Rainbow connection in 3-connected graphs*

Xueliang Li, Yongtang Shi
Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, China
lxl@nankai.edu.cn, shi@nankai.edu.cn

Abstract

An edge-colored graph G is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection number of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. In this paper, we proved that $rc(G) \leq 3(n+1)/5$ for all 3-connected graphs.

Keywords: rainbow connection; connectivity; the fan lemma

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the notation and terminology of Bondy and Murty [1]. An edge-colored graph G is *rainbow connected* if any two vertices are connected by a path whose edges have distinct colors. Obviously, if G is rainbow connected, then it is also connected. This concept of rainbow connection in graphs was introduced by Chartrand et al. in [5]. The *rainbow connection number* of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. An easy observation is

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that if G is of order n then $rc(G) \leq n-1$, since one may color the edges of one spanning tree of G with different colors and the remaining edges with colors already used. It is easy to verify that $rc(G) = 1$ if and only if G is a complete graph, that $rc(G) = n-1$ if and only if G is a tree. Notice that for the cycle C_n of order n, $rc(C_n) = \lceil n/2 \rceil$. It was shown that computing the rainbow connection number of an arbitrary graph is NP-hard [2].

There are some approaches to study the bounds of $rc(G)$ with respect to the minimum degree $\delta(G)$. In [2] Caro et al. have shown that if G is a graph of order n with minimum degree δ, then $rc(G) < \min\{(\ln \delta/\delta)n(1 + o_\delta(1)), (4 \ln \delta + 3)n/\delta\}$. By employing the method of 2-step dominating set, Krivelevich and Yuster [6] have shown that a connected graph G with n vertices and minimum degree δ has $rc(G) < 20n/\delta$. Schiermeyer [8] proved that $rc(G) < 3n/4$ for graphs with minimum degree three. Very recently, Chandran et al. [4] have improved the upper bound of Krivelevich and Yuster by showing that for every connected graph G of order n and minimum degree δ, $rc(G) \leq 3n/(\delta + 1) + 3$. For more details on various rainbow connections we refer the reader to a new book [7].

With respect to the the relation between $rc(G)$ and the connectivity $\kappa(G)$, in [8], the author mentioned that Hajo Broersma asked a question at the IWOCA workshop:

Problem 1 *What happens with the value $rc(G)$ for graphs with higher connectivity.* ∎

Schiermeyer [8] have shown that if G is a graph of order n with $\kappa(G) = 1$ and $\delta \geq 3$, then $rc(G) \leq (3n-1)/4$. In [2] Caro et al. proved that if $\kappa(G) = 2$ then $rc(G) \leq 2n/3$. From the result of Chandran et al. [4], we can easily obtain an upper bound of the rainbow connection number:

$$rc(G) \leq \frac{3n}{\delta + 1} + 3 \leq \frac{3n}{\kappa(G) + 1} + 3.$$

Therefore, for $\kappa(G) = 3$, $rc(G) \leq 3n/4 + 3$, and $\kappa(G) = 4$, $rc(G) \leq 3n/5 + 3$. In this paper, motivated by the results in [2], we will improve this bound by showing the following result.

Theorem 1 *If G is a 3-connected graph with n vertices, then $rc(G) \leq 3(n+1)/5$.*

Before proceeding, we recall the fan lemma, which will be used frequently in the proof of Theorem 1.

Lemma 1 (The Fan Lemma) *Let G be a k-connected graph, x a vertex of G, and let $Y \subseteq V - \{x\}$ be a set of at least k vertices of G. Then there exists a k-fan in G from x to Y, namely there exists a family of k internally disjoint (x, Y)-paths whose terminal vertices are distinct in Y.*

2 Proof of Theorem 1.

Let H be a maximal connected subgraph of G satisfying that $rc(H) \leq 3h/5 - 1/5$, where h is the number of vertices of H.

We first claim the existence of H. If G contains a triangle C_3, then we can choose the triangle as H, since $rc(C_3) = 1 < 8/5$. If G contains C_k ($k \geq 4$ and $k \neq 5$) as a subgraph, then we take $H = C_k$, since $rc(C_k) = \lceil k/2 \rceil \leq 3k/5 - 1/5$. Now suppose all the cycles contained in G are of length 5, then we can take H as the graph obtained by adding one pendent edge to C_5. Observe that $h = 6$ and $rc(H) = 3 < 17/5$.

We next claim that $h \geq n - 3$. By contradiction. Suppose there are four distinct vertices outside of H, denoted by x_1, x_2, x_3, x_4. Then by the fan lemma, each of them has three internally disjoint paths to H.

We assume first that each of x_1, x_2, x_3, x_4 has three neighbors in H. Let f_{ij} be the edges incident to the vertex x_i, $j = 1, 2, 3$. We can add x_1, x_2, x_3, x_4 to H, and form a larger subgraph H' with $h + 4$ vertices. Now we use only two new colors 1 and 2 to color the 12 edges. Assigning color 1 to edges f_{i1} for $i = 1, 2, 3$ and color 2 to other 9 edges. Then we have

$$rc(H') \leq rc(H) + 2 \leq 3h/5 - 1/5 + 2 < 3(h + 4)/5 - 1/5,$$

contradicting to the choice of H.

