Rainbow connection number, bridges and radius[1]

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Abstract

Let G be a connected graph. The notion of rainbow connection number $rc(G)$ of a graph G was introduced by Chartrand et al. Basavaraju et al. proved that for every bridgeless graph G with radius r, $rc(G) \leq r(r+2)$ and the bound is tight. In this paper, we show that for a connected graph G with radius r and center vertex u, if we let $D^r = \{u\}$, then G has $r-1$ connected dominating sets $D^{r-1}, D^{r-2}, \cdots, D^1$ such that $D^r \subset D^{r-1} \subset D^{r-2} \cdots \subset D^1 \subset D^0 = V(G)$ and $rc(G) \leq \sum_{i=1}^{r} \max\{2i+1, b_i\}$, where b_i is the number of bridges in $E[D^i, N(D^i)]$ for $1 \leq i \leq r$. From the result, we can get that if $b_i \leq 2i+1$ for all $1 \leq i \leq r$, then $rc(G) \leq \sum_{i=1}^{r}(2i+1) = r(r+2)$; if $b_i > 2i+1$ for all $1 \leq i \leq r$, then $rc(G) = \sum_{i=1}^{r} b_i$, the number of bridges of G. This generalizes the result of Basavaraju et al. In addition, an example is given to show that there exist infinitely graphs with bridges whose $rc(G)$ is only dependent on the radius of G, and another example is given to show that there exist infinitely graphs with bridges whose $rc(G)$ is only dependent on the number of bridges in G.

Keywords: edge-colored graph, rainbow connection number, bridge, radius.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. Let $G = (V(G), E(G))$ be a graph. For two subsets X and Y of $V(G)$, an (X, Y)-path is a path which connects a vertex of X and a vertex of Y and whose internal vertices belong to neither X nor Y. We use $E[X, Y]$ to denote the set of edges of G with one end in X and the other end in Y and use $e(X, Y)$ to denote $|E[X, Y]|$. The subgraph $G[Y]$ of G induced by Y is the graph with vertex set Y and edge set consisting of the edges of G with both ends

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in Y. The *distance between two vertices* u and v in G, denoted by $d(u,v)$, is the length of a shortest path between them in G, and the *distance between a vertex* u *and a set* $S \subseteq V(G)$ is defined as $d(u,S) = \min\{d(u,x)|x \in S\}$. The *eccentricity* of a vertex v in G is defined as $ecc(v) := \max_{x\in V(G)} d(v,x)$. The *radius* of G is defined as $rad(G) = \min_{x\in V(G)} ecc(x)$, and the *diameter* of G is defined as $diam(G) = \max_{x\in V(G)} ecc(x)$. The k*-step open neighborhood* of S is $N^k(S) = \{v \in V(G)|d(v,S) = k, k \in Z, k \geq 0\}$. Generally speaking, $N^1(S) = N(S), N^0(S) = S$.

Let $c : E(G) \rightarrow \{1, 2, \cdots, k\}, k \in N$ be an edge-coloring, where adjacent edges may be colored the same. A graph G is called *rainbow connected* if every two vertices are connected by a path whose edges have different colors. The *rainbow connection number* of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. These concepts were introduced by Chartrand et al. in [4], where they determined the rainbow connection numbers of wheels, complete graphs and all complete multipartite graphs. Many upper bounds for the rainbow connection number were obtained, involving other graph parameters. Results involving the minimum degree were obtained in [3, 8, 7, 5], whereas results involving the parameters σ_2 and σ_k were obtained in [9, 6].

In [1], Basavaraju et al. proved that for every bridgeless graph G with radius r, $rc(G) \leq r(r + 2)$ and the bound is tight. However, we know that bridges are important objects in a rainbow coloring since every two bridges in a rainbow coloring must receive different colors. If G is a graph with $b \geq 1$ bridges, then $rc(G) \geq b$, and from their result in [1], we can only say that $rc(G) \leq r(r+2)+b$. On the other hand, to save colors, one may use the colors appeared on bridges to color some 2-connected blocks. As a consequence, the upper bound $r(r + 2) + b$ could be far from a good bound. In fact, in the following we give an example to show that there exist infinitely graphs with bridges whose $rc(G)$ is only dependent on the radius of G but independent of the number of bridges of G, and we also give another example to show that there exist infinitely graphs with bridges whose $rc(G)$ is only dependent on the number of bridges G but independent of the radius of G. This paper is to study the rainbow connection number of a graph with bridges. The main difference of our proof technique different from that in [1] is to divide the bridges into suitable classes, and then to color these bridges and the remaining non-bridge edges, so that we can get a new upper bound of $rc(G)$ better than $r(r + 2) + b$.

