/N is not qusiprimitive on either U_N or W_N. By Lemma 3.4, G/N has a minimal normal subgroup, says M/N, which is intransitive on both U_N and W_N. Then, by Theorem 3.3, we know that M/N is semiregular on both U_N and W_N, which yields $|M/N| = 3$. Then M is a normal subgroup of G of order 9 and acts semiregularly on both U and W.

Thus, without loss of generality, we suppose that G/N is qusiprimitive on U_N. Then G/N is primitive on U_N by Lemma 4.7. Thus either $G/N \leq \mathrm{AGL}(2, 3)$, or $\mathsf{soc}(G/N)$ is one of A_9 and $\mathrm{PSL}(2, 8)$.

Assume that $G/N \leq \mathrm{AGL}(2, 3)$. Then for $\alpha \in U_N$ the stabilizer $(G/N)_\alpha$ has order dividing $|\mathrm{GL}(2, 3)| = 48$, yielding $k = 3$ as k is an odd prime dividing $|(G/N)_\alpha|$. Thus Γ has valency 3 and order 54. By [20] or [22], G has a normal subgroup of order 9 acting semiregularly on both U and W.

Suppose that $\mathsf{soc}(G/N) = \mathrm{A}_9$ or $\mathrm{PSL}(2, 8)$. Then G/N is 2-transitive on U_N. Note G/N has no a permutation representation of degree 3. It follows that G/N is also 2-transitive on W_N. Then, it is easily shown that Σ is isomorphic to the complete bipartite graph $\mathsf{K}_{9,9}$ with a complete matching deleted; in particular, Σ has non-prime valency 8, a contradiction.

Case 2. Assume that $|N| = 3$ and $p > 3$. In this case G/N is a transitive permutation group of degree p^2 on both U_N and W_N.

Subcase 2.1. Suppose first that G/N is not qusiprimitive on either U_N or W_N. Then, by Lemma 3.4 and Theorem 3.3, G/N has a minimal normal subgroup, says M/N, which is intransitive and semiregular on both U_N and W_N. Then M/N has order p. Then M is normal in G and of order $3p$. Since $p > 3$, we know that $M \cong \mathbb{Z}_{3p}$. Clearly M is intransitive on both U and W. By [14, Lemma 5.1], Γ is a normal cover of Γ_M which has order $2p$. Let U_M and W_M be the M-orbits on U and W, respectively.

Assume that G/M is faithful on one of U_M and W_M. Then G/M is listed in Table 4.1; in particular, p^2 is not a divisor of $|G/M|$. Let R/M be a Sylow subgroup of G/M. Then $|R/M| = p$ and it is easily shown that R/M is transitive on both U_M and W_M. Thus R has order $3p^2$ and is regular on both U and W. Then Γ is a bi-Cayley graph of R. Recalling that $N \lhd R$ and N has order 3, it follows that N

lies in the center of R. Noting that $R/N \cong \mathbb{Z}_p^2$ or $\mathbb{Z}_{p^2}$, it follows that $R \cong \mathbb{Z}_3 \times \mathbb{Z}_p^2$ or $\mathbb{Z}_3 \times \mathbb{Z}_{p^2}$. Thus, R is an abelian group.

Assume that G/M is unfaithful on both U_M and W_M. Then $\mathit{\Gamma}_M \cong \mathsf{K}_{p,p}$ and $G/M \le \mathsf{Aut}\mathsf{K}_{p,p} = (\mathrm{S}_p \times \mathrm{S}_p){:}\mathbb{Z}_2$, hence $G/M \le \mathrm{S}_p \times \mathrm{S}_p$ as G/M is intransitive on $V\mathit{\Gamma}_M$. It follows that G/M has a subgroup of order p, says R/M, which is transitive on both U_M and W_M. Then arguing similarly as above implies that $\mathit{\Gamma}$ is a bi-Cayley graph of an abelian group of order $3p^2$.