It follows that at least one of these four vertices, say x, has the property that one of the three internally disjoint (x, H)-paths P_0, P_1, P_2 has length at least two. Furthermore, among all vertices satisfying the above property, we choose vertex x such that one of the three paths has length one, say $P_0 = e_0$, and that the sum of lengths of P_1 and P_2 is as large as possible. Denote $P_1 = au_1u_2 \ldots u_sx$ and $P_2 = xv_1v_2 \ldots v_tb$ with a, $b \in H$ and u_i, $v_j \notin H$ for all i and j. With loss of generality, we assume $t \geq s$, and then $t \geq 1$. We first assume $s+t \geq 3$. We can add v_1, v_2, $\ldots$, v_s, x, u_1, u_2, $\ldots$, u_t to H and form a larger subgraph H' with $h + s + t + 1$ vertices. If $s + t$ is even, then we can color the $s+t+2$ edges of path $au_1u_2 \ldots u_sxv_1v_2 \ldots v_tb$ with $(s+t+2)/2$ new colors. In the first half of the path the colors are all distinct, and the same ordering of colors is repeated in the second half of the path. We can color edge e_0 with any color already appeared in H, and then it is straightforward to verify that H' is rainbow connected. If $s+t$ is odd, then we can color the $s+t+2$ edges of path $au_1u_2\dots u_sxv_1v_2\dots v_tb$ with $(s+t+1)/2$ new colors as follows. The middle edge of the path and edge e_0 get any color that already used in H. The first $(s+t+1)/2$ edges of the path all receive distinct new colors, and in the last $(s+t+1)/2$ edges of the path this coloring is repeated in the same order. Again it is straightforward to verify that H' is rainbow connected. We now have

$$\begin{aligned} rc(H') &\le rc(H) + \lceil (s+t+1)/2 \rceil \\ &\le 3h/5 - 1/5 + \lceil (s+t+1)/2 \rceil \le 3(h+s+t+1)/5 - 1/5, \end{aligned}$$

contradicting the maximality of H. Hence, we only assume $1 \le s+t \le 2$. We consider three cases as follows.

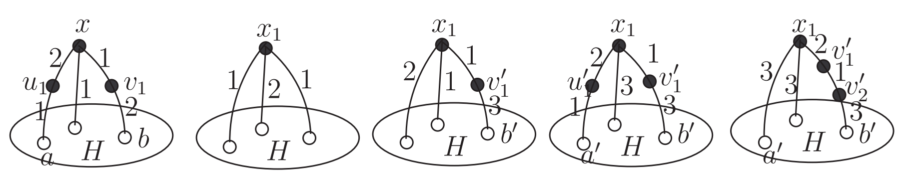

Figure 1: $s+t=2$ and $P_1 = au_1x$, $P_2 = xv_1b$

Case 1. $s+t=2$ and $P_1 = au_1x$, $P_2 = xv_1b$ (see Figure 1(A)).

Since there are at least 4 vertices outside of H, there exists at least one vertex different from x, u_1 and v_1, say x_1. By the choice of x, there is no (x_1, x)-path, (x_1, u_1)-path and (x_1, v_1)-path without using any vertex of H except one case: there is one path of length two joining x to H through x_1, say $P_3 = xx_1c$ with $c \in H$. In this case, we only consider the three paths P_1, P_2 and P_3 (as shown in Figure 2(A)). We can add vertices x, u_1, v_1 and x_1 to H and form a larger subgraph H' with $h+4$ vertices. By assigning color 1 to edges au_1, bv_1, color 2 to edges u_1x, cx_1, and one color already appeared in H to edges v_1x, xx_1, we have a contradiction as

$$rc(H') \le rc(H) + 2 \le 3h/5 - 1/5 + 2 < 3(h+4)/5 - 1/5.$$

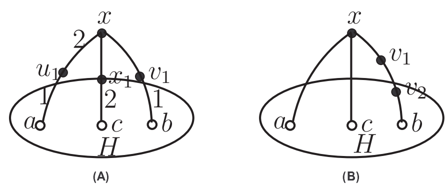

Figure 2: Graphs used in Case 1 and Case 2.

Now by the fan lemma, there are three internally disjoint (x_1, H)-paths P_0', P_1', P_2'. By the choice of x, the lengths of P_0', P_1', P_2' only have four possibilities:

Subcase 1.1. 1, 1, 1. Let $P_0' = e_0'$, $P_1' = e_1'$, $P_2' = e_2'$. We can add x, u_1, v_1 and x_1 to H, and form a larger graph H' of order $h + 4$. By assigning color 1 to au_1, e_0, xv_1, e_0', e_1', and color 2 to u_1x, v_2b, e_2', as shown in Figure 1(1), we can obtain that $rc(H') \leq rc(H) + 2 \leq 3h/5 - 1/5 + 2 < 3(h + 4)/5 - 1/5$, a contradiction.