Theorem 1 *Let G be a connected graph with radius r and center vertex u. If we let $D^r = \{u\}$, then G has $r - 1$ connected dominating sets $D^{r-1}, D^{r-2}, \cdots, D^1$ such that $D^r \subset D^{r-1} \subset D^{r-2} \cdots \subset D^1 \subset D^0 = V(G)$ and $rc(G) \leq \sum_{i=1}^{r} \max\{2i + 1, b_i\}$, where b_i is the number of bridges in $E[D^i, N(D^i)]$ for $1 \leq i \leq r$, and the number of bridges in G is equal to $\sum_{i=1}^{r} b_i$.*

Note that if $b_i \leq 2i+1$ for all $1 \leq i \leq r$, then $rc(G) \leq \sum_{i=1}^{r}(2i+1) = r(r+2)$, which is independent of the number of bridges in G, whereas if $b_i > 2i+1$ for all $1 \leq i \leq r$, then $rc(G) = \sum_{i=1}^{r} b_i$, the number of bridges of G. This substantially generalizes the result of Basavaraju et al.

In the following we give two examples, in which Example 1 constructs a graph G with $rad(G) = r$, such that the number of bridges is not more than $r(r+2)$ and $rc(G) = r(r+2)$, whereas Example 2 constructs a graph G with $rad(G) = r$, such that the number of bridges b is at least $r(r+2)+1$ and $rc(G) = b$.

Example 1. Let $C_{1,1}, C_{1,2}, \cdots, C_{1,r}$ be r cycles such that for $1 \leq i, j \leq r$ with $|i-j| \geq 2$, we have $|C_{1,i}| = 2(r+1-i)+1$ and $|V(C_{1,i}) \cap V(C_{1,j})| = \emptyset$; for $1 \leq i \leq r-1$, $|V(C_{1,i}) \cap V(C_{1,i+1})| = 1$ and let $V(C_{1,i}) \cap V(C_{1,i+1}) = \{u_{1,i+1}\}$, $u_{1,1} \in V(C_{1,1})$ and $u_{1,1}u_{1,2} \in E(G)$. Let $u_{1,r}u_{1,r+1}$ be an edge of $C_{1,r}$, and $P = u_{1,1} \cdots u_{1,r+1}$ be a path of length r. For $1 \leq i \leq r$, $u_{1,i}$ is incident to $b_i \leq 2(r+1-i)+1$ pendant edges. The resulting graph is denoted by H. Now, we take t^r+1 copies H, denoted by $H_1, \cdots, H_{t^r+1}$, such that $\bigcap_{i=1}^{t^r+1} V(H_i) = \{u_{1,1}\} = \{u\}$. The resulting graph is denoted by G.

Now, we use $t \geq r(r+2)$ colors to color each $u - u_{i,r+1}$ (length of r) path $P_i = uu_{i,2}u_{i,3} \cdots u_{i,r+1}$ for $1 \leq i \leq t^r+1$. There are at most t^r different ways of colorings. Hence, there exist two paths P_k, P_ℓ such that for any $1 \leq j \leq r$, $c(e_{k,j}) = c(e_{\ell,j})$ where $e_{k,j} = u_{k,j}u_{k,j+1}, e_{\ell,j} = u_{\ell,j}u_{\ell,j+1}$. Consider any rainbow path R between $u_{\ell,r+1}$ and $u_{k,r+1}$. One can see that $|R| \geq r + \sum_{i=1}^{r} 2i = r(r+2)$. Hence $rc(G) \geq r(r+2)$. In the following we use $r(r+2)$ different colors to give G a coloring. Let C_{2k+1} be a cycle of length $2k+1$, we use $2k+1$ different colors to color every edge of C_{2k+1}. Call a coloring of C_{2k+1} *appropriate* if $c(v_iv_{i+1}) = i$ for $1 \leq i \leq 2k$ and $c(v_{2k+1}v_1) = 2k+1$. Now, for a given i and $1 \leq h \leq [r(r+2)]^r + 1$, we use $2(r+1-i)+1$ different colors, denoted by a set $c(C_{i,h})$ of the colors, to give $C_{i,h}$ an *appropriate* coloring and use b_i different colors from $c(C_{i,h})$ to color every bridge incident to $u_{i,h}$. For any i and j with $1 \leq i \neq j \leq r$, we color $C_{i,h}$ and $C_{j,h}$ in the above way separately such that $c(C_{i,h}) \cap c(C_{j,h}) = \emptyset$. Now one can see that G is rainbow connected and the number of used colors is $\sum_{i=1}^{r}(2(r+1-i)+1) = r(r+2)$. Hence, $rc(G) \leq r(r+2)$ and so $rc(G) = r(r+2)$.

