Brändén's Conjectures on the Boros-Moll Polynomials

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Abstract. We prove two conjectures of Brändén on the real-rootedness of the polynomials $Q_n(x)$ and $R_n(x)$ which are related to the Boros-Moll polynomials $P_n(x)$. In fact, we show that both $Q_n(x)$ and $R_n(x)$ form Sturm sequences. The first conjecture implies the 2-log-concavity of $P_n(x)$, and the second conjecture implies the 3-log-concavity of $P_n(x)$.

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1 Introduction

In this paper, we prove two conjectures of Brändén [4] concerning the Boros-Moll polynomials. Brändén introduced two polynomials based on the coefficients of the Boros-Moll polynomials and conjectured that these polynomials have only real roots. As pointed out by Brändén, the first conjecture implies the 2-fold log-concavity, or 2-log-concavity, for short, of the Boros-Moll polynomials, whereas the second conjecture implies the 3-log-concavity.

Let us start with some definitions. Given a finite nonnegative sequence $\{a_i\}_{i=0}^n$, we say that it is unimodal if there exists an integer $m \geq 0$ such that

$$a_0 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots \geq a_n,$$

and we say that it is log-concave if

$$a_i^2 - a_{i+1}a_{i-1} \geq 0$$

for $1 \leq i \leq n-1$. Define $\mathcal{L}$ to be an operator acting on the sequence $\{a_i\}_{i=0}^n$ as given by

$$\mathcal{L}(\{a_i\}_{i=0}^n) = \{b_i\}_{i=0}^n,$$

where $b_i = a_i^2 - a_{i+1}a_{i-1}$ for $0 \leq i \leq n$ under the convention that $a_{-1} = 0$ and $a_{n+1} = 0$. Clearly, the sequence $\{a_i\}_{i=0}^n$ is log-concave if and only if the sequence $\{b_i\}_{i=0}^n$ is nonnegative. Given a sequence $\{a_i\}_{i=0}^n$, we say that it is

k-fold log-concave, or k-log-concave, if $\mathcal{L}^j(\{a_i\}_{i=0}^n)$ is a nonnegative sequence for any $1 \leq j \leq k$. A sequence $\{a_i\}_{i=0}^n$ is said to be infinitely log-concave if it is k-log-concave for all $k \geq 1$. Given a polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

we say that $f(x)$ is log-concave (or k-log-concave, or infinitely log-concave) if the sequence $\{a_i\}_{i=0}^n$ is log-concave (resp., k-log-concave, infinitely log-concave). Throughout this paper, we shall be concerned with polynomials with real coefficients.

The notion of infinite log-concavity was introduced by Boros and Moll [3] in their study of the following quartic integral

$$\int_0^\infty \frac{1}{(t^4 + 2xt^2 + 1)^{n+1}} \mathrm{d}t.$$

For any $x > -1$ and any nonnegative integer n, they obtained the following formula,

$$\int_0^\infty \frac{1}{(t^4 + 2xt^2 + 1)^{n+1}} \mathrm{d}t = \frac{\pi}{2^{n+3/2}(x+1)^{n+1/2}} P_n(x),$$

where

$$P_n(x) = \sum_{j,k} \binom{2n+1}{2j} \binom{n-j}{k} \binom{2k+2j}{k+j} \frac{(x+1)^j (x-1)^k}{2^{3(k+j)}}$$

are the Boros-Moll polynomials. Using Ramanujan's Master Theorem, they derived an alternative expression of $P_n(x)$,

$$P_n(x) = 2^{-2n} \sum_j 2^j \binom{2n-2j}{n-j} \binom{n+j}{j} (x+1)^j. \tag{1.1}$$

For other proofs of (1.1), see Amdeberhan and Moll [1]. Write

$$P_n(x) = \sum_{i=0}^n d_i(n) x^i. \tag{1.2}$$

We call $\{d_i(n)\}_{i=0}^n$ a Boros-Moll sequence.

The log-concavity of $\{d_i(n)\}_{i=0}^n$ was conjectured by Moll [17], and it was proved by Kauers and Paule [13] by establishing the following recurrence relations of $d_i(n)$:

$$d_i(n+1) = \frac{n+i}{n+1} d_{i-1}(n) + \frac{4n+2i+3}{2(n+1)} d_i(n), \quad 0 \le i \le n+1, \tag{1.3}$$

$$\begin{aligned} d_i(n+1) = {} & \frac{(4n-2i+3)(n+i+1)}{2(n+1)(n+1-i)} d_i(n) \\ & - \frac{i(i+1)}{(n+1)(n+1-i)} d_{i+1}(n), \quad 0 \le i \le n, \end{aligned} \tag{1.4}$$

$$\begin{aligned} d_i(n+2) = {} & \frac{8n^2+24n+19-4i^2}{2(n+2-i)(n+2)} d_i(n+1) \\ & - \frac{(n+i+1)(4n+3)(4n+5)}{4(n+2-i)(n+1)(n+2)} d_i(n), \quad 0 \le i \le n+1, \end{aligned} \tag{1.5}$$

$$\begin{aligned} d_{i-2}(n) = {} & \frac{(i-1)(2n+1)}{(n+2-i)(n+i-1)} d_{i-1}(n) \\ & - \frac{i(i-1)}{(n+2-i)(n+i-1)} d_i(n), \quad 0 \le i \le n. \end{aligned} \tag{1.6}$$

In fact, (1.5) and (1.6) can be derived from (1.3) and (1.4). Note that Moll [18] independently derived the relation (1.5) and (1.6) via the WZ-method.

Chen and Xia [7] showed that the polynomials $P_n(x)$ are ratio monotone. A sequence of positive real numbers $\{a_i\}_{0\le i\le n}$ is said to be ratio monotone if

$$\frac{a_n}{a_0} \le \frac{a_{n-1}}{a_1} \le \cdots \le \frac{a_{n-i}}{a_i} \le \cdots \le \frac{a_{n-[\frac{n-1}{2}]}}{a_{[\frac{n-1}{2}]}} \le 1 \tag{1.7}$$

and

$$\frac{a_0}{a_{n-1}} \le \frac{a_1}{a_{n-2}} \le \cdots \le \frac{a_{i-1}}{a_{n-i}} \le \cdots \le \frac{a_{[\frac{n}{2}]-1}}{a_{n-[\frac{n}{2}]}} \le 1. \tag{1.8}$$

Notice that for a positive sequence, the ratio monotone property implies both log-concavity and the spiral property. It is worth mentioning that there are approaches to proving log-concavity without using recurrence relations. Llamas and Martínez-Bernal [15] proved that if $f(x)$ is a polynomial with nondecreasing and nonnegative coefficients, then $f(x+1)$ is log-concave. Furthermore, Chen, Yang and Zhou [9] proved that if $f(x)$ is a polynomial with nondecreasing and nonnegative coefficients, then $f(x+1)$ is ratio monotone. From (1.1) it is easily seen that the coefficients of $P_n(x-1)$ are nondecreasing and nonnegative. Hence $P_n(x)$ are log-concave and ratio monotone. A combinatorial interpretation of the log-concavity of $P_n(x)$ has been found by Chen, Pang and Qu [6].

Boros and Moll [3] also proposed the following conjecture.

Conjecture 1.1 *The sequence* $\{d_i(n)\}_{i=0}^{n}$ *is infinitely log-concave.*

The infinite log-concavity of the Boros-Moll polynomials seems to be a difficult problem. As remarked by Kauers and Paule [13], it seems that there is little hope to prove the 2-log-concavity of $\{d_i(n)\}_{i=0}^n$ using recurrence relations. By constructing an intermediate function, Chen and Xia [8] proved the 2-log-concavity of $P_n(x)$ by applying recurrence relations. Based on a technique of McNamara and Sagan [16], Kauers verified the infinite log-concavity of $P_n(x)$ for $n \leq 129$.

Brändén [4] presented an approach to Conjecture 1.1 by relating higher-order log-concavity to real-rooted polynomials. Boros and Moll [3] conjectured that for any nonnegative integer n the sequence $\{\binom{n}{k}\}_{k=0}^n$ is infinitely log-concave. Fisk [12], McNamara and Sagan [16] and Stanley independently made the following conjecture which implies the conjecture of Boros and Moll. This conjecture has been proved by Brändén [4].

Theorem 1.2 *If $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is a real-rooted polynomial with nonnegative coefficients, the polynomial*

$$a_0^2 + (a_1^2 - a_0a_2)x + \cdots + (a_{n-1}^2 - a_{n-2}a_n)x^{n-1} + a_n^2x^n$$

is also real-rooted.

Brändén's proof is based on a symmetric function identity and the Grace-Walsh-Szegö theorem concerning the location of zeros of multi-affine and symmetric polynomials. Moreover, Brändén obtained a general result on the characterization of nonlinear transformations preserving real-rootedness, in the spirit of the characterization of linear transformations preserving stability given by Borcea and Brändén [2]. From the viewpoint of total positivity, Cardon and Nielsen [5] proposed a conjecture that implies Theorem 1.2. Although the Boros-Moll polynomials $P_n(x)$ are not real-rooted, Brändén [4] introduced two polynomials related to $P_n(x)$, and conjectured that they are real-rooted.

Conjecture 1.3 ([4, Conjecture 8.5]) *For any $n \geq 1$, the polynomial*

$$Q_n(x) = \sum_{i=0}^{n} \frac{d_i(n)}{i!} x^i \tag{1.9}$$

has only real zeros.

Conjecture 1.4 ([4, Conjecture 8.6]) *For any $n \geq 1$, the polynomial*

$$R_n(x) = \sum_{i=0}^{n} \frac{d_i(n)}{(i+2)!} x^i \tag{1.10}$$

has only real zeros.

As pointed out by Brändén [4], by Craven and Csordas's results on iterated Turán inequalities obtained in [10], the real-rootedness of $Q_n(x)$ implies the 2-log-concavity of $P_n(x)$, and the real-rootedness of $R_n(x)$ implies the 3-log-concavity of $P_n(x)$. Brändén's approach suggests that it might be possible to prove the k-log-concavity of $P_n(x)$ for $k \geq 4$ by using the higher iterated Turán inequalities for real entire functions in the Laguerre-Pólya class. However, little is known about the k-th iterated Turán inequalities when $k \geq 4$. It is worth mentioning that Csordas [11] proved the real-rootedness of some polynomials related to $Q_n(x)$.

In this paper, we shall prove the above conjectures by showing that the polynomials $Q_n(x)$ and $R_n(x)$ form Sturm sequences. We say that a polynomial is standard if it is zero or its leading coefficient is positive. Let RZ denote the set of polynomials with only real zeros. Suppose that $f(x) \in \mathrm{RZ}$ is a polynomial of degree n with zeros $\{r_k\}_{k=1}^n$, and $g(x) \in \mathrm{RZ}$ is a polynomial of degree m with zeros $\{s_k\}_{k=1}^m$. We say that $g(x)$ interlaces $f(x)$ if $n = m+1$ and

$$r_n \leq s_{n-1} \leq r_{n-1} \leq \cdots \leq r_2 \leq s_1 \leq r_1,$$

and we say that $g(x)$ strictly interlaces $f(x)$ if, in addition, they have no common zeros. We use $g(x) \preceq f(x)$ to denote that $g(x)$ interlaces $f(x)$, and use $g(x) \prec f(x)$ to denote that $g(x)$ strictly interlaces $f(x)$. For any real numbers a, b and c, we assume that $a \in \mathrm{RZ}$ and $a \prec bx + c$. A sequence $\{f_n(x)\}_{n\geq 0}$ of standard polynomials is said to be a Sturm sequence if, for $n \geq 0$, we have $\deg f_n(x) = n$ and

$$f_n(x) \in \mathrm{RZ} \text{ and } f_n(x) \prec f_{n+1}(x).$$

To prove that $Q_n(x)$ and $R_n(x)$ are Sturm sequences, we shall use the following sufficient condition, due to Liu and Wang [14], for a polynomial sequence $\{f_n(x)\}_{n\geq 0}$ to form an interlacing sequence.

Theorem 1.5 ([14, Corollary 2.4]) *Let $\{f_n(x)\}_{n\geq 0}$ be a sequence of polynomials with nonnegative coefficients and $\deg f_n(x) = n$, which satisfy the following recurrence relation:*

$$f_{n+1}(x) = a_n(x)f_n(x) + b_n(x)f_n'(x) + c_n(x)f_{n-1}(x), \tag{1.11}$$

where $a_n(x), b_n(x), c_n(x)$ are some polynomials with real coefficients. Assume that, for some $n \geq 1$, the following conditions hold:

(i) $f_{n-1}(x), f_n(x) \in \mathrm{RZ}$ and $f_{n-1}(x) \prec f_n(x)$; and

(ii) for any $x \leq 0$ both of $b_n(x)$ and $c_n(x)$ are nonpositive, and at least one of them is nonzero.

Then we have $f_{n+1}(x) \in \mathrm{RZ}$ and $f_n(x) \prec f_{n+1}(x)$.

2 Proofs of Brändén's Conjectures

We first derive recurrence relations for $Q_n(x)$ and $R_n(x)$ based on the recurrence relations (1.3) and (1.5) of the coefficients $d_i(n)$ of the Boros-Moll polynomials $P_n(x)$.

Theorem 2.1 *For $n \geq 1$, we have the following recurrence relation*

$$\begin{aligned} Q_{n+1}(x) = &\left(\frac{(2n+1)x}{(n+1)^2} + \frac{8n^2+8n+3}{2(n+1)^2}\right) Q_n(x) \\ &- \frac{(4n-1)(4n+1)}{4(n+1)^2} Q_{n-1}(x) + \frac{x}{(n+1)^2} Q_n'(x). \end{aligned} \tag{2.1}$$

Proof. For $n \geq 1$, relation (2.1) can be rewritten as

$$\begin{aligned} 4(n+1)^2 d_i(n+1) = &\, 2(8n^2+8n+3+2i)d_i(n) + 4i(2n+1)d_{i-1}(n) \\ &- (16n^2-1)d_i(n-1), \end{aligned} \tag{2.2}$$

where $0 \leq i \leq n+1$. From (1.3) it follows that

$$d_{i-1}(n) = \frac{n+1}{n+i} d_i(n+1) - \frac{4n+2i+3}{2(n+i)} d_i(n). \tag{2.3}$$

Substituting (2.3) into (2.2), we get

$$\begin{aligned} d_i(n+1) = &\frac{8n^2+8n+3-4i^2}{2(n+1-i)(n+1)} d_i(n) \\ &- \frac{(n+i)(4n-1)(4n+1)}{4n(n+1)(n+1-i)} d_i(n-1). \end{aligned} \tag{2.4}$$

It is easily checked that the above relation (2.4) coincides with (1.5) with n replaced by $n-1$. This completes the proof. ∎

Using the above recurrence relation and the criterion of Liu and Wang, we can deduce that the polynomials $Q_n(x)$ form a Sturm sequence. This leads to an affirmative answer to Conjecture 1.3.

Theorem 2.2 *The polynomial sequence $\{Q_n(x)\}_{n\geq 0}$ is a Sturm sequence.*

Proof. Clearly, we have $\deg(Q_n(x)) = n$. It suffices to prove that $Q_n(x) \in \mathrm{RZ}$ and $Q_n(x) \prec Q_{n+1}(x)$ for any $n \geq 0$. We use induction on n. By convention,

$$Q_0(x), Q_1(x) \in \mathrm{RZ} \quad \text{and} \quad Q_0(x) \prec Q_1(x).$$

Assume that

$$Q_{n-1}(x), Q_n(x) \in \mathrm{RZ} \quad \text{and} \quad Q_{n-1}(x) \prec Q_n(x).$$

We proceed to verify that

$$Q_{n+1}(x) \in \mathrm{RZ} \quad \text{and} \quad Q_n(x) \prec Q_{n+1}(x).$$

We see that the recurrence relation (2.1) of $Q_n(x)$ is of the form (1.11) in Theorem 1.5, where the polynomials $a_n(x), b_n(x), c_n(x)$ are given by

$$\begin{aligned}
a_n(x) &= \frac{(2n+1)x}{(n+1)^2} + \frac{8n^2+8n+3}{2(n+1)^2}, \\
b_n(x) &= \frac{x}{(n+1)^2}, \\
c_n(x) &= -\frac{(4n-1)(4n+1)}{4(n+1)^2}.
\end{aligned}$$

For $n \geq 1$ and $x \leq 0$, one can check that

$$b_n(x) \leq 0 \quad \text{and} \quad c_n(x) < 0.$$

In view of Theorem 1.5, we find that $Q_{n+1}(x) \in \mathrm{RZ}$ and $Q_n(x) \prec Q_{n+1}(x)$. This completes the proof. ∎

The following recurrence relation for $R_n(x)$ can be proved in a way similar to the proof of Theorem 2.1.

Theorem 2.3 *For $n \geq 1$, we have*

$$\begin{aligned}
R_{n+1}(x) = &\left(\frac{(2n+1)x}{(n+1)(n+3)} + \frac{8n^2+8n+7}{2(n+1)(n+3)} \right) R_n(x) \\
&- \frac{(4n-1)(4n+1)(n-2)}{4n(n+1)(n+3)} R_{n-1}(x) + \frac{5x}{(n+1)(n+3)} R_n'(x).
\end{aligned} \tag{2.5}$$

Using the above recurrence relation, we obtain the following theorem, which leads to an affirmative answer to Conjecture 1.4.

Theorem 2.4 *The polynomial sequence $\{R_n(x)\}_{n \geq 0}$ is a Sturm sequence.*

Proof. The proof is analogous to that of Theorem 2.2. It is routine to verify that

$$R_0(x), R_1(x), R_2(x), R_3(x) \in \mathrm{RZ} \quad \text{and} \quad R_0(x) \prec R_1(x) \prec R_2(x) \prec R_3(x).$$

It remains to show that $R_n(x) \in \mathrm{RZ}$ and $R_{n-1}(x) \prec R_n(x)$ for $n \geq 3$. We use induction n. Assume that

$$R_{n-1}(x), R_n(x) \in \mathrm{RZ} \quad \text{and} \quad R_{n-1}(x) \prec R_n(x).$$

We wish to prove that

$$R_{n+1}(x) \in \mathrm{RZ} \quad \text{and} \quad R_n(x) \prec R_{n+1}(x).$$

The recurrence relation (2.5) of $R_n(x)$ is of the form (1.11) in Theorem 1.5, and the polynomials $a_n(x), b_n(x), c_n(x)$ are given by

$$\begin{aligned} a_n(x) &= \frac{(2n+1)x}{(n+1)(n+3)} + \frac{8n^2+8n+7}{2(n+1)(n+3)}, \\ b_n(x) &= \frac{5x}{(n+1)(n+3)}, \\ c_n(x) &= -\frac{(4n-1)(4n+1)(n-2)}{4n(n+1)(n+3)}. \end{aligned}$$

For $n \geq 3$ and $x \leq 0$, we find that

$$b_n(x) \leq 0 \quad \text{and} \quad c_n(x) < 0.$$

By Theorem 1.5, we conclude that $R_{n+1}(x) \in \mathrm{RZ}$ and $R_n(x) \prec R_{n+1}(x)$. This completes the proof. ∎

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