Subcase 1.2. 1, 1, 2. Let $P_0' = e_0'$, $P_1' = e_1'$, $P_2' = x_1v_1'b'$. We can add x, u_1, v_1, x_1 and v_1' to H, and form a larger graph H' of order $h + 5$. By coloring all edges of paths P_0, P_1, P_2 the same as Subcase 1.1 and assigning color 1 to e_0', x_1v_1', color 2 to e_1', and color 3 to $v_1'b'$, as shown in Figure 1(2), we can obtain that $rc(H') \leq rc(H) + 3 \leq 3h/5 - 1/5 + 3 = 3(h + 5)/5 - 1/5$, a contradiction.

Subcase 1.3. 1, 2, 2. Let $P_0' = e_0'$, $P_1' = a'u_1'x_1$, $P_2' = x_1v_1'b'$. We can add x, u_1, v_1, x_1, u_1' and v_1' to H, and form a larger graph H' of order $h + 6$. By coloring all edges of paths P_0, P_1, P_2 the same as Subcase 1.1 and assigning color 1 to $a'u_1'$, x_1v_1', color 2 to $u_1'x_1$, and color 3 to e_0', $v_1'b'$, as shown in Figure 1(3), we can obtain that $rc(H') \leq rc(H) + 3 \leq 3h/5 - 1/5 + 3 < 3(h + 6)/5 - 1/5$, a contradiction.

Subcase 1.4. 1, 1, 3. Let $P_0' = e_0'$, $P_1' = e_1'$, $P_2' = x_1v_1'v_2'b'$. We can add x, u_1, v_1, x_1, v_1' and v_2' to H, and form a larger graph H' of order $h + 6$. By coloring all edges of paths P_0, P_1, P_2 the same as Subcase 1.1 and assigning color 1 to $v_1'v_2'$, color 2 to x_1v_1', and color 3 to e_0', e_1', $v_2'b'$, as shown in Figure 1(4), we can obtain that $rc(H') \leq rc(H) + 3 \leq 3h/5 - 1/5 + 3 < 3(h + 6)/5 - 1/5$, a contradiction.

Case 2. $s + t = 2$ and $P_1 = ax$, $P_2 = xv_1v_2b$ (see Figure 2(B)).

Since $v_1 \notin H$, there are three disjoint (v_1, H)-paths by the fan lemma.

Then there is at least one (v_1, H)-path P_3 in addition to the paths v_1xa and v_1v_2b. By the choice of x, the length of P_3 must be at most two. If P_3 is of length two, then this is the case pictured in Figure 2(A), which was addressed in Case 1. If P_3 is of length one, then the paths axv_1, v_1v_2b and P_3 from the same structure as in Case 1 and we are done.

Case 3 $s + t = 1$.

Since $t \geq s$, we have $t = 1$. Now we can assume $P_1 = e_1$ and $P_2 = xv_1b$. Then there are at least two distinct vertices outside of H different from x and v_1, say x_1 and x_2. Similarly, for $i = 1, 2$, there is no (x_i, x)-path and (x_i, v_1)-path without using any vertex of H. So there are also three internally disjoint (x_1, H)-paths P'_0, P'_1, P'_2 and (x_2, H)-paths P''_0, P''_1, P''_2, respectively. If all these paths are of length one, then we can add x, v_1, x_1 and x_2 to H, and form a larger graph H' of order $h + 4$. By assigning color 1 to edges e_0, xv_1, P'_0, P'_1, P''_0, P''_1, color 2 to edges e_1, v_1b, P'_2, P''_2, we can obtain that $rc(H') \leq rc(H) + 2 \leq 3h/5 - 1/5 + 2 < 3(h + 4)/5 - 1/5$, a contradiction. Otherwise, without loss of generality, we assume one of the three (x_1, H)-paths P'_0, P'_1, P'_2 has length 2. Let $P'_0 = e'_0$, $P'_1 = e'_1$, $P'_2 = x_1v'_1b'$. We can add x, v_1, x_1 and v'_1 to H, and form a larger graph H' of order $h + 4$. By assigning color 1 to edges e_0, e_1, xv_1, v_1b, color 2 to edges e'_0, e'_1, $x_1v'_1$, v'_1b', we can obtain that $rc(H') \leq rc(H) + 2 \leq 3h/5 - 1/5 + 2 < 3(h + 4)/5 - 1/5$, a contradiction.

Now we have proved that $h \geq n - 3$. By considering some cases, we can easily obtain that $rc(G) \leq 3(n+1)/5$: if $h = n-3$, then $rc(G) \leq rc(H)+2 \leq 3(h - 3)/5 - 1/5 + 2 < 3(n + 1)/5$; if $h = n - 2$, then $rc(G) \leq rc(H) + 2 \leq 3(h - 2)/5 - 1/5 + 2 = 3(n + 1)/5$; if $h = n - 1$, then $rc(G) \leq rc(H) + 1 \leq 3(h - 1)/5 - 1/5 + 1 < 3(n + 1)/5$.

The proof is completed. ∎

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