Example 2. Similar to the construction of H, we only change the sentence "for $1 \leq i \leq r$, $u_{1,i}$ is incident to $b_i \leq 2(r+1-i)+1$ pendant edges" in Example 1 into "for $1 \leq i \leq r$, $u_{1,i}$ is incident to $b_{1,i} \geq 2(r+1-i)+2$ pendant edges". The resulting graph is denoted by H'. Now, we take t^r+1 copies of H', denoted by $H'_1, \cdots, H'_{t^r+1}$, such that $\bigcap_{i=1}^{t^r+1} V(H'_i) = \{u_{1,1}\} = \{u\}$. The resulting graph is denoted by G. For a given i and $1 \leq h \leq [r(r+2)]^r + 1$, we use b_i different colors, denoted by a set $c(B_i)$ of the colors, to color every bridge incident to $u_{i,h}$ and use $2(r+1-i)+1$ different colors from $c(B_i)$ to give $C_{i,h}$ an *appropriate* coloring. For any i and j with $1 \leq i \neq j \leq r$, we color $C_{i,h}$ and

$C_{j,h}$ in the above way separately such that $c(C_{i,h}) \cap c(C_{j,h}) = \emptyset$. One can see that G is rainbow connected and the number of used colors is $\sum_{i=1}^{r} b_i = b$ and so $rc(G) = b$.

2 Proof of Theorem 1

At first we give some definitions which are needed in our proof. Let S be a subset of $V(G)$. If every vertex in G is at a distance at most k from S, we say that S is a *k-step dominating set*. If $G[S]$ is connected, then S is a *connected k-step dominating set*. Let D^k be a connected k-step dominating set. A *D^k-ear* is a path $P = v_0v_1 \cdots v_p$ in G such that $P \cap D^k = \{v_0, v_p\}$. When $v_0 = v_p$, P is a *closed D^k-ear*. Moreover, we say that P is an *acceptable D^k-ear*, if P is a shortest D^k-ear containing v_0v_1. Given $2k+1$ distinct colors $1, 2, 3, \cdots, 2k+1$, we call that P is *evenly colored* if either the edges of P are colored by this way: $c(v_0v_1) = 1, c(v_1v_2) = 2, c(v_2v_3) = 3, \cdots, c(v_{\lceil \frac{p}{2} \rceil - 1}v_{\lceil \frac{p}{2} \rceil}) = \lceil \frac{p}{2} \rceil, c(v_{\lceil \frac{p}{2} \rceil}v_{\lceil \frac{p}{2} \rceil + 1}) = 2k+2-\lfloor \frac{p}{2} \rfloor, \cdots, c(v_{p-2}v_{p-1}) = 2k, c(v_{p-1}v_p) = 2k+1$, or the edges of P are colored by the contrary way: $c(v_0v_1) = 2k+1, c(v_1v_2) = 2k, \cdots, c(v_{p-2}v_{p-1}) = 2, c(v_{p-1}v_p) = 1$ where p is the length of P. For example, let $P = v_0v_1v_2v_3v_4v_5$. Then $c(v_0v_1) = 1, c(v_1v_2) = 2, c(v_2v_3) = 3, c(v_5v_4) = 2k+1, c(v_4v_3) = 2k$ is an evenly colored of P.

Before the proof of our Theorem, we need the following lemma and two claims.