Subcase 2.2. Thus, without loss of generality, we may assume that G/N is qusiprimitive on U_N. Then G/N is primitive on U_N by Lemma 4.7, hence G/N is known as in Theorem 4.5. Thus, by Lemma 4.3, a Sylow p-subgroup of G/N must have order p^2. Let R/N be a Sylow p-subgroup of G/N. It is easily shown that R/N is transitive on both U_N and W_N. Then a similar argument as above implies that $\mathit{\Gamma}$ is a bi-Cayley graph of an abelian group of order $3p^2$.

Case 3. Assume $|N| = p > 3$. In this case G/N is a transitive permutation group of degree $3p$ on both U_N and W_N.

Subcase 3.1. Suppose that G/N is not qusiprimitive on either U_N or W_N. Then, by Lemma 3.4 and Theorem 3.3, G/N has a minimal normal subgroup, says M/N, which is intransitive and semiregular on both U_N and W_N. Then M/N has order 3 or p. In particular, M is intransitive on both U and N. Then, by [14], $\mathit{\Gamma}$ is a normal cover of $\mathit{\Gamma}_M$; in particular, $\mathit{\Gamma}$ and $\mathit{\Gamma}_M$ have the same valency k.

If $|M/N| = p$ then M has order p^2, and the lemma follows. Thus we assume that $|M| = 3p$. If M is cyclic then M has a unique Sylow 3-subgroup which has order 3 and is normal in G, so the lemma holds by the above case. Thus, we assume further M is not cyclic; in particular, $p \equiv 1 \,(\mathsf{mod}\ 3)$.

Assume that G/M is faithful on one of U_M and W_M, where U_M and W_M denote the M-orbits on U and W, respectively. Then G/M is known as in Table 4.1, $\mathsf{soc}(G/M) = \mathbb{Z}_p$, A_p or $\mathrm{PSL}(d,q)$ (with $p = \frac{q^d-1}{q-1}$, yielding d a prime) as $p \equiv 1 \,(\mathsf{mod}\ 3)$.

Suppose that $\mathsf{soc}(G/M) = \mathrm{A}_p$ or $\mathrm{PSL}(d,q)$. Then it is easily shown that $\mathit{\Gamma}$ is either $\mathsf{K}_{p,p}$ with a complete matching deleted or the point-hyperplane incidence graph of the projective geometry $\mathrm{PG}(d-1,q)$. For the former case, $\mathit{\Gamma}$ has degree $p-1$ which is not a prime, a contradiction. For the latter case, $\mathit{\Gamma}$ has valency $k = \frac{q^{d-1}-1}{q-1}$. Since k is an odd prime, $d-1$ must be a prime. So $d = 3$, and $q = 2^{2^s}$ for $s \ge 0$. Since $p = q^2 + q + 1$ and 3 divides $p-1$, we get $q = 2$, $p = 7$ and $G/M = \mathrm{PSL}(3,2)$. In particular, $k = 3$. Then G has a normal subgroup of order $p^2 = 49$ by [20]; however, $G = (\mathbb{Z}_7{:}\mathbb{Z}_3).\mathrm{PSL}(3,2)$ has no such a normal subgroup, a contradiction.

Therefore, $\mathsf{soc}(G/M) = \mathbb{Z}_p$. Then $G/M \le \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$. Let R/M be the Sylow p-subgroup of G/M. Then $R/M \lhd G/M$ and so $R \lhd G$. Since $p > 3$, we know that R has a unique Sylow p-subgroup P, which has order p^2. Then $P \lhd R \lhd G$, and so P is normal in G as P is also a Sylow p-subgroup of G.

Assume that G/M is unfaithful on both of U_M and W_M. Then $\mathit{\Gamma}_M \cong \mathsf{K}_{p,p}$ and $G/M \le \mathrm{S}_p \times \mathrm{S}_p$ as G/M is faithful on $V\mathit{\Gamma}_M$. Let H/M be a Sylow p-subgroup of G/M. Then $H/M \cong \mathbb{Z}_p^2$ and $\mathit{\Gamma}_M$ is (H/M)-semisymmetric. Thus $\mathit{\Gamma}$ is H-semisymmetric, and hence $\mathit{\Gamma}_N$ is (H/N)-semisymmetric. Moreover, $\mathit{\Gamma}$, $\mathit{\Gamma}_N$ and $\mathit{\Gamma}_M$ has the same valency $k = p$.

