THE EDGE-TRANSITIVE TETRAVALENT CAYLEY GRAPHS OF SQUARE-FREE ORDER

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ABSTRACT. A classification is given of connected edge-transitive tetravalent Cayley graphs of square-free order. The classification shows that, with a few exceptions, a connected edge-transitive tetravalent Cayley graph of square-free order is either arc-regular or edge-regular. It thus provides a generic construction of half-transitive graphs of valency 4.

1. INTRODUCTION

Let $\Gamma = (V, E)$ be a graph with vertex set $V\Gamma = V$ and edge set $E\Gamma = E$. The number of vertices $|V|$ is called the *order* of the graph Γ. We say Γ to be *edge-transitive* or *edge-regular* if the automorphism group $\mathsf{Aut}\Gamma$ is transitive or regular on E, respectively. An *arc* of Γ is an ordered pair of adjacent vertices. Thus, an edge $\{u, v\}$ corresponds to two arcs (u, v) and (v, u). If $\mathsf{Aut}\Gamma$ is transitive or regular on the set of arcs of Γ, then Γ is called *arc-transitive* or *arc-regular*, respectively.

Edge-transitive graphs of square-free order have been extensively studied in some special cases. For example, see [1, 26, 27, 31, 32] for those with order a product of two distinct primes, see [18] for a characterization of edge-transitive circulant graphs of square-free order, and [19] for a classification of pentavalent arc-regular graphs of square-free order.

A graph Γ is called a *Cayley graph* if its vertex set can be identified with a group G which has a subset $S \subset G$ such that two vertices g, h are adjacent whenever $gh^{-1} \in S$. In this case Γ is denoted by $\mathsf{Cay}(G, S)$. For the graph $\mathsf{Cay}(G, S)$ to be simple and undirected, $S = S^{-1} := \{x^{-1} \mid x \in S\}$ must hold and S must not contain the identity of G.

In this paper, we classify connected edge-transitive Cayley graphs of square-free order and of valency 4. Before stating our classification, we introduce some notation.

Throughout this paper, for two groups A and B, denote by $A\times B$ the direct product of A and B, by $A.B$ an extension of A by B, and by $A{:}B$ a semi-direct product of A by B, that is, a split extension of A by B. For example, the dihedral group D_{2m} of order $2m$ is a semi-direct product of $\mathbb{Z}_m$ by $\mathbb{Z}_2$. For a group G and a subgroup $N \le G$, by $N \trianglelefteq G$ we mean that N is a normal subgroup of G.

For an integer $m \ge 3$, we denote by $\mathbf{C}_{m[2]}$ the *lexicographic product* of the empty graph $2\mathsf{K}_1$ of order 2 by a cycle $\mathbf{C}_m$ of size m, which has vertex set $\{(i, j) \mid 1 \le i \le m, 1 \le j \le 2\}$ such that (i, j) and (i', j') are adjacent if and only if $i - i' \equiv \pm 1 \pmod{m}$.

Our main result is stated as follows.

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Line	$\mathsf{Aut}\Gamma$	G (up to isomorphism)	Γ
1	$\mathrm{PGL}(2,7)$	D_{14}	Example 2.5 (1)
2	$\mathrm{PGL}(2,7)$	$\mathbb{Z}_7{:}\mathbb{Z}_6$	Example 2.8
3	$\mathrm{PSL}(3,3){:}\mathbb{Z}_2$	D_{26}	Example 2.5 (2)
4	$\mathrm{PSL}(2,23)$	$\mathbb{Z}_{23}{:}\mathbb{Z}_{11}$	Example 2.7
5	$\mathrm{PSL}(2,23)\times\mathbb{Z}_2$	$(\mathbb{Z}_{23}{:}\mathbb{Z}_{11})\times\mathbb{Z}_2$	Example 2.10 (3)
6	$\mathrm{PSL}(2,23)\times\mathbb{Z}_2$	$(\mathbb{Z}_{23}{:}\mathbb{Z}_{11})\times\mathbb{Z}_2$	Example 2.11
7	$\mathrm{PSL}(2,23)\times\mathrm{S}_3$	$(\mathbb{Z}_{23}{:}\mathbb{Z}_{11})\times\mathrm{S}_3$	Example 2.14
8	$\mathrm{PGL}(2,11)$	$\mathbb{Z}_{11}{:}\mathbb{Z}_5$	Example 2.7
9	$\mathrm{PGL}(2,11)\times\mathbb{Z}_2$	$(\mathbb{Z}_{11}{:}\mathbb{Z}_5)\times\mathbb{Z}_2, \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$	Example 2.10 (2)
10	$(\mathrm{PSL}(2,11)\times\mathbb{Z}_3){:}\mathbb{Z}_2$	$(\mathbb{Z}_{11}{:}\mathbb{Z}_5)\times\mathbb{Z}_3$	Example 2.12
11	$\mathbb{Z}_2\times((\mathrm{PSL}(2,11)\times\mathbb{Z}_3){:}\mathbb{Z}_2)$	$\mathbb{Z}_{33}{:}\mathbb{Z}_{10}, (\mathbb{Z}_{11}{:}\mathbb{Z}_5)\times\mathbb{Z}_6$	Example 2.13

Table 1

Line	$\mathsf{Aut}\Gamma$	G (up to isomorphism)	Γ
1	S_5	$\mathbb{Z}_5$	K_5
2	$\mathrm{S}_5\times\mathbb{Z}_2$	$\mathbb{Z}_{10}, \mathrm{D}_{10}$	$\mathsf{K}_{5,5}-5\mathsf{K}_2$
3	$\mathrm{S}_5\times\mathrm{S}_3$	$\mathrm{S}_3\times\mathbb{Z}_5, \mathrm{D}_{30}$	Example 2.14
4	$\mathrm{PGL}(2,7)$	$\mathbb{Z}_7{:}\mathbb{Z}_3$	Example 2.7
5	$\mathrm{PGL}(2,7)\times\mathbb{Z}_2$	$(\mathbb{Z}_7{:}\mathbb{Z}_3)\times\mathbb{Z}_2, \mathbb{Z}_7{:}\mathbb{Z}_6$	Example 2.10 (1)
6	$\mathrm{PGL}(2,7)\times\mathrm{D}_{2l}$	$(\mathbb{Z}_7{:}\mathbb{Z}_6)\times\mathbb{Z}_l, (\mathbb{Z}_7{:}\mathbb{Z}_3)\times\mathrm{D}_{2l}$	Example 2.15

Table 2

Theorem 1.1. *Let G be a group of square-free order, and let Γ be a connected edge-transitive tetravalent Cayley graph of G. Then one of the following statements holds.*

(1) *$\Gamma\cong\mathbf{C}_{m[2]}$, $\mathsf{Aut}\Gamma\cong\mathbb{Z}_2^m{:}\mathrm{D}_{2m}$ and $G\cong\mathbb{Z}_{2m}$ or D_{2m}, where $m\geq 3$;*

(2) *Γ is arc-regular, $\mathsf{Aut}\Gamma=G{:}\mathbb{Z}_2^2$ or $G{:}\mathbb{Z}_4$, and either G is cyclic or $G\cong\mathrm{D}_{2m}\times\mathbb{Z}_l$; Γ is constructed as in Constructions* 2.3 *and* 2.4*;*

(3) *Γ is edge-regular, $\mathsf{Aut}\Gamma=G{:}\mathbb{Z}_2$ and $G\cong(\mathbb{Z}_m{:}\mathbb{Z}_n)\times\mathbb{Z}_l$, where the center $\mathbf{Z}(G)\cong\mathbb{Z}_l$, and $n\geq 3$; Γ is constructed as in Construction* 2.6*;*

(4) *Γ is isomorphic to one of the graphs listed in Tables* 1 *and* 2.

A Cayley graph $\Gamma=\mathsf{Cay}(G,S)$ is said to be *normal* (with respect to G) if G is normal in $\mathsf{Aut}\Gamma$, refer to [35]; and Γ is said to be *normal-edge-transitive* (with respect to G) if the normalizer $\mathbf{N}_{\mathsf{Aut}\Gamma}(G)$ is transitive on the edges of Γ, refer to [25]. It was suggested in [25] to study the Cayley graphs which are not normal-edge-transitive or are normal-edge-transitive but not normal. Our classification gives several examples in this topic. The next corollary is proved at the end of this paper.

Corollary 1.2. *The graphs in Table* 1 *are not normal-edge-transitive, and those in Table* 2 *are normal-edge-transitive.*

An edge-transitive graph Γ is said to be *half-transitive* if $\mathsf{Aut}\Gamma$ is transitive on the vertices but not on the arcs of Γ. Studying half-transitive graphs was initiated by Tutte [30], and has received much attention in the literature, see [21] for references, and see [4, 7, 16, 17, 20, 22, 23, 28, 29, 33] for some recent development in this topic.

Let Γ be a graph described as in Construction 2.6. By Theorem 1.1 and Corollary 1.2, Γ is either edge-regular or isomorphic to one of the graphs in Examples

2.7, 2.10 (1) and 2.15. Note that edge-regular Cayley graphs are half-transitive. A straightforward consequence of our classification is the following corollary.

Corollary 1.3. *Let $\Gamma_{j,k}$ be described as in Construction* 2.6*. Then, with a few exceptions, $\Gamma_{j,k}$ is half-transitive.*

2. Examples

In this section we study the graphs appearing in Theorem 1.1.

Let Γ be a graph. For a subgroup $X \leq \mathsf{Aut}\Gamma$, we say Γ to be X*-edge-transitive* or X*-arc-transitive* if X is transitive on the edges or the arcs of Γ, respectively. For a vertex u of Γ, denote by $\Gamma(u)$ the set of neighbors of u in Γ.

2.1. **Group automorphisms.** For a given group G, a simple method to construct edge-transitive Cayley graphs is by a suitable subgroup of the automorphism group $\mathsf{Aut}(G)$ of G. Let $\Gamma = \mathsf{Cay}(G,S)$, and let

$$\mathsf{Aut}(G,S) = \{\sigma \in \mathsf{Aut}(G) \mid S^\sigma = S\}.$$

Then each element of $\mathsf{Aut}(G,S)$ induces an automorphism of Γ in the natural action on G. Moreover, if Γ is connected, i.e., $\langle S\rangle = G$, then $\mathsf{Aut}(G,S)$ can be identified with a subgroup of $\mathsf{Aut}\Gamma$ which fixes the vertex corresponding to the identity of G. Each $g \in G$ induces an automorphism, denoted by $\hat{g}$ sometimes, of Γ by the right multiplication on the elements of G. Then G can be identified with a subgroup of $\mathsf{Aut}\Gamma$ which acts regularly on $V\Gamma$.

Lemma 2.1. *Let G be a finite group, and let $H \leq \mathsf{Aut}(G)$. Let $S = \{g^h, (g^{-1})^h \mid h \in H\}$, where $g \in G$. If $\langle S\rangle = G$, then $\Gamma = \mathsf{Cay}(G,S)$ is a connected edge-transitive graph.*

This provides us with a generic method for constructing edge-transitive Cayley graphs, refer to [13] for more examples.

Let G be a group of square-free order. We first determine the automorphisms of G. It is well-known and easily shown that $G = C\times(A{:}B)$, where $A = \langle a\rangle \cong \mathbb{Z}_m$, $B = \langle b\rangle \cong \mathbb{Z}_n$ and $C = \langle c\rangle \cong \mathbb{Z}_l$, such that C is the center of G. If G is not cyclic, then $A{:}B$ has the presentation

$$A{:}B = \langle a, b \mid a^m = b^n = 1, b^{-1}ab = a^r\rangle,$$

where r is such that $r^n \equiv 1 \pmod{m}$ and $r^k \not\equiv 1 \pmod{m}$ for $1 \leq k < n$. Write $D = A{:}B$. Since $(|C|, |D|) = 1$ and $G = C\times D$, we have $\mathsf{Aut}(G) = \mathsf{Aut}(C)\times\mathsf{Aut}(D)$. Each automorphism $\sigma \in \mathsf{Aut}(A)$ can be extended to an automorphism of D such that $a \mapsto a^\sigma$ and $b \mapsto b$. Since D has trivial center, $D \cong \mathsf{Inn}(D)$, the inner automorphism group of D. Let $\bar{A}$ denote the subgroup of $\mathsf{Inn}(D)$ induced by A. Then $\bar{A} \trianglelefteq \mathsf{Inn}(D)$ and $\bar{A}$ is a Hall subgroup of $\mathsf{Inn}(D)$, so $\bar{A}$ is a characteristic subgroup of $\mathsf{Inn}(D)$. Since $\mathsf{Inn}(D) \trianglelefteq \mathsf{Aut}(D)$, we have $\bar{A} \trianglelefteq \mathsf{Aut}(D)$. Set $\mathbf{C}_{\mathsf{Aut}(G)}(B) = \{\rho \in \mathsf{Aut}(G) \mid b^\rho = b\}$. Then $\mathbf{C}_{\mathsf{Aut}(G)}(B) \geq \mathsf{Aut}(C)\times\mathsf{Aut}(A)$. Further, $\mathsf{Aut}(G)$ is given in the next lemma.

Lemma 2.2. $\mathsf{Aut}(G) = \mathsf{Aut}(C)\times(\bar{A}{:}\mathsf{Aut}(A))$ *and* $\mathbf{C}_{\mathsf{Aut}(G)}(B) = \mathsf{Aut}(C)\times\mathsf{Aut}(A)$.

Proof. It suffices to show $\mathsf{Aut}(D) = \bar{A}{:}\mathsf{Aut}(A)$. By the above discussion, we have $\mathsf{Aut}(D) \geq \bar{A}{:}\mathsf{Aut}(A)$. Note that $|\bar{A}| = m$ and $b^\tau = a^ib$ for $\tau \in \bar{A}$. It follows that for each value $i \in \{0, 1, \cdots, m-1\}$ there exists $\tau \in \bar{A}$ such that $b^\tau = a^ib$.

Now let $\alpha \in \mathsf{Aut}(D)$. Since A is a normal Hall subgroup of D, we have $a^{\alpha}, a^{\alpha^{-1}} \in A$. Then $a^{b^{\alpha}b^{-1}} = b(b^{-1}a^{\alpha^{-1}}b)^{\alpha}b^{-1} = b((a^{\alpha^{-1}})^{r})^{\alpha}b^{-1} = ba^{r}b^{-1} = a$. Thus $b^{\alpha}b^{-1} \in \mathbf{C}_D(\langle a\rangle) = \langle a\rangle$, and so $b^{\alpha} = a^{t}b$ for some t. Take $\tau \in \bar{A}$ with $b^{\tau} = a^{t}b$. Take $\sigma \in \mathsf{Aut}(A)$ with $a^{\sigma} = a^{\alpha}$, and extend σ to an automorphism of D by assigning $b^{\sigma} = b$. Then $\alpha = \sigma\tau$. Therefore, $\mathsf{Aut}(D) = \bar{A}{:}\mathsf{Aut}(A)$, and the result follows. □

Note that G is metacyclic, namely, G has a cyclic normal subgroup such that the corresponding quotient group is also cyclic. A special case is that G is cyclic.

Construction 2.3. Let $G = \langle c\rangle \cong \mathbb{Z}_l$, where l is square-free.

(i) Assume that there is an integer k with $k^2 \equiv -1 \pmod{l}$. Then G has an automorphism ρ such that $c^{\rho} = c^{k}$. Let $S = \{c, c^{k}, c^{k^2}, c^{k^3}\} = \{c, c^{k}, c^{-1}, c^{-k}\}$ and $X = G{:}\langle\rho\rangle \cong \mathbb{Z}_l{:}\mathbb{Z}_4$.

(ii) Assume that l has two distinct odd prime divisors. Let $\tau \in \mathsf{Aut}(G)$ be such that $c^{\tau} = c^{-1}$. Then $\mathsf{Aut}(G)$ contains an involution $\sigma \in \mathsf{Aut}(G) \setminus \{\tau\}$ such that $\sigma\tau = \tau\sigma$. Let $c^{\sigma} = c^{k}$, where $k^2 \equiv 1 \pmod{l}$. Let $S = \{c, c^{-1}, c^{k}, c^{-k}\}$ and $X = G{:}\langle\sigma,\tau\rangle \cong \mathbb{Z}_l{:}\mathbb{Z}_2^2$.

Then the Cayley graph $\mathsf{Cay}(G, S)$ is connected and X-arc-regular. □

We next consider the case where G has a dihedral direct factor.

Construction 2.4. Let $G = (\langle a\rangle{:}\langle b\rangle)\times\langle c\rangle \cong \mathrm{D}_{2m}\times\mathbb{Z}_l$, where ml is odd square-free.

(i) Assume that $l = 1$. Suppose that there is an integer i with $1 < i < m-1$ and $i^3 + i^2 + i + 1 \equiv 0 \pmod{m}$. Take $\rho \in \mathsf{Aut}(G)$ with $a^{\rho} = a^{i}$ and $b^{\rho} = b$. Let $S = \{ab, a^{i}b, a^{i^2}b, a^{i^3}b\}$ and $X = G{:}\langle\rho\rangle \cong \mathrm{D}_{2m}{:}\mathbb{Z}_4$.

(ii) Assume that $l = 1$ and m is not a prime. Let $\sigma, \tau \in \mathsf{Aut}(G)$ be involutions, say $a^{\sigma} = a^{i_1}$, $a^{\tau} = a^{i_2}$ and $b^{\sigma} = b^{\tau} = b$, where $i_1 \not\equiv i_2 \pmod{m}$ and $(i_1 - 1, i_2 - 1, m) = 1$. Then $\langle\sigma,\tau\rangle = \mathbb{Z}_2^2$. Let $S = \{ab, a^{i_1}b, a^{i_2}b, a^{i_1 i_2}b\}$ and $X = G{:}\langle\sigma,\tau\rangle$.

(iii) Assume that $l > 1$. Suppose that there is an integer k with $k^2 \equiv -1 \pmod{l}$. Let $\rho \in \mathsf{Aut}(G)$ be such that $a^{\rho} = a^{-1}$, $b^{\rho} = b$ and $c^{\rho} = c^{k}$. Let $S = \{abc, a^{-1}bc^{k}, abc^{k^2}, a^{-1}bc^{k^3}\} = \{abc, a^{-1}bc^{k}, abc^{-1}, a^{-1}bc^{-k}\}$ and $X = G{:}\langle\rho\rangle \cong (\mathrm{D}_{2m}\times\mathbb{Z}_l){:}\mathbb{Z}_4$.

(iv) Assume that $l > 1$. Set $S = \{abc, abc^{-1}, a^{-1}bc^{k}, a^{-1}bc^{-k}\}$, where $1 \le k < m$ and $k^2 \equiv 1 \pmod{l}$. Take σ, $\tau \in \mathsf{Aut}(G)$ with $a^{\sigma} = a^{-1}$, $a^{\tau} = a$, $b^{\sigma} = b^{\tau} = b$, $c^{\sigma} = c^{k}$ and $c^{\tau} = c^{-1}$. Then $\langle\sigma,\tau\rangle \cong \mathbb{Z}_2^2$, and $X = G{:}\langle\sigma,\tau\rangle = (\mathrm{D}_{2m}\times\mathbb{Z}_l){:}\mathbb{Z}_2^2$.

Then the Cayley graph $\mathsf{Cay}(G, S)$ is connected and X-arc-regular. □

For $m = 7$ or 13, a Cayley graph of the dihedral group D_{2m} can be constructed geometrically.

Example 2.5. Let $\mathbb{F} = \mathrm{GF}(p)$ be the Galois field of size p. Let U and W consist of 1-subspaces and 2-subspaces of $\mathbb{F}^3$, respectively.

(1) Let $p = 2$. Define a bipartite graph Γ with biparts U and W such that $u \in U$ and $w \in W$ are adjacent if and only if $u + w = \mathbb{F}^3$. This is the point-line non-incidence graph of the Fano plane $\mathrm{PG}(2, 2)$. Further, $\mathsf{Aut}\Gamma = \mathrm{PGL}(3, 2).\mathbb{Z}_2$, and Γ is a Cayley graph of $G = \mathrm{D}_{14}$. See [24], for example.

(2) Let $p = 3$. Define a bipartite graph Γ with biparts U and W such that $u \in U$ and $w \in W$ are adjacent if and only if u is a subspace of w. Then Γ is the point-line

incidence graph of the projective plane $\mathrm{PG}(2,3)$. Further, $\mathsf{Aut}\Gamma = \mathrm{PGL}(3,3).\mathbb{Z}_2$, and Γ is a Cayley graph of $G = \mathrm{D}_{26}$. See [14, 15], for example. □

Next, we consider the case where $G = (\langle a\rangle{:}\langle b\rangle) \times \langle c\rangle \cong (\mathbb{Z}_m{:}\mathbb{Z}_n) \times \mathbb{Z}_l$ such that the center $\mathbf{Z}(G) = \langle c\rangle \cong \mathbb{Z}_l$ and $n \geq 3$. In particular, m is odd.

Construction 2.6. Let j be a positive integer such that $(j,n) = 1$. Let k be an integer with $k^2 \equiv 1 \,(\mathsf{mod}\ l)$, and let

$$\Gamma_{j,k} = \mathsf{Cay}(G, S_{j,k}),\ \text{where } S_{j,k} = \{ab^jc, (ab^jc)^{-1}, a^{-1}b^jc^k, (a^{-1}b^jc^k)^{-1}\}.$$

Note that $\langle S_{j,k}\rangle = \langle ab^jc, a^{-1}b^jc^k\rangle = \langle ab^j, a^{-1}b^j, c\rangle = \langle a^2, ab^j, c\rangle = \langle a,b,c\rangle = G$. Then $\Gamma_{j,k}$ is connected. By Lemma 2.2, there exists an involution $\tau \in \mathbf{C}_{\mathsf{Aut}(G)}(\langle b\rangle)$ such that $a^\tau = a^{-1}$, $b^\tau = b$ and $c^\tau = c^k$. So $\Gamma_{j,k}$ is X-edge-regular, where $X = G{:}\langle\tau\rangle$. □

2.2. **Coset graphs.** Let X be a group and H a core-free subgroup of X, that is, H does not contain non-trivial normal subgroups of X. Take $g \in X \setminus H$ and define the *coset graph*

$$\Gamma = \mathsf{Cos}(X, H, H\{g, g^{-1}\}H)$$

with vertex set $[X : H] := \{Hx \mid x \in X\}$ such that Hx and Hy are adjacent whenever $yx^{-1} \in H\{g,g^{-1}\}H$. Then Γ is well-defined, and X induces a subgroup of $\mathsf{Aut}\Gamma$ acting on $[X : H]$ by right multiplication, namely, $a : Hx \mapsto Hxa$ for $x, a \in X$. Label v, w to be the vertices of Γ corresponding to H and Hg, respectively. Then

(a) $\Gamma(v) = \{Hgh \mid h \in H\} \cup \{Hg^{-1}h \mid h \in H\}$;
(b) Γ is X-edge-transitive and X is transitive on the vertices of Γ;
(c) Γ is connected if and only if $X = \langle g, H\rangle$;
(d) $H^g \cap H = X_{vw}$, the stabilizer of the arc (v,w), where H^g is the conjugate of H by g;
(e) Γ is X-arc-transitive if and only if $HgH = Hg^{-1}H$, which yields that $HgH = HoH$ for some (2-element) $o \in \mathbf{N}_X(X_{vw}) \setminus H$ with $o^2 \in X_{vw}$, refer to [16]. (An element o in the group X is a 2-element if its order is a power of 2.)

Moreover, for any X-edge-transitive graph Σ, if X is transitive on $V\Sigma$ then the map $u^x \mapsto Hx$, $x \in X$ gives an isomorphism form Σ to $\mathsf{Cos}(X, H, H\{g,g^{-1}\}H)$, where $u \in V\Sigma$, $H = X_u$ and $g \in X \setminus H$ with $u^g \in \Gamma(u)$.

Here are a few of examples, that appear in our classification.

Example 2.7. (1) Let $X = \mathrm{S}_5$, $\mathrm{PGL}(2,11)$ or $\mathrm{PSL}(2,23)$. Then X has a maximal subgroup $H \cong \mathrm{S}_4$. Let $K \leq H$ and $K \cong \mathrm{S}_3$. Checking the subgroups of X in the Atlas [5], we conclude that $\mathbf{N}_X(K) = \langle o\rangle \times K \cong \mathrm{D}_{12}$, where $o \in X \setminus H$ is an involution. Set $\Gamma = \mathsf{Cos}(X, H, HoH)$. Since H is a maximal subgroup of X, $\langle o, H\rangle = X$. Then Γ is a connected X-arc-transitive graph of valency 4. Moreover, X has a subgroup G which is regular on the vertices, where $G = \mathbb{Z}_5$, $\mathbb{Z}_{11}{:}\mathbb{Z}_5$ or $\mathbb{Z}_{23}{:}\mathbb{Z}_{11}$, respectively. We denote by $\mathrm{P}_{11,5}$ and $\mathrm{P}_{23,11}$ the graphs associated with $\mathrm{PGL}(2,11)$ and $\mathrm{PSL}(2,23)$, respectively. By [16], $\mathsf{Aut}\mathrm{P}_{11,5} = \mathrm{PGL}(2,11)$ and $\mathsf{Aut}\mathrm{P}_{23,11} = \mathrm{PSL}(2,23)$.

(2) Let $X = \mathrm{PGL}(2,7)$. Then X has a maximal subgroup $H \cong \mathrm{D}_{16}$. Take a subgroup $K \leq H$ with $K \cong \mathbb{Z}_2^2$. Then $\mathrm{D}_8 \cong \mathbf{N}_H(K) \leq \mathbf{N}_X(K) \cong \mathrm{S}_4$. Take an involution $o \in \mathbf{N}_H(K) \setminus K$ and an element $z \in \mathbf{N}_X(K)$ of order 3 such that $z^o = z^{-1}$. Then $\mathbf{N}_X(K) = K{:}\langle o, z\rangle$, and $H\{g,g^{-1}\}H = HozH$ for any $g \in \mathbf{N}_G(K) \setminus H$. Set

$\mathrm{P}_{7,3} = \mathsf{Cos}(X, H, HozH)$. By [16], $\mathsf{Aut}\mathrm{P}_{7,3} = \mathrm{PGL}(2,7)$, and $\mathrm{P}_{7,3}$ is a connected tetravalent arc-transitive Cayley graph of $\mathbb{Z}_7{:}\mathbb{Z}_3$. □

Example 2.8. Let $X = \mathrm{PGL}(2,7)$, $T = \mathrm{PSL}(2,7)$ and $\mathrm{D}_8 \cong H \leq T$. Let $o \in H$ be an involution which is not in the center of H. Then $\mathbf{N}_H(\langle o\rangle) \cong \mathbb{Z}_2^2$, $\mathbf{N}_T(\langle o\rangle) \cong \mathrm{D}_8$ and $\mathbf{N}_X(\langle o\rangle) \cong \mathrm{D}_{16}$. Write $\mathbf{N}_X(\langle o\rangle) = \mathbf{N}_T(\langle o\rangle){:}\langle z\rangle$ for an involution $z \in X \setminus T$. Let y be an element of order 4 in $\mathbf{N}_T(\langle o\rangle)$. Set $\Gamma = \mathsf{Cos}(X, H, HxH)$, where $x = z$ or yz.

Let M be a maximal subgroup of T such that $H \leq M \cong \mathrm{S}_4$. If $M^{xt} = M$ for some $t \in T$, then $xt \in \mathbf{N}_X(M) = M$ by checking the subgroups of X, so $x \in M \leq T$, a contradiction. Thus M^x and M are not conjugate in T. By the information given in the Atlas [5], T contains exactly two conjugation classes of subgroups isomorphic to S_4. Enumerating the Sylow 2-subgroups of T, we conclude that two subgroups in the same conjugation class do not contain a common Sylow 2-subgroup. Thus $\langle H, H^x\rangle \ncong \mathrm{S}_4$, yielding $H = H^x$ or $\langle H, H^x\rangle = T$.

Since $\mathbf{N}_H(\langle o\rangle) \cong \mathbb{Z}_2^2$, we know that $\mathbf{N}_H(\langle o\rangle)$ is normal in $\langle H, \mathbf{N}_T(\langle o\rangle)\rangle$. Then $\langle H, \mathbf{N}_T(\langle o\rangle)\rangle \cong \mathrm{S}_4$ by checking the subgroups of T. If $H^x = H$, then x normalizes $\langle H, \mathbf{N}_T(\langle o\rangle)\rangle$, so $x \in \langle H, \mathbf{N}_T(\langle o\rangle)\rangle \leq T$, a contradiction. Thus $\langle H, H^x\rangle = T$, and so $\langle H, x\rangle = \langle H, H^x, x\rangle = X$. If $|H \cap H^x| = 4$, then $H \cap H^x \trianglelefteq \langle H, H^x\rangle = T$, a contradiction. Then $|H \cap H^x| = 2$, and so $|H : (H \cap H^x)| = 4$. Therefore, Γ is connected, X-arc-transitive and of valency 4. It is easily shown that Γ is bipartite and T-edge-transitive, and that X has a regular subgroup isomorphic to $\mathbb{Z}_7{:}\mathbb{Z}_6$. □

2.3. **Normal covers.** Let $\Gamma = (V, E)$ be a connected graph. Assume that $X \leq \mathsf{Aut}\Gamma$ is transitive on both V and E. Let $N \trianglelefteq X$, and let V_N be the set of N-orbits on V. The *normal quotient* Γ_N (with respect to N and X) is defined as the graph with vertex set V_N such that $B_1, B_2 \in V_N$ are adjacent if and only if some $u \in B_1$ and $v \in B_2$ are adjacent in Γ. It is easily shown that the valency of Γ_N is a divisor of the valency of Γ. The graph Γ is a *normal cover* or an *N-cover* of Γ_N (with respect to X and N) if Γ and Γ_N have the same valency. Let K be the kernel of X acting on V_N. Then X/K, viewed as a subgroup of $\mathsf{Aut}\Gamma_N$, is transitive on both the vertices and the edges of Γ_N. If Γ is a normal cover of Γ_N, then it is easily shown that $N = K$ is semiregular on V, and Γ is X-arc-transitive if and only if Γ_N is (X/N)-arc-transitive.

Lemma 2.9. *If Γ is of valency 4 and X/N is insoluble, then Γ is an N-cover of Γ_N.*

Proof. Let $u \in V$ and let B the N-orbit containing u. Then, by [3], the stabilizer X_u is a $\{2,3\}$-group, that is, $|X_u| = 2^i3^j$. In particular, X_u is soluble. Let K be the kernel of X acting on V_N. Then $K_u \trianglelefteq X_u$, so K_u is soluble. Since K is transitive on B, we have $K = NK_u$. So $K/N = NK_u/N \cong K_u/(N \cap K_u)$ is soluble. Then $X/K \cong (X/N)/(K/N)$ is insoluble as X/N is insoluble, so $\mathsf{Aut}\Gamma_N$ is insoluble, hence Γ_N is not a cycle. Note that Γ is connected and the valency of Γ_N is a divisor of the valency of Γ. This implies that Γ_N has valency 4, and the lemma follows. □

We now construct the normal covers of several known graphs.

Example 2.10. Let $\Gamma = (V, E)$ be a connected arc-transitive Cayley graph. The *standard double cover* $\Gamma^{(2)}$ is the graph with vertex set $V \cup \{u' \mid u \in V\}$ such that $\{u, v'\} \in E\Gamma^{(2)}$ whenever $\{u, v\} \in E$. For each $x \in \mathsf{Aut}\Gamma$, define $\tilde{x} : u \mapsto u^x,\ u' \mapsto (u^x)'$. Then $\mathsf{Aut}\Gamma$ can be viewed as a subgroup of $\mathsf{Aut}\Gamma^{(2)}$ in the above way. Define

$\epsilon : u \mapsto u',\ u' \mapsto u$. Then $\epsilon \in \mathsf{Aut}\Gamma^{(2)}$. Set $X = \langle \mathsf{Aut}\Gamma, \epsilon\rangle$. Then $X = \mathsf{Aut}\Gamma \times \langle \epsilon \rangle$, and $\Gamma^{(2)}$ is an X-arc-transitive Cayley graph. For example,

(1) $\mathrm{P}_{7,3}^{(2)}$ is a Cayley graph of $(\mathbb{Z}_7{:}\mathbb{Z}_3)\times\mathbb{Z}_2$ and of $\mathbb{Z}_7{:}\mathbb{Z}_6$;
(2) $\mathrm{P}_{11,5}^{(2)}$ is a Cayley graph of $(\mathbb{Z}_{11}{:}\mathbb{Z}_5)\times\mathbb{Z}_2$ and of $\mathbb{Z}_{11}{:}\mathbb{Z}_{10}$;
(3) $\mathrm{P}_{23,11}^{(2)}$ is a Cayley graph of $(\mathbb{Z}_{23}{:}\mathbb{Z}_{11})\times\mathbb{Z}_2$.

Here we just explain (2) briefly. Note that $\mathsf{Aut}\mathrm{P}_{11,5}^{(2)} \geq \mathsf{Aut}\mathrm{P}_{11,5}\times\langle\epsilon\rangle$. Take a subgroup $R \cong \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$ of $\mathsf{Aut}\mathrm{P}_{11,5} = \mathrm{PGL}(2,11)$, and let L be the $2'$-Hall subgroup of R. Then $L \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5$ and $R = L{:}\langle z\rangle$ for an involution $z \in R$. Then $\mathsf{Aut}\Gamma^{(2)}$ contains two regular subgroups $L\times\langle\epsilon\rangle \cong (\mathbb{Z}_{11}{:}\mathbb{Z}_5)\times\mathbb{Z}_2$ and $L{:}\langle z\epsilon\rangle \cong \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$. □

Next we construct the $\mathbb{Z}_2$-covers of $\mathrm{P}_{23,11}$ which are not the standard double cover.

Example 2.11. Let $X = T\times K$ with $T = \mathrm{PSL}(2,23)$ and $K = \langle z_2\rangle \cong \mathbb{Z}_2$. Take $\mathrm{A}_4 \cong H_1 \leq T$ and an involution $z_1 \in T$ with $\langle H_1, z_1\rangle \cong \mathrm{S}_4$. Set $z = z_1z_2$ and $H = H_1{:}\langle z\rangle$. Then $H \cong \mathrm{S}_4$ and $H\cap T = H_1$. Let $x \in H_1$ be of order 3 with $x^{z_1} = x^{-1}$. Then $\mathbf{N}_T(\langle x\rangle) \cong \mathrm{D}_{24}$ and $\langle x, z_1\rangle \cong \mathrm{S}_3 \cong \langle x, z\rangle \leq H$. Let o be the involution in the center of $\mathbf{N}_T(\langle x\rangle)$. Checking the maximal subgroups of $\mathrm{PSL}(2,23)$, we conclude that $\langle H_1, o\rangle = \langle H_1, oz_1\rangle = \mathrm{PSL}(2,23)$. Then $\langle H, o\rangle = \langle H_1, o, z_1z_2\rangle = X$ and $\langle H, oz_2\rangle = \langle H_1, oz_1, z_1z_2\rangle = X$. Thus we get two connected graphs $\Gamma_1 := \mathsf{Cos}(X, H, HoH)$ and $\Gamma_2 := \mathsf{Cos}(X, H, Hoz_2H)$. Note that $\mathrm{S}_3 \cong \langle x, z\rangle \leq H \cap H^o$. Then Γ_1 has valency $|H : (H\cap H^o)|$ dividing $|H : \langle x, z\rangle| = 4$. Since Γ_1 is connected, Γ_1 is not a cycle as $X \leq \mathsf{Aut}\Gamma_1$ is insoluble. Thus Γ_1 has valency 4. Similarly, Γ_2 has valency 4.

Let $\Gamma = \Gamma_1$ or Γ_2. Then, by Lemma 2.9, Γ is a normal cover of Γ_K. Then Γ_K is an X/K-arc-transitive graph of order 253 and valency 4. Since $X/K \cong \mathrm{PSL}(2,23)$, we have $\Gamma_K \cong \mathrm{P}_{23,11}$ by [16]. Take a subgroup $\mathbb{Z}_{23}{:}\mathbb{Z}_{11} \cong L < T$. Then X contains a regular subgroup $L\times K \cong (\mathbb{Z}_{23}{:}\mathbb{Z}_{11})\times\mathbb{Z}_2$. □

Next we construct the $\mathbb{Z}_3$- and $\mathbb{Z}_6$-covers of $\mathrm{P}_{11,5}$.

Example 2.12. Let $X = (T\times K){:}\langle z\rangle$ with $T = \mathrm{PSL}(2,11)$, $K = \langle y\rangle \cong \mathbb{Z}_3$, $Y := T{:}\langle z\rangle = \mathrm{PGL}(2,11)$ and $y^z = y^{-1}$. Take $\mathrm{S}_4 \cong H_1 \leq Y$. Let P be the normal subgroup of order 4 in H_1. Then $P \cong \mathbb{Z}_2^2$, $\mathbf{N}_Y(P) = H_1$ and $H_1 = P{:}\langle x, z\rangle$ for some $x \in T$ of order 3 with $x^z = x^{-1}$. Set $H = P{:}\langle xy, z\rangle$. Then $H \cong \mathrm{S}_4$ and $\langle xy, z\rangle \cong \mathrm{S}_3$.

For $g \in \mathbf{N}_X(\langle xy, z\rangle)$, we have $(xy)^g = (xy)^{\pm1}$ and $z^g = z^{(xy)^i}$ for some i, yielding $g \in \mathbf{N}_X(\langle x\rangle) = K{:}\mathbf{N}_Y(\langle x\rangle)$ and $g(xy)^{-i} \in \mathbf{C}_X(z)$. Thus $g(xy)^{-i} \in \mathbf{C}_X(z)\cap(K{:}\mathbf{N}_Y(\langle x\rangle))$. Computation shows that $\mathbf{C}_X(z)\cap(K{:}\mathbf{N}_Y(\langle x\rangle)) = \langle z, o\rangle$, where o is the involution in the center of $\mathbf{N}_Y(\langle x\rangle) \cong \mathrm{D}_{24}$, and so $o \in T$. Thus $\mathbf{N}_X(\langle xy, z\rangle) = \langle z, o\rangle\langle xy\rangle = \langle xy, z\rangle\times\langle o\rangle$, so $HgH = HoH$ for $g \in \mathbf{N}_X(\langle xy, z\rangle)\setminus H$. Let $\Gamma = \mathsf{Cos}(X, H, HoH)$.

Note that $o \in T$, $z \in Y$ and $P \leq T$. Suppose that $M := \langle o, P, z\rangle \neq Y$. Then $M \neq \langle P, z\rangle$; otherwise, $o \in H \cong \mathrm{S}_4$ and o centralizes $xy \in H$, which is impossible. Thus M contains two distinct Sylow 2-subgroups $\langle P, z\rangle$ and $\langle P, z\rangle^o$ of Y. Checking the subgroups of $\mathrm{PGL}(2,11)$ in the Atlas [5], we know that $M \cong \mathrm{S}_4$ or D_{24}, and M is maximal in Y. If $x \in M$, then $\mathrm{S}_4 \cong H_1 = \langle P, x, z\rangle \leq M$, so $H_1 = M$, hence $o \in H_1$ and o centralizes the element $x \in H$, which is impossible. Then $Y = \langle M, x\rangle = \langle o, P, z, x\rangle$, yielding $\langle o, P\rangle \trianglelefteq Y$. Thus $\langle o, P\rangle = T$, so $M = \langle o, P, z\rangle \geq \langle o, P\rangle = T$, a contradiction. Then $\langle o, P, z\rangle = Y$, so $\langle o, H\rangle = \langle o, P, z, xy\rangle = \langle Y, xy\rangle = \langle Y, y\rangle = X$. Therefore, $\Gamma = \mathsf{Cos}(X, H, HoH)$ is connected.

Noting that $H \cap H^o \geq \langle xy, z\rangle$, the index $|H : (H \cap H^o)|$ divides 4. Since $X \leq \mathsf{Aut}\Gamma$ is insoluble, Γ is not a cycle. Thus Γ is X-arc-transitive and of valency 4. Take a subgroup $L \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5$ of T. Then X contains a regular subgroup $L{\times}K \cong (\mathbb{Z}_{11}{:}\mathbb{Z}_5){\times}\mathbb{Z}_3$. □

Example 2.13. For the graph Γ in Example 2.12, the standard double cover $\Gamma^{(2)}$ is a Cayley graph of $(\mathbb{Z}_{11}{:}\mathbb{Z}_5){\times}\mathbb{Z}_6$ and of $\mathbb{Z}_{33}{:}\mathbb{Z}_{10}$. In fact, if ϵ is defined as in Example 2.10, then $\mathsf{Aut}\Gamma^{(2)} \geq ((K{\times}T){:}\langle z\rangle){\times}\langle\epsilon\rangle$. Take $L \leq R \leq T{:}\langle z\rangle = Y$ with $z \in R$, $L \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5$ and $R \cong \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$. Then $\mathsf{Aut}\Gamma^{(2)}$ has regular subgroups $(K{\times}L){\times}\langle\epsilon\rangle$ and $(K{\times}L){:}\langle z\epsilon\rangle$ isomorphic to $(\mathbb{Z}_{11}{:}\mathbb{Z}_5){\times}\mathbb{Z}_6$ and $\mathbb{Z}_{33}{:}\mathbb{Z}_{10}$, respectively. □

The next example gives the S_3-covers of K_5 and of $\mathrm{P}_{23,11}$.

Example 2.14. Let $X = Y{\times}K$ with $Y = \mathrm{S}_5$ or $\mathrm{PSL}(2, 23)$ and $K = \langle y_2\rangle{:}\langle z_2\rangle \cong \mathrm{S}_3$. Take a subgroup $H_1 \cong \mathrm{S}_4$ of Y. Then H_1 has a normal subgroup $P \cong \mathbb{Z}_2^2$. Write $H_1 = P{:}(\langle y_1\rangle{:}\langle z_1\rangle)$ with $\langle y_1\rangle{:}\langle z_1\rangle \cong \mathrm{S}_3$. Set $y = y_1y_2$, $z = z_1z_2$ and $H = P{:}(\langle y\rangle{:}\langle z\rangle)$. Then $H \cong \mathrm{S}_4$ and $\langle y, z\rangle \cong \mathrm{S}_3$.

It is easily shown that $\mathbf{N}_X(\langle y, z\rangle) \leq \mathbf{N}_X(\langle y_1\rangle) = \mathbf{N}_Y(\langle y_1\rangle){\times}K$, where $\mathbf{N}_Y(\langle y_1\rangle) \cong \mathrm{D}_{12}$ or D_{24} for $Y = \mathrm{S}_5$ or $\mathrm{PSL}(2, 23)$, respectively. Note that $\langle y, z\rangle$ contains exactly three involutions, say z, z^y and z^{y^2}. Assume that $g \in \mathbf{N}_X(\langle y_1\rangle)$ normalizes $\langle y, z\rangle$. Then $z^g = z^{y^i}$ for some $0 \leq i \leq 2$, yielding that y^ig^{-1} centralizes $z = z_1z_2$. Further computation shows that $y^ig^{-1} \in \langle o, z\rangle = \langle o\rangle{\times}\langle z\rangle$, where o is the involution in the center of $\mathbf{N}_Y(\langle y_1\rangle)$. It follows that $\mathbf{N}_X(\langle y, z\rangle) = \langle o\rangle{\times}(\langle y, z\rangle)$.

It is easily shown that $\langle o, H\rangle = X$. Then $\Gamma = \mathsf{Cos}(X, H, HoH)$ is connected, X-arc-transitive and of valency 4. Moreover, X contains a regular subgroup isomorphic to $\mathbb{Z}_5{\times}\mathrm{S}_3$ or $(\mathbb{Z}_{23}{:}\mathbb{Z}_{11}){\times}\mathrm{S}_3$, respectively.

For $Y = \mathrm{S}_5$, we may take $P = \langle(1\,2)(3\,4), (1\,3)(2\,4)\rangle$, $y_1 = (1\,2\,3)$, $z_1 = (1\,2)$ and $o = (4\,5)$. Let $g = (1\,2\,3\,4\,5)$ and $h = (1\,3)(2\,4)$. Then X has two regular subgroups $\langle g\rangle{\times}K \cong \mathbb{Z}_5{\times}\mathrm{S}_3$ and $\langle gy_2\rangle{:}\langle hz_2\rangle \cong \mathrm{D}_{30}$. □

We finally give a normal cover of $\mathrm{P}_{7,3}^{(2)}$. An X-edge-transitive graph Γ is said to be *X-half-transitive* if X is transitive on the vertices but not on the arcs of Γ.

Example 2.15. (1) Let $Y = \mathrm{PGL}(2, 7)$, $T = \mathrm{PSL}(2, 7)$ and $\mathrm{D}_8 \cong H \leq T$. Then $\mathbf{N}_Y(H) \cong \mathrm{D}_{16}$. Let o be the involution in the center of H. It is easily shown that o lies in the center of $\mathbf{N}_Y(H)$. Take $M \leq T$ with $H \leq M \cong \mathrm{S}_4$, and take an element $y \in M$ of order 3 with $y^o = y^{-1}$. Then $\langle y, H\rangle = M$ and $H \cap H^y \cong \mathbb{Z}_2^2$. Let $z \in \mathbf{N}_Y(H) \setminus T$ be an involution. Then $\langle M, z\rangle = Y$. Set $x = zy$. Then $x \notin T$ and $H{\cap}H^x = H{\cap}H^y \cong \mathbb{Z}_2^2$, so $|H : (H{\cap}H^x)| = 2$. Note that $\langle H, x\rangle = \langle H, (zy)^o, zy\rangle = \langle H, zy^{-1}, zy\rangle = \langle H, y, z\rangle = \langle M, z\rangle = Y$. Then $\Sigma := \mathsf{Cos}(Y, H, H\{x, x^{-1}\}H)$ is connected, X-half-transitive and of valency 4. Further, Y has a regular subgroup isomorphic to $\mathbb{Z}_7{:}\mathbb{Z}_6$.

(2) Let Σ be as in (1). Let $X = Y{\times}\langle c\rangle$, where $\langle c\rangle = \mathbb{Z}_l$ with odd l coprime to 21. Define a graph

$$\Gamma = \mathsf{Cos}(X, H, H\{cx, (cx)^{-1}\}H).$$

Then Γ is a connected X-edge-transitive tetravalent Cayley graph of $(\mathbb{Z}_7{:}\mathbb{Z}_6){\times}\mathbb{Z}_l$. □

Lemma 2.16. *Let Σ and $\mathit{\Gamma}$ be as in Example* 2.15. *Then $\Sigma \cong \mathrm{P}_{7,3}^{(2)}$, $(Y\times\langle c\rangle){:}\langle\theta\rangle \cong \mathrm{PGL}(2,7)\times\mathrm{D}_{2l}$ for an involution $\theta \in \mathsf{Aut}\mathit{\Gamma}$, and $\mathit{\Gamma}$ is isomorphic to an arc-transitive Cayley graph of $(\mathbb{Z}_7{:}\mathbb{Z}_3)\times\mathrm{D}_{2l}$.*

Proof. Recall that $z \in \mathbf{N}_Y(H)\setminus T$ is an involution. Define $\tilde{z} : Hg \mapsto Hzg$, $g \in Y$. Then $\tilde{z}$ centralizes Y. Since $y^o = y^{-1}$ and $o \in H$ lies in the center of $\mathbf{N}_Y(H)$, we have $zH\{x,x^{-1}\}Hz = H^z\{yz,(yz)^{-1}\}^z H^z = H\{yz,(yz)^{-1}\}^{zo}H = H\{zy,y^{-1}z\}^o H = H\{zy^{-1},yz\}H = H\{x,x^{-1}\}H$. Then it is easily shown that $\tilde{z}$ is an automorphism of Σ. Set $\tilde{Y} = T{:}\langle\tilde{z}z\rangle$. Then $\tilde{Y} \cong \mathrm{PGL}(2,7)$, and $\tilde{Y}$ has exactly two orbits on $V\Sigma$, say $\{Ht \mid t\in T\}$ and $\{Hzt \mid t \in T\}$. Let u be the vertex corresponding to H. Then $\Sigma(u) = \{Hg \mid g \in H\{yz,zy^{-1}\}H\}$, and $\tilde{Y}_u = H{:}\langle\tilde{z}z\rangle \cong \mathrm{D}_{16}$ is a Sylow 2-subgroup of $\tilde{Y}$. It is easily shown that $\tilde{Y}_u$ is transitive on $\Sigma(u)$. Thus Σ is $\tilde{Y}$-edge-transitive. Note that $\tilde{Y}$ is normal in $Y\times\langle\tilde{z}\rangle$. For an arbitrary vertex $v = Hg$, we have $\tilde{Y}_v = (Y\times\langle\tilde{z}\rangle)_v \cap \tilde{Y} = (Y\times\langle\tilde{z}\rangle)_u^g \cap \tilde{Y} = \tilde{Y}_u^g \cong \mathrm{D}_{16}$, so $\tilde{Y}_v$ and $\tilde{Y}_u$ are conjugate in $\tilde{Y}$. Then, by [8, Lemma 3.4], Σ is the standard double cover of a $\tilde{Y}$-arc-transitive graph Σ_1 of order 21 and valency 4. By [16], $\Sigma_1 \cong \mathrm{P}_{7,3}$, so $\Sigma \cong \mathrm{P}_{7,3}^{(2)}$.

Now we extend $\sigma := \tilde{z}z$ to an automorphism of $\mathit{\Gamma}$. Let $\tau \in \mathsf{Aut}(\langle c\rangle)$ with $c^\tau = c^{-1}$. Consider the direct product $\tilde{X} := Y\times\langle\tilde{z}\rangle\times(\langle c\rangle{:}\langle\tau\rangle)$. Then the element $\theta := \tau\sigma$ is an involution which normalizes both Y and H. Thus θ induces an automorphism of $X = Y\times\langle c\rangle$ by conjugation. Moreover, $(HcxH)^\theta = H(cx)^\theta H = H(cx)^{-1}H$. Then it is easily shown that $Hg \mapsto Hg^\theta$ gives an automorphism of $\mathit{\Gamma}$, and $X{:}\langle\theta\rangle$ is transitive on the arcs of $\mathit{\Gamma}$. Moreover, $\theta z = \tau\tilde{z}$ centralizes Y. Let L be a subgroup of Y with $L \cong \mathbb{Z}_7{:}\mathbb{Z}_3$. Then $X{:}\langle\theta\rangle$ contains a regular subgroup $L\times(\langle c\rangle{:}\langle\theta z\rangle) \cong (\mathbb{Z}_7{:}\mathbb{Z}_3)\times\mathrm{D}_{2l}$. Noting that $\mathsf{Aut}\mathit{\Gamma} \geq \langle Y,c,\theta\rangle = \langle Y,c,\theta z\rangle \cong \mathrm{PGL}(2,7)\times\mathrm{D}_{2l}$, the lemma follows. $\Box$

3. Soluble automorphism groups

In this section we determine the graphs having soluble edge-transitive automorphism groups. We first list two basic facts about edge-transitive (Cayley) graphs.

Lemma 3.1. *Let $\mathit{\Gamma} = (V,E)$ be a connected regular X-edge-transitive graph, and let $N \trianglelefteq X$. Then, for any given vertex $u \in V$, all N_u-orbits on $\mathit{\Gamma}(u)$ have the same length. If further X is transitive on V, then the following statements hold:*

- (i) *$|N_u : N_{uv}|$ is constant while $\{u,v\}$ runs over E; in particular, $|N_u : N_{uv}| \neq 1$ if N is not semiregular on V;*
- (ii) *N has at most two orbits on V provided that N_u is transitive on $\mathit{\Gamma}(u)$.*

Proof. Since $\mathit{\Gamma}$ is X-edge-transitive, either X is transitive on V, or X is intransitive on V and X_u is transitive on $\mathit{\Gamma}(u)$ for each $u \in V$. If X_u is transitive on $\mathit{\Gamma}(u)$ then, since $N_u \trianglelefteq X_u$, all N_u-orbits on $\mathit{\Gamma}(u)$ have the same length. Thus, to complete the proof, we assume that X is transitive on V in the following.

Let Δ be an N_u-orbit on $\mathit{\Gamma}(u)$. Then $|\Delta| = |N_u : N_{uv}|$ for $v \in \Delta$. Let $x \in X$ with $v = u^x$. Then $N_v = X_{u^x} \cap N = (N_u)^x$; in particular, $|N_u| = |N_v|$, and so $|\Delta| = |N_v : N_{uv}|$. Let $\{u',v'\}$ be an arbitrary edge of $\mathit{\Gamma}$. Since X is transitive on E, there is $y \in X$ with $\{u',v'\}^y = \{u,v\}$, so $(u',v')^y = (u,v)$ or (v,u). Thus $(N_{u'})^y = X_{u'^y} \cap N = N_{u'^y} = N_u$ or N_v, and $N_{uv} = N_{u'^y v'^y} = X_{u'^y v'^y} \cap N = (N_{u'v'})^y$. Then $|N_{u'} : N_{u'v'}| = |(N_{u'})^y : (N_{u'v'})^y| = |N_{u'^y} : N_{u'^y v'^y}| = |\Delta|$.

Assume that $|N_u : N_{uv}| = 1$ for some edge $\{u, v\} \in E$. Then $N_{u'} = N_{v'}$ for any $\{u', v'\} \in E$. It follows from the connectedness that $N_u = N_w$ for any $w \in V$. Thus $N_u = 1$ as $N_u \leq \mathsf{Aut}\Gamma$, so N is semiregular. Then (i) follows.

Assume further that N_u is transitive on $\Gamma(u)$ but N is intransitive on V. Let B and B' be two N-orbits such that $u \in B$ is adjacent to some $u' \in B'$. By (i), since B' is N_u-invariant, the subgraph $[B, B']$ induced by $B \cup B'$ is regular and has the same valency as Γ. Since Γ is connected, $\Gamma = [B, B']$, and so (ii) follows. □

Lemma 3.2. *Let $\Gamma = \mathsf{Cay}(G, S)$ be a connected Cayley graph and $G \leq X \leq \mathsf{Aut}\Gamma$. Let v be the vertex corresponding to the identity of G. Then Γ is X-half-transitive if and only if S consists of two X_v-orbits S_1 and S_2 with $S_2^{-1} = S_1$, in particular, S contains no involutions.*

Proof. Note that $\{1, s\}^{\widehat{s^{-1}}} = \{1, s^{-1}\}$ for $s \in S$. Then the sufficiency follows.

Assume that Γ is X-half-transitive. Then X_v has exactly two orbits S_1 and S_2 on S, and $|S_1| = |S_2|$. Thus there is some $x \in X$ such that $\{1, s_1\}^x = \{1, s_2\}$, where $s_1 \in S_1$ and $s_2 \in S_2$. Since $X = GX_v = X_vG$, write $x = x_1\hat{g}$ for $x_1 \in X_v$ and $1 \neq g \in G$. Then $\{g, s_1'g\} = \{1, s_1\}^x = \{1, s_2\}$ for some $s_1' \in S_1$ with $x_1 : s_1 \mapsto s_1'$. Thus $g = s_2$ and $s_1'g = 1$; in particular, $s_2^{-1} = s_1' \in S_1$. Then $S_2^{-1} \subseteq S_1$, and so $S_2^{-1} = S_1$. Since $S_1 \cap S_2 = \emptyset$, there are no involutions in S. □

Let G be a group of square-free order, and $\Gamma = (V, E)$ be a connected tetravalent X-edge-transitive Cayley graph of G, where $G \leq X \leq \mathsf{Aut}\Gamma$ and X is soluble.

Lemma 3.3. *Either $\Gamma \cong \mathbf{C}_{m[2]}$, or X has a normal regular subgroup R.*

Proof. For an arbitrary prime divisor p of $|X|$, let $\mathbf{O}_p(X)$ be the largest normal p-subgroup of X. Set $M = \mathbf{O}_p(X)$. Since Γ is of square-free order, either $M = 1$ or the orbits of M are of size p. Suppose that M is not semiregular on V. Then $1 \neq M_u \trianglelefteq X_u$ for $u \in V$. Since Γ has valency 4, the stabilizer X_u is a $\{2, 3\}$-group. By Lemma 3.1, we know that $p = 2$, and so the orbits of M are of size 2. Since M is not semiregular, we have that $\Gamma \cong \mathbf{C}_{m[2]}$, where $m = \frac{|V|}{2}$.

Assume now that $\mathbf{O}_p(X)$ is semiregular on V for all primes p. (Since $X \neq 1$ is soluble, there exists a prime p such that $\mathbf{O}_p(X)$ is nontrivial.) Then $\mathbf{O}_p(X)$ has order 1 or p, so $\mathbf{O}_p(X)$ is cyclic. Let F be the Fitting subgroup of X, that is, $F = \langle \mathbf{O}_p(X) \mid p \text{ divides } |G| \rangle$. Then F is cyclic and acts semiregularly on V; in particular, $|F|$ is a divisor of $|G|$. Since $|G|$ is square-free, there exists a subgroup $L \leq G$ of order $|G|/|F|$. Set $R = F{:}L$. Then $|R| = |G| = |V|$. Let B be an F-orbit on V. Then G_B is regular on B, and so $|F| = |B| = |G_B|$. Thus $|G| = |G_B||L|$, yielding $G = G_BL$. It follows that L acts transitively on the set of all F-orbits. Then R is transitive on V, and so R is a regular subgroup of X.

Since X is soluble, $\mathbf{C}_X(F) \leq F$, yielding $\mathbf{C}_X(F) = F$. Thus $X/F = \mathbf{N}_X(F)/\mathbf{C}_X(F)$ is isomorphic to a subgroup of $\mathsf{Aut}(F)$. Since F is cyclic, $\mathsf{Aut}(F)$ is abelian, so X/F is abelian. Then $R/F \trianglelefteq X/F$, and so $R \trianglelefteq X$. □

This lemma allows us to assume that X contains a normal regular subgroup R. Set $\Gamma = \mathsf{Cay}(R, S)$ for some $S \subset R$. Choose v to be the vertex corresponding to the identity of R. Then we have a subgroup of $\mathsf{Aut}(R)$:

$$\mathsf{Aut}(R,S)=\{\sigma\in\mathsf{Aut}(R)\mid x^\sigma\in S\text{ for all } x\in S\},$$

which is contained in the stabilizer of v in $\mathsf{Aut}\Gamma$. Moreover, by [9, Lemma 2.1], the normalizer $\mathbf{N}_{\mathsf{Aut}\Gamma}(R)=R{:}\mathsf{Aut}(R,S)$. Since $X\le\mathbf{N}_{\mathsf{Aut}\Gamma}(R)$, we have $X_v\le\mathsf{Aut}(R,S)$.

The next lemma determines $\mathsf{Aut}(R,S)$.

Lemma 3.4. *The subgroup* $\mathsf{Aut}(R,S)$ *is isomorphic to* $\mathbb{Z}_2$*,* $\mathbb{Z}_4$ *or* $\mathbb{Z}_2^2$*.*

Proof. Since Γ is connected, $\langle S\rangle=R$, so $\mathsf{Aut}(R,S)$ acts faithfully on S. Since $|S|=4$, we have $\mathsf{Aut}(R,S)\le \mathrm{S}_4$.

Write $R=(A{:}B)\times C$, where A, B and $C=\mathbf{Z}(R)$ are cyclic. Then $|A|$ is odd. By Lemma 2.2, $\bar{A}\trianglelefteq\mathsf{Aut}(R)$ and $\mathsf{Aut}(R)/\bar{A}\cong\mathsf{Aut}(C)\times\mathsf{Aut}(A)$ is abelian. Then the commutator subgroup of $\mathsf{Aut}(R)$ has order dividing $|\bar{A}|=|A|$. Thus the commutator subgroup of every subgroup of $\mathsf{Aut}(R)$ is of odd order. Then $\mathsf{Aut}(R,S)$ has no subgroups isomorphic to D_8, A_4 or S_4, so $\mathsf{Aut}(G,S)\cong\mathbb{Z}_2$, $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$. □

The following lemma allows us to choose $R=G$.

Lemma 3.5. *Assume that* X *contains a normal regular subgroup. Then* $G\trianglelefteq X$*.*

Proof. Let R be a normal regular subgroup of X. Then RG is a subgroup of $X=R{:}X_v$, so $|RG|=|R||G|/|R\cap G|$ is a divisor of $|X|=2|R|$ or $4|R|$. Then either $R=G$ or $R\cap G$ is the $2'$-Hall subgroup of R (and of G). Assume the latter case occurs. Then $R\cap G$ is characteristic in R, and so $R\cap G\trianglelefteq X$. Let P be a Sylow 2-subgroup of X with $X_v\le P$. Then $P=(RX_v)\cap P=(R\cap P)X_v$. Since $R\trianglelefteq X$ and $|R|$ is square-free, $R\cap P\trianglelefteq P$ and $|R\cap P|=2$. It follows that $P=(R\cap P)X_v=(R\cap P)\times X_v$ is abelain. Thus $X/(R\cap G)=(RX_v)/(R\cap G)=(R\cap G)P/(R\cap G)\cong P$ is abelian, so $G/(R\cap G)\trianglelefteq X/(R\cap G)$, and hence $G\trianglelefteq X$. □

Thus we assume that $G\trianglelefteq X$ in the following. Write $\Gamma=\mathsf{Cay}(G,S)$ and $G=(\langle a\rangle{:}\langle b\rangle)\times\langle c\rangle\cong(\mathbb{Z}_m{:}\mathbb{Z}_n)\times\mathbb{Z}_l$ with center $\mathbf{Z}(G)=\langle c\rangle\cong\mathbb{Z}_l$.

Lemma 3.6. *There exists* $\rho\in\mathsf{Aut}(G)$ *such that* $ab^jc\in S^\rho$ *and* $X_v^\rho\le\mathbf{C}_{\mathsf{Aut}(G)}(\langle b\rangle)$*, where* $(j,n)=1$ *and* v *is the vertex corresponding to the identity of* G*.*

Proof. Let $A=\langle a\rangle$, $B=\langle b\rangle$ and $C=\langle c\rangle$. By Lemma 2.2, since $|\bar{A}|=|A|=m$ is odd, $\mathbf{C}_{\mathsf{Aut}(G)}(B)=\mathsf{Aut}(C)\times\mathsf{Aut}(A)$ contains a Sylow 2-subgroup of $\mathsf{Aut}(G)$. Recall that $X_v\le\mathsf{Aut}(G,S)\cong\mathbb{Z}_2$, $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$. Then there is some $\alpha\in\mathsf{Aut}(G)$ such that $X_v^\alpha\le\mathbf{C}_{\mathsf{Aut}(G)}(B)$. Note that α induces an isomorphism from $\mathsf{Cay}(G,S)$ to $\mathsf{Cay}(G,S^\alpha)$ such that $v^\alpha=v$, and that X^α is transitive on the edges of $\mathsf{Cay}(G,S^\alpha)$. Clearly, X^α contains G as a normal regular subgroup. Take $x=a^ib^jc^k\in S^\alpha$. Since $\mathsf{Cay}(G,S^\alpha)$ is connected and X^α is transitive on the edges of $\mathsf{Cay}(G,S^\alpha)$, we have $\langle x^\sigma\mid\sigma\in X_v^\alpha\rangle=G$. Then $G=\langle a^ib^jc^k,a^{ii'}b^jc^{kk'}\rangle$ or $\langle a^ib^jc^k,a^{ii'}b^jc^{kk'},a^{ii''}b^jc^{kk''},a^{ii'''}b^jc^{kk'''}\rangle$, where i', i'', i''', k', k'' and k''' are integers. It follows that $\langle c^k\rangle=\langle c\rangle$, $\langle a^i\rangle=\langle a\rangle$ and $\langle b^j\rangle=\langle b\rangle$, which implies that $(k,l)=1$, $(i,m)=1$ and $(j,n)=1$, respectively. Take $\beta\in\mathbf{C}_{\mathsf{Aut}(G)}(B)$ with $(c^k)^\beta=c$ and $(a^i)^\beta=a$. Set $\rho=\alpha\beta$. Then $cab^j\in S^\rho$ and $X_v^\rho\le\mathbf{C}_{\mathsf{Aut}(G)}(B)$, as desired. □

Now we determine the graphs when X is soluble and G is normal in X.

Lemma 3.7. *Assume that G is normal in X. Then one of the following holds.*

(1) $\mathsf{Aut}(G,S)\cong\mathbb{Z}_2^2$ *or* $\mathbb{Z}_4$, *and either G is cyclic or* $G\cong\mathbb{Z}_l\times\mathrm{D}_{2m}$; $\mathit{\Gamma}$ *is constructed as in Construction* 2.3 *and* 2.4.
(2) $\mathsf{Aut}(G,S)\cong\mathbb{Z}_2$ *and* $G\cong\mathbb{Z}_l\times(\mathbb{Z}_m{:}\mathbb{Z}_n)$, *where* $n\geq 3$ *and the center* $\mathbf{Z}(G)\cong\mathbb{Z}_l$; $\mathit{\Gamma}$ *is constructed as in Construction* 2.6.

Proof. Since G is normal in X, we have $X\leq\mathbf{N}_{\mathsf{Aut}\mathit{\Gamma}}(G)=G{:}\mathsf{Aut}(G,S)$ and $X_v\leq\mathsf{Aut}(G,S)$. Note that $G{:}\mathsf{Aut}(G,S)$ is transitive on the edges of $\mathit{\Gamma}$ as $\mathit{\Gamma}$ is X-edge-transitive. To complete the proof of Lemma 3.7, we may assume that $X_v=\mathsf{Aut}(G,S)$ and $X=G{:}\mathsf{Aut}(G,S)$. By Lemma 3.6, up to isomorphism of graphs, we may assume that $ab^jc\in S$ and $X_v=\mathsf{Aut}(G,S)\leq\mathbf{C}_{\mathsf{Aut}(G)}(\langle b\rangle)$, where $1\leq j\leq n-1$ with $(j,n)=1$.

Assume first that $G=\langle c\rangle=\mathbb{Z}_l$. Then $a=b=1$ and $c\in S$. Since $\sigma:\ \ c^i\mapsto c^{-i}$ is an automorphism of G, we have $\sigma\in\mathsf{Aut}(G,S)$. By Lemma 3.2, $\mathit{\Gamma}$ is X-arc-transitive, so $\mathsf{Aut}(G,S)\cong\mathbb{Z}_4$ or $\mathbb{Z}_2^2$. Thus the four elements of S are conjugate under $\mathsf{Aut}(G,S)$, and $\mathit{\Gamma}$ is given as in Construction 2.3.

Assume that $n=2$ and $l=1$. Then $G=\langle a\rangle{:}\langle b\rangle\cong\mathrm{D}_{2m}$, and S contains the involution ab. By Lemma 3.2, $\mathit{\Gamma}$ is X-arc-transitive, hence $\mathsf{Aut}(G,S)$ is transitive on S, and so $\mathsf{Aut}(G,S)\cong\mathbb{Z}_4$ or $\mathbb{Z}_2^2$. Suppose first that $\mathsf{Aut}(G,S)=\langle\rho\rangle\cong\mathbb{Z}_4$. Then $a^\rho=a^i$ and $b^\rho=b$ for $i^4\equiv 1\,(\mathsf{mod}\ m)$, and $S=\{ab,a^ib,a^{i^2}b,a^{i^3}b\}$. Since $\mathit{\Gamma}$ is connected, $G=\langle S\rangle=\langle a^{i-1},a^{i^2-1},a^{i^3-1},ab\rangle=\langle a^{i-1}\rangle{:}\langle ab\rangle$. Thus $\langle a^{i-1}\rangle=\langle a\rangle$, so $(i-1,m)=1$, yielding $i^3+i^2+i+1\equiv 0\,(\mathsf{mod}\ m)$. Thus $\mathit{\Gamma}$ is given as in Construction 2.4 (i). Now let $\mathsf{Aut}(G,S)=\langle\sigma\rangle\times\langle\tau\rangle\cong\mathbb{Z}_2^2$. Set $a^\sigma=a^{i_1}$ and $a^\tau=a^{i_2}$, where $i_1^2\equiv i_2^2\equiv 1\,(\mathsf{mod}\ m)$. Then $S=\{ab,a^{i_1}b,a^{i_2}b,a^{i_1i_2}b\}$. Since $G=\langle S\rangle=\langle a^{i_1-1},a^{i_2-1},a^{i_1i_2-1},ab\rangle=\langle a^{i_1-1},a^{i_2-1}\rangle{:}\langle ab\rangle$, we have $\langle a\rangle=\langle a^{i_1-1},a^{i_2-1}\rangle$, yielding $(i_1-1,i_2-1,m)=1$. Then $\mathit{\Gamma}$ is given as in Construction 2.4 (ii).

Assume that $n=2$ and $l>1$. Then $abc\in S$, l is odd and abc has order $2l$. By Lemma 3.2, since $\mathit{\Gamma}$ is X-edge-transitive, there is $\rho\in X_v=\mathsf{Aut}(G,S)$ such that $(abc)^\rho\neq(abc)^{-1}$. Noting that ρ has order 2 or 4, we set $a^\rho=a^i$ and $c^\rho=c^k$, where $i^4\equiv 1\,(\mathsf{mod}\ m)$ and $k^4\equiv 1\,(\mathsf{mod}\ l)$. Then $S^{-1}=S=\{abc,a^ibc^k,abc^{-1},a^ibc^{-k}\}$. Since $\mathit{\Gamma}$ is connected, $G=\langle S\rangle=\langle abc,a^ibc^k\rangle=\langle a^{i-1},ab,c\rangle=(\langle a^{i-1}\rangle{:}\langle ab\rangle)\times\langle c\rangle$. It follows that $\langle a^{i-1}\rangle=\langle a\rangle$, so $(i-1,m)=1$. Suppose that ρ has order 4, then $S=\{abc,a^ibc^k,a^{i^2}bc^{k^2},a^{i^3}bc^{k^3}\}$, so $abc^{-1}=(abc)^{-1}=a^{i^2}bc^{k^2}$ or $a^{i^3}bc^{k^3}$, yielding $i^2\equiv 1\,(\mathsf{mod}\ m)$ and $k^2\equiv -1\,(\mathsf{mod}\ l)$. Moreover, $i\equiv -1\,(\mathsf{mod}\ m)$ as $(i-1,m)=1$. Thus $\mathit{\Gamma}$ is given as in Construction 2.4 (iii). Now let $\sigma:=\rho$ be of order 2. Then $i^2\equiv 1\,(\mathsf{mod}\ m)$ and $k^2\equiv 1\,(\mathsf{mod}\ l)$. Thus $i\equiv -1\,(\mathsf{mod}\ m)$, and $\mathit{\Gamma}$ is given as in Construction 2.4 (iv). Take $\tau\in\mathsf{Aut}(G)$ such that $a^\tau=a$, $b^\tau=b$ and $c^\tau=c^{-1}$. Then $\sigma\neq\tau\in\mathsf{Aut}(G,S)$, $\sigma\tau=\tau\sigma$ and $\tau^2=1$, so $\mathsf{Aut}(G,S)=\langle\sigma,\tau\rangle\cong\mathbb{Z}_2^2$.

Finally, let $n\geq 3$. Recall that $ab^jc\in S$. Since $\mathit{\Gamma}$ is X-edge-transitive, by Lemma 3.2, there is $\tau\in X_v=\mathsf{Aut}(G,S)$ such that $(ab^jc)^\tau\neq(ab^jc)^{-1}$. Set $a^\tau=a^i$ and $c^\tau=c^k$. Then $S=\{ab^jc,a^ib^jc^k,b^{-j}a^{-1}c^{-1},b^{-j}a^{-i}c^{-k}\}$. It is easily shown that $\{ab^jc,a^ib^jc^k\}^\sigma\neq\{b^{-j}a^{-1}c^{-1},b^{-j}a^{-i}c^{-k}\}$ for any $\sigma\in\mathbf{C}_{\mathsf{Aut}(G)}(\langle b\rangle)$. Thus $\mathsf{Aut}(G,S)$ is not transitive on S, and so $\mathsf{Aut}(G,S)=\langle\tau\rangle\cong\mathbb{Z}_2$. Then $i^2\equiv 1\,(\mathsf{mod}\ m)$ and $k^2\equiv 1\,(\mathsf{mod}\ l)$. Since $\mathit{\Gamma}$ is connected, $G=\langle S\rangle=\langle ab^jc,a^ib^jc^k\rangle=\langle a^{i-1},ab^j,c\rangle=(\langle a^{i-1}\rangle{:}\langle ab^j\rangle)\times\langle c\rangle$. Thus $\langle a^{i-1}\rangle=\langle a\rangle$, so $(i-1,m)=1$, hence $i\equiv -1\,(\mathsf{mod}\ m)$ as $i^2\equiv 1\,(\mathsf{mod}\ m)$. Then $\mathit{\Gamma}$ is given as in Construction 2.6. ☐

4. Insoluble automorphism groups

In this section, we study the case where the automorphism groups are insoluble.

An s-arc of $\Gamma = (V, E)$ is a sequence of $s + 1$ vertices $v_0, v_1, \ldots, v_s$ such that v_i is adjacent to v_{i+1} and $v_i \neq v_{i+2}$. For a subgroup $X \leq \mathsf{Aut}\Gamma$, the graph Γ is said to be (X, s)-*arc-transitive* if X acts transitively on V and on the set of all s-arcs of Γ, and (X, s)-*transitive* if further X is intransitive on the set of all $(s + 1)$-arcs of Γ.

The vertex stabilizer for s-arc-transitive graphs of valency 4 is known, refer to [34].

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Lemma 4.1. *Let $\Gamma = (V, E)$ be a connected (X, s)-transitive graph of valency* 4*. Then, for $u \in V$, the stabilizer X_u and s are listed in the following table,*

s	2	3	4	7
X_u	A_4, S_4	$\mathbb{Z}_3\times A_4$, $(\mathbb{Z}_3\times A_4).\mathbb{Z}_2$, $S_3\times S_4$	$\mathbb{Z}_3^2{:}\mathrm{GL}(2,3)$	$[3^5]{:}\mathrm{GL}(2,3)$

where $[3^5]$ is a 3*-group of order 3^5.*

For a finite group X, the *socle* of X, denoted by $\mathsf{soc}(X)$, is the subgroup generated by all minimal normal subgroups of X. The group X is said to be *almost simple* if its socle $\mathsf{soc}(X)$ is a non-abelian simple group.

In the rest of this section, assume that $\Gamma = (V, E)$ is a connected tetravalent graph of square-free order such that a subgroup $X \leq \mathsf{Aut}\Gamma$ is transitive on both V and E.

Lemma 4.2. *If Γ has order $|V| = 21$ then $X \neq \mathrm{PSL}(2, 7)$.*

Proof. Suppose that $X = \mathrm{PSL}(2, 7)$ and Γ is a connected X-edge-transitive graphs of valency 4 and order 21. Then X is transitive on V and, for $u \in V$, the stabilizer $X_u \cong D_8$ is a Sylow 2-subgroup of X. Let $v \in \Gamma(u)$. Then $|X_u : X_{uv}| = 2$ or 4. Set $v = u^x$ for some $x \in X$. Since Γ is connected, $\langle X_u, x\rangle = X$; in particular, $x \notin X_u$.

Let $|X_u : X_{uv}| = 4$. Then X_u is transitive on $\Gamma(u)$, so Γ is X-arc-transitive. We may choose x such that $(u, v)^x = (v, u)$, yielding $x \in \mathbf{N}_X(X_{uv}) \cong D_8$. In particular, $\mathbf{N}_X(X_{uv}) \neq X_u$. Then $|\mathbf{N}_{X_u}(X_{uv})| = 4$. Thus $\mathbf{N}_{X_u}(X_{uv})$ is normal in both $\mathbf{N}_X(X_{uv})$ and X_u, so $\mathbf{N}_{X_u}(X_{uv}) \trianglelefteq \langle X_u, \mathbf{N}_X(X_{uv})\rangle$. Checking the subgroups of $\mathrm{PSL}(2, 7)$, we get $\langle X_u, \mathbf{N}_X(X_{uv})\rangle \cong S_4$, which contradicts $\langle X_u, x\rangle = X$.

Let $|X_u : X_{uv}| = 2$. Then $|X_{uv}| = 4$, so $X_{uv} \trianglelefteq M := \langle X_u, X_v\rangle$, and hence $M \cong S_4$. Noting that X_u and X_v are two Sylow 2-subgroups of M, there is some $y \in M$ such that $X_u^y = X_v = X_u^x$. Thus $xy^{-1} \in \mathbf{N}_X(X_u) = X_u$, so $\langle X_u, x\rangle \leq \langle X_u, xy^{-1}, y\rangle \leq M$, again a contradiction. Then the lemma follows. □

Lemma 4.3. *Assume that X is almost simple and contains a regular subgroup G. Then, for $u \in V$, the triple (X, G, X_u) is one of the triples listed in Table* 3*.*

Proof. By the assumption, $X = GX_u$, so $|X| = |G||X_u|$ for $u \in V$. Since Γ is of valency 4 and $|G|$ is square-free, either

(i) X_u is a 2-group, and hence r^2 does not divide $|X|$ for any odd prime r; or

(ii) X_u is given in Lemma 4.1, and hence none of 2^6, 3^8 and r^2 is a divisor of $|X|$, where r is a prime with $r \geq 5$.

In particular, $|X|$ is not divisible by $2^6 \cdot 3^2$. Next we consider the socle T of X. Since T is normal in X, the T-orbits on V have the same length $|T : T_u|$. Thus $|T : T_u|$ is square-free, and T has a $\{2, 3\}$-subgroup of square-free index.

X	G	X_u
A_5	$\mathbb{Z}_5$	A_4
S_5	$\mathbb{Z}_5$	S_4
$\mathrm{PGL}(2,7)$	D_{14}	S_4
	$\mathbb{Z}_7{:}\mathbb{Z}_3$	D_{16}
	$\mathbb{Z}_7{:}\mathbb{Z}_6$	D_8
$\mathrm{PSL}(2,11)$	$\mathbb{Z}_{11}{:}\mathbb{Z}_5$	A_4
$\mathrm{PGL}(2,11)$	$\mathbb{Z}_{11}{:}\mathbb{Z}_5$	S_4
$\mathrm{PGL}(2,11)$	$\mathbb{Z}_{11}{:}\mathbb{Z}_{10}$	A_4
$\mathrm{PSL}(2,23)$	$\mathbb{Z}_{23}{:}\mathbb{Z}_{11}$	S_4
$\mathrm{PSL}(3,3){:}\mathbb{Z}_2$	D_{26}	$\mathbb{Z}_3^2{:}\mathrm{GL}(2,3)$

Table 3

Suppose that T is a sporadic simple group. Since $|T|$ is not divisible by $2^6 \cdot 3^2$, we have $X = T = \mathrm{M}_{11}$ or J_1, and further, by the Atlas [5], J_1 does not have a proper subgroup of index a $\{2,3\}$-number. Thus $X = \mathrm{M}_{11}$, and then $3 \cdot 2^3$ divides $|X_u|$ and $|X_u|$ divides $2^4 \cdot 3^2$. By Lemma 4.1, $X_u \cong \mathrm{S}_4$, $(3{\times}\mathrm{A}_4).2$ or $\mathrm{S}_4{\times}\mathrm{S}_3$. Checking the subgroups of M_{11} in the Atlas [5], we get $X_u \cong \mathrm{S}_4$, so $|V| = |X : X_u| = 330$. Then $|G| = 330$ as G is regular on V; however, M_{11} has no such a subgroup, a contradiction.

Let $T = \mathrm{A}_n$. Since 5^2 does not divide $|X|$, we have $n \le 9$. The groups A_8 and A_9 are excluded as their orders are divisible by $2^6 \cdot 3^2$. For A_7, neither A_7 nor S_7 has a subgroup of index dividing $|X_v|$ other than A_7. Suppose that $T = \mathrm{A}_6$. Then $X \le \mathsf{Aut}(T) \cong \mathrm{A}_6.\mathbb{Z}_2^2$, so $|X_u|$ is divisible by 3 but not by 3^3. Examming the maximal subgroups of X in the Atlas [5], it follows that $X_u \cong \mathrm{A}_4$ or S_4; however, X does not have a subgroup of order divisible by 15, a contradiction. Thus $T = \mathrm{A}_5$, and $G \cong \mathbb{Z}_5$.

Assume now that T is a simple group of Lie type defined over $\mathrm{GF}(q)$, where $q = p^f$ is a power of a prime p. Then we can get T by checking the orders of finite simple groups of Lie type (see [12, Table 5.1.A and Table 5.1.B], for example). Since r^2 does not divide $|T|$ for any prime $r \ge 5$, either $T = \mathrm{PSL}(2,p)$, or $p \in \{2,3\}$.

Case 1. Let $T = \mathrm{PSL}(2,p)$ for a prime $p \ge 5$. In this case, $X = T$ or $\mathrm{PGL}(2,p)$, a Sylow 2-subgroup of X