Lemma 1 *If G is a connected graph and D^k is a connected k-step dominating set of G, then G has a connected $(k-1)$-step dominating set $D^{k-1} \supset D^k$ such that $rc(G[D^{k-1}]) \leq rc(G[D^k]) + \max\{2k+1, b_k\}$, where b_k is the number of bridges of G in $E[D^k, N(D^k)]$.*

Proof of Lemma 1: If G is a tree, then every edge of G is a bridge, and the result is obvious. Hence we assume that G is not a tree.

In the following, we let D^k be a connected k-step dominating set of G. Then G has k mutually disjoint subsets $N^1(D^k), N^2(D^k), \cdots, N^k(D^k)$ such that $V(G) = \bigcup_{i=0}^{k} N^i(D^k)$.

Claim 1. If there exist $x \in D^k$ and $y \in N(D^k)$ such that xy is a bridge, then y has only one neighbor x in D^k and $d_{G[N(D^k)]}(y) = 0$.

Proof of Claim 1: If there exist $x' \in D^k$ with $x' \neq x$ such that $yx' \in E(G)$, since $G[D^k]$ is connected, then $G[D^k]$ has a path connecting x and x'. So xy is in a cycle, a contradiction to that xy is a bridge. Hence y has only one neighbor x in D^k. If there exists $y_1 \in N(D^k)$ such that $yy_1 \in E(G)$, since D^k is a dominating set of $N(D^k)$, then there exists a vertex $x_1 \in D^k$ such that $x_1y_1 \in E(G)$ (x_1 may be x). Then xyy_1x_1 is a path, and so we can also get that xy is in some cycle, a contradiction. Hence, $d_{G[N(D^k)]}(y) = 0$.

■

Now, we let $x_1y_1, x_2y_2, \cdots, x_{b_k}y_{b_k}$ be all the bridges in $E[D^k, N(D^k)]$, where $x_i \in D^k$ and $y_i \in N(D^k)$ for $1 \leq i \leq b_k$, and let $B = \{y_1, \cdots, y_{b_k}\}$, $B_E = \{x_1y_1, x_2y_2, \cdots, x_{b_k}y_{b_k}\}$ and $D_1 = D^k \cup B$. We rainbow color $G[D^k]$ with $rc(G[D^k])$ colors. If $N(D^k) = B$, then $D_1 = D^k \cup B$ is a connected $(k-1)$-step dominating set. Thus, we let $D^{k-1} = D_1$ and use b_k fresh colors to color these b_k bridges, respectively. Hence $rc(G[D^{k-1}]) \leq rc(G[D^k]) + b_k$, and so the theorem follows. So we assume $N(D^k) \neq B$, and then $N(D^k) \setminus B \neq \emptyset$. In the following we will construct a connected $(k-1)$-step dominating set D^{k-1} and color every edge of $G[D^{k-1}]$ such that $G[D^{k-1}]$ is rainbow connected. Since $E[D^k, N(D^k) \setminus B]$ has no bridges, for each edge e of $E[D^k, N(D^k) \setminus B]$, e must be in some cycle. So there exists an acceptable D^k-ear P containing e. We use $2k+1$ fresh colors, different from the used colors of $G[D^k]$, to evenly color the edges of P. So we can construct a sequence of sets $D_1 \subset D_2 \subset D_3 \subset \cdots \subset D_t = D^{k-1}$, where $D_1 = D^k \cup B, D_2 = D_1 \cup P_1$, $D_3 = D_2 \cup P_2, \cdots, D_t = D_{t-1} \cup P_{t-1}$, $P_1, \cdots, P_{t-1}$ are all acceptable D^k-ears. We color the new edges in every induced subgraph $G[D_i]$ such that every $x \in D_i \setminus D_1$ lies in an evenly colored acceptable D^k-ear for all $1 \leq i \leq t$. If for some D_i we have $N(D^k) \subset D_i$, note that $N(D^k) \not\subset D_j$ for $1 \leq j \leq i-1$, then D_i is a connected $(k-1)$-step dominating set. Now we stop the procedure, and set $D^{k-1} = D_i$. Then we evenly color the edges of P_{i-1} and color the remaining uncolored new edges of $G[D_i]$ with a used color. Otherwise, we will construct D_{i+1} as follows: We choose any edge $vw \in E[D^k, N(D^k) \setminus D_i]$ with $v \in D^k, w \in N(D^k) \setminus D_i$. If P is an acceptable D^k-ear containing vw and $P \cap (D_i \setminus D_1) = \emptyset$, then we let $D_{i+1} = D_i \cup P$ and evenly color P. For the uncolored new edges of $G[D_{i+1}]$, we color them with a used colors. Otherwise, the acceptable D^k-ear P containing vw must satisfy $P \cap (D_i \setminus D_1) \neq \emptyset$. Assume $P_1 \subset P$, and let $P_1 = vw(v_1) \cdots v_\ell$ and $P_1 \cap (D_i \setminus D_1) = \{v_\ell\}$. Since $v_\ell \in D_i \setminus D_1$, v_ℓ is in an evenly colored acceptable D^k-ear Q. Let Q_1 be the shorter segment of Q respect to v_ℓ. Then $P = P_1 \cup Q_1$ is an acceptable D^k-ear containing vw. If Q_1 is evenly colored by the colors from $\{2k+1, 2k, 2k-1, \cdots\}$, then we will evenly color P where $c(vw) = 1$. If Q_1 is evenly colored by the colors from $\{1, 2, 3, \cdots$, then we will evenly color P where $c(vw) = 2k+1$. Hence, P is evenly colored, and we let $D_{i+1} = D_i \cup P$. For the uncolored new edges of $G[D_{i+1}]$, we color them with a used color. Clearly, every $x \in D_{i+1} \setminus D_1$ lies in an evenly colored acceptable D^k-ear in $G[D_{i+1}]$. Thus, we have constructed a connected $(k-1)$-step dominating set D^{k-1} and every edge of $G[D^{k-1} \setminus B]$ is colored.

Now, we are ready for coloring B_E: If $b_k \leq 2k+1$, then we use b_k different colors from $\{1, 2, \cdots, 2k+1\}$ to color each edge of B_E, respectively. If $b_k > 2k+1$, then we first use colors $1, 2, \cdots, 2k+1$ to color any $2k+1$ edges of B_E, respectively, we then use $b_k - (2k+1)$ fresh colors to color the remaining uncolored edges of B_E, respectively.

In the following we show that $G[D^{k-1}]$ is rainbow connected. For any two vertices

$x, y \in D_1$, we know that x and y are rainbow connected. For $x \in D^{k-1} \setminus D_1, y \in D^k$, since x is in an acceptable D^k-ear P, $P \cap D^k$ has at least one vertex, say y_1. Since in $G[D^k]$ there exists a rainbow path connecting y and y_1, we can get that there is a rainbow path connecting x and y. For $x \in D^{k-1} \setminus D_1$, $y \in B$, we know that x is in an evenly colored acceptable D^k-ear P. Let $c(P)$ be the set of all colors used for the edges of P. If the bridge $yy_1 \in B_E(y_1 \in D^k)$ is colored by a color c_{yy_1} from $c(P)$, then we choose the segment (which does not contain the color c_{yy_1}) from x to D^k. So there is a rainbow path connecting x and y. If the bridge $yy_1 \in B_E(y_1 \in D^k)$ is colored by a color c_{yy_1} not in $c(P)$, then we arbitrarily choose a segment of P from x to D^k, and we can also find a $x - y$ rainbow path.

Since for any $v \in D^{k-1} \setminus D_1$, v is in an evenly colored acceptable D^k-ear. For any $x \in D^{k-1} \setminus D_1$ and $y \in D^{k-1} \setminus D_1$ and $x \in P$ and $y \in Q$, P and Q are evenly colored acceptable D^k-ears. If $P = Q$, then x and y are rainbow connected. Hence we may assume $P \neq Q$. Let $P = x_0x_1 \cdots x_i(x)x_{i+1} \cdots x_p$ and $Q = y_0y_1 \cdots y_j(y)y_{j+1} \cdots y_q$. We distinguish two cases to show that x and y are rainbow connected.

Case 1. P and Q are internally disjoint.

Without loss of generality, we assume that $x_0x_1 \cdots x_{\lceil \frac{p}{2} \rceil}$ and $y_0y_1 \cdots y_{\lceil \frac{q}{2} \rceil}$ are evenly colored by the colors from $\{1, 2, 3, \cdots, k+1\}$, respectively. We distinguish four subcases to show that there is an $x - y$ rainbow path. Since $G[D^k]$ is rainbow connected, there exists a rainbow path of $G[D^k]$ connecting any two vertices of D^k.

Subcase 1.1. $i \leq \lfloor \frac{p}{2} \rfloor$, $j > \lfloor \frac{q}{2} \rfloor$.

Let R_1 be a rainbow path of $G[D^k]$ connecting x_0 and y_0. Since the edges of $x = x_ix_{i-1} \cdots x_0$ are colored by the colors from $\{1, 2, \cdots, k+1\}$, the edges of $y_qy_{q-1} \cdots y_j = y$ are colored by the colors from $\{2k+1, 2k, \cdots, k+2\}$. Hence $x_ix_{i-1} \cdots x_1R_1y_{q-1} \cdots y_j$ is an $x - y$ rainbow path.

Subcase 1.2. $i > \lfloor \frac{p}{2} \rfloor$, $j \leq \lfloor \frac{q}{2} \rfloor$.

Let R_2 be a rainbow path of $G[D^k]$ connecting x_0 and y_0. Then $x_ix_{i+1} \cdots x_{p-1}R_2y_1 \cdots y_j$ is an $x - y$ rainbow path.

Subcase 1.3. $i \leq \lfloor \frac{p}{2} \rfloor$, $j \leq \lfloor \frac{q}{2} \rfloor$.

If $i < j$, let R_3 be a rainbow path of $G[D^k]$ connecting x_0 and y_q, then $x_ix_{i-1} \cdots x_1R_3y_{q-1} \cdots y_j$ is a $x - y$ rainbow path. If $i \geq j$, let R_4 be a rainbow path of $G[D^k]$ connecting x_p and y_0, then $x_ix_{i+1} \cdots x_{p-1}R_4y_1 \cdots y_j$ is an $x - y$ rainbow path.

Subcase 1.4. $i > \lfloor \frac{p}{2} \rfloor$, $j > \lfloor \frac{q}{2} \rfloor$.

If $p-i \leq q-j$, let R_5 be a rainbow path of $G[D^k]$ connecting x_p and y_0, then $x_ix_{i+1}\cdots x_{p-1}R_5y_1\cdots y_j$ is an $x-y$ rainbow path. If $p-i > q-j$, let R_6 be a rainbow path of $G[D^k]$ connecting x_0 and y_q, then $x_ix_{i-1}\cdots x_1R_6y_{q-1}\cdots y_j$ is an $x-y$ rainbow path.

Case 2. P intersects Q at some of their internal vertices.

According to the construction and the coloring of D^{k-1}, we may assume that $P \subset D_{i_1}$ and $Q \subset D_{i_2}$ with $i_1 > i_2$, and x_ℓ is the first internal vertex of P in Q. If $x_px_{p-1}\cdots x_{l+1}x_l = y_qy_{q-1}\cdots y_{l+1}y_l$, then the case is similar to Case 1 in essence. So we assume $x_px_{p-1}\cdots x_{l+1}x_l$
$= y_0y_1,\cdots,y_{p-l}$. We also distinguish four subcases to show that there is an $x-y$ rainbow path.

Without loss of generality, assume that the edges of $y_0y_1\cdots y_{\lceil\frac{q}{2}\rceil}$ are colored by $1,2,\cdots,\lceil\frac{q}{2}\rceil$. According to the coloring of D^{k-1}, the edges of $x_px_{p-1}\cdots x_{\lfloor\frac{p}{2}\rfloor}$ are also colored by the colors from $\{1,2,\cdots,k+1\}$ and the edges of $x_0x_1\cdots x_{\lceil\frac{p}{2}\rceil}$ are colored by the colors from $\{2k+1,2k,\cdots,k+2\}$.

Subcase 2.1. $i \leq \lfloor\frac{p}{2}\rfloor$, $j > \lfloor\frac{q}{2}\rfloor$.

If $i < q-j$, let P_1 be a rainbow path of $G[D^k]$ connecting x_0 and y_0, then $x_ix_{i-1}\cdots x_1P_1y_1$ $\cdots y_j$ is an $x-y$ rainbow path. If $i \geq q-j$, let P_2 be a rainbow path of $G[D^k]$ connecting x_p and y_q, then $x_ix_{i+1}\cdots x_{p-1}P_2y_{q-1}\cdots y_j$ is an $x-y$ rainbow path.

Subcase 2.2. $i > \lfloor\frac{p}{2}\rfloor$, $j \leq \lfloor\frac{q}{2}\rfloor$.

If $p-i \leq j$, let P_3 be a rainbow path of $G[D^k]$ connecting x_p and y_q, then $x_ix_{i+1}\cdots x_{p-1}P_3$ $y_{q-1}\cdots y_j$ is an $x-y$ rainbow path. If $p-i > j$, let P_4 be a rainbow path of $G[D^k]$ connecting x_0 and y_0, then $x_ix_{i-1}\cdots x_1P_4y_1$
$\cdots y_j$ is an $x-y$ rainbow path.

Subcase 2.3. $i \leq \lfloor\frac{p}{2}\rfloor$, $j \leq \lfloor\frac{q}{2}\rfloor$.

Let P_5 be a rainbow path of $G[D^k]$ connecting c_0 and y_0. Then $x_ix_{i-1}\cdots x_1P_5y_1\cdots y_j$ is an $x-y$ rainbow path.

Subcase 2.4. $i > \lfloor\frac{p}{2}\rfloor$, $j > \lfloor\frac{q}{2}\rfloor$.

Let P_6 be a rainbow path of $G[D^k]$ connecting x_p and y_q. Then $x_ix_{i+1}\cdots x_{p-1}P_6y_{q-1}\cdots y_j$ is an $x-y$ rainbow path.

Hence, for any two vertices $x, y \in D^{k-1}\setminus D_1$, there is a rainbow path connecting x and y. Thus, we have constructed a connected $(k-1)$-step dominating set D^{k-1} from D^k, and

$rc(G[D^{k-1}]) \leq rc(G[D^k]) + \max\{2k+1, b_k\}$. The proof of Lemma 1 is now complete. ∎

Claim 2. $G[D^{k-1} \setminus D^k]$ has no bridges.

Proof of Claim 2: Since the bridges in B_E are incident to the vertices $x_1, \cdots, x_{b_k}$, $G[D^{k-1} \setminus D^k]$ does not contain any edge of B_E. Suppose that xy is a bridge with $xy \in E(G[D^{k-1} \setminus D^k])$. We know that $xy \notin E(G[B])$, or else xy is in a cycle. If $xy \in E[B, D^{k-1} \setminus D^1]$, then xy is also in some cycle. Hence we assume $x, y \in D^{k-1} \setminus D^1$. If xy is in some acceptable D^k-ear, then xy is in a cycle, a contradiction. If x is in some acceptable D^k-ear P and y is in some acceptable D^k-ear Q, we still can get that xy is in a cycle, a contradiction. Hence Claim 2 is true. ∎

Let u be a center vertex of G and set $D^r = \{u\}$. Then D^r is an r-step dominating set of G and $rc(G[D^r]) = 0$. By making use of Lemma 1, we can construct $D^{r-1}, D^{r-2}, \cdots, D^2, D^1$ such that $D^r \subset D^{r-1} \subset D^{r-2} \cdots \subset D^1 \subset D^0 = V(G)$, and so we have

$$rc(G[D^{r-1}]) \leq rc(G[D^r]) + \max\{2r+1, b_r\},$$

$$rc(G[D^{r-2}]) \leq rc(G[D^{r-1}]) + \max\{2(r-1)+1, b_{r-1}\},$$

$$\cdots,$$

$$rc(G[D^0]) \leq rc(G[D^1]) + \max\{2+1, b_1\},$$

where $rc(G[D^0]) = rc(G)$, and for $1 \leq i \leq r$, b_i is the number of bridges in $E[D^i, N(D^i)]$. Thus we get that $rc(G) \leq rc(G[D^r]) + \sum_{i=1}^{r} \max\{2i+1, b_i\} = \sum_{i=1}^{r} \max\{2i+1, b_i\}$.

From Claim 2, we can see that the number of bridges of G is equal to $\sum_{i=1}^{r} b_i$.

This completes the proof of Theorem 1. ∎

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