It is easily shown that H/M has a subgroup R/M of order p which is regular on both parts of $\mathit{\Gamma}_M$. Then $R \lhd H$, $\bar{R} := R/N \lhd H/N := \bar{H}$, and R is regular on both U and W, which yields that $\bar{R}$ is regular on both parts of $\mathit{\Gamma}_N$. Thus $\mathit{\Gamma}_N$ is a bi-Cayley graph of $\bar{R}$. Let α be a vertex of $\mathit{\Gamma}_N$. Then $\bar{H}_\alpha \cong \mathbb{Z}_p$. Let $\bar{H} = \langle \sigma \rangle$ and β be a neighbor of α. Then, by Lemma 3.1, there are $t \in \bar{R}$ and $\tau \in \mathsf{Aut}(\bar{R})$ such that $(\beta^x)^\sigma = (\beta^t)^{x^\tau}$ for all $x \in \bar{R}$. Thus the neighborhood of α is $\{\beta^s \mid s \in S\}$, where $S = \{t, tt^\tau, tt^\tau t^{\tau^2}, \ldots, tt^\tau t^{\tau^2} \cdots t^{\tau^{p-1}} = 1\}$.

Recalling that $\bar{R}$ has a normal subgroup M/N of order 3, it follows that $\bar{R}$ is a cyclic group of order $3p$. Since $\tau \in \mathsf{Aut}(\bar{R})$, there is a positive integer i such that $t^\tau = t^i$. It follows from $tt^\tau t^{\tau^2} \cdots t^{\tau^{p-1}} = 1$ that $i^{p-1} + \cdots + i + 1 \equiv 0 \,(\mathsf{mod}\ 3p)$, so $i^p \equiv 1 \,(\mathsf{mod}\ 3p)$. On the other hand, since $\mathit{\Gamma}_N$ is connected, $\bar{R} = \langle S \rangle = \langle t \rangle$. Thus τ is an automorphism of $\bar{R} \cong \mathbb{Z}_{3p}$ of order dividing p. Since $\mathsf{Aut}(\bar{R}) \cong \mathbb{Z}_2 \times \mathbb{Z}_{p-1}$, we have $\tau = 1$. Then $i \equiv 1 \,(\mathsf{mod}\ 3p)$, and so $p \equiv i^{p-1} + \cdots + i + 1 \equiv 0 \,(\mathsf{mod}\ 3p)$, a contradiction.

Subcase 3.2. Thus, without loss of generality, we may assume that G/N is qusiprimitive on U_N. Then G/N is known as in Table 4.2 or (1)-(5) of Theorem 4.9; in particular, G/N is almost simple. Suppose that $\mathsf{soc}(G/N)$ is intransitive on W_N. Then it has 3 orbits with size p on W_N, and so $\mathit{\Gamma}$ has valency $k = 3$. By [20], G has a normal subgroup of order p^2. It follows that G/N has solvable normal subgroup, which is impossible as G/N is almost simple. Thus $\mathsf{soc}(G/N)$ is transitive on W_N. Since $\mathit{\Gamma}_N$ has prime valency k, then $\mathit{\Gamma}_N$ is $\mathsf{soc}(G/N)$-semisymmetric. Let $\alpha \in V\mathit{\Gamma}_N$. If $\mathsf{soc}(G/N)_\alpha$ is a $\{2,3\}$ group, then $\mathit{\Gamma}_N$ must have valency 2 or 3, which is not the case. Thus $\mathsf{soc}(G/N)_\alpha$ is not a $\{2,3\}$-group. Then, by Theorems 4.4 and 4.9, $\mathsf{soc}(G/N)$ is listed in the following table.

Line	$3p$	$\mathsf{soc}(G/N)$	$\mathsf{soc}(G/N)_\alpha$ or actions
1	$3p$	A_{3p}	A_{3p-1}
2	21	A_7	S_5
3	57	$\mathrm{PSL}(2,19)$	A_5, two actions
4	15	A_7	$\mathrm{PSL}(2,7)$, two actions
5	15	$\mathrm{PSL}(4,2)$	$\mathbb{Z}_2^3{:}\mathrm{PSL}(3,2)$, two actions
6	2^e+1	$\mathrm{PSL}(2,2^e)$	$\mathbb{Z}_2^e{:}\mathbb{Z}_{2^e-1}$, e odd prime
7	q^2+q+1	$\mathrm{PSL}(3,q)$	on points or hyperplanes, $q \equiv 1 \,(\mathsf{mod}\ 3)$
8	$3(1+2^{2^s})$	$\mathrm{PSL}(2,2^{2^s})$	$\mathbb{Z}_2^{2^s}{:}\mathbb{Z}_{\frac{2^{2^s}-1}{3}}$
9	$3\frac{q^d-1}{q-1}$	$\mathrm{PSL}(d-1,q)$	$[q^{d-1}].\mathbb{Z}_{(q-1,d-1)}.\mathrm{PSL}(d-1,q).\mathbb{Z}_{\frac{q-1}{3(q-1,d)}}$ $q \equiv 1 \,(\mathsf{mod}\ 3)$

We set $\mathsf{soc}(G/N) = M/N$. Then M is a central extension of N by $soc(G/N)$. Let T be the derived subgroup of M. Suppose that $T = M$. Then $N = \mathbb{Z}_p$ is (isomorphic to) a subgroup of the Schur multiplier of $\mathsf{soc}(G/N)$. By [17, Theorem 5.1.4], the Schur multiplier of $\mathsf{soc}(G/N)$ is one of 1, $\mathbb{Z}_2$, $\mathbb{Z}_3$ (for lines 1-8) and $\mathbb{Z}_{(q-1,d)}$ (for line 9); however, they all have order less than p, a contradiction. It follows that $T \neq M$, and hence $T \cong M/N$ and $M = N \times T$.

Suppose that T is as in line 1 of the above table. Then Γ_N is the complete bipartite graph $\mathsf{K}_{3p,3p}$ with a complete matching deleted. So Γ_N and hence Γ has valency $3p-1$, which is not a prime, a contradiction.

Thus we assume that T is one of the simple groups from line 2 to line 9. Clearly, for each of lines 2-5, $|T|$ is not divisible by p^2. For each of lines 6-9, by Lemma 4.3, we conclude that p^2 is not a divisor of $|T|$. Then T, as a normal subgroup of M, is intransitive on both U and W. Recalling that Γ_N is $\mathsf{soc}(G/N)$-semisymmetric and Γ is a normal cover of Γ_N, we conclude that Γ is M-semisymmetric. Thus, by Theorem 3.3 and Lemma 3.2, T is semiregular on both U and W, so $|T|$ is a divisor of $3p^2$, a contradiction. This complete the proof. ∎

Now we are ready to give a proof of Theorem 1.2.

Proof of Theorem 1.2 Let Γ be a connected semisymmetric graph, with bipartition $V\Gamma = U \cup W$, of order $6p^2$ and valency k for odd primes p and k. Let $G = \mathsf{Aut}\Gamma$. Then, by Lemma 5.2, either Γ is a bi-Cayley graph of an abelian group or G contains a normal subgroup of order p^2. By [11], the former case implies that Γ is vertex transitive, which is not the case. Thus we assume that G has a normal subgroup N of order p^2. Clearly, N is intransitive on both U and W. Note that $3p^2$ is not a prime. Thus $\Gamma \not\cong \mathsf{K}_{3p^2,3p^2}$. It follows from [14, Lemma 5.1] that Γ is a normal cover of Γ_N; in particular, N is semiregular. Then $\Gamma_N \cong \mathsf{K}_{3,3}$, and so Γ is of valency 3. Thus Theorem 1.2 follows from Theorem 2.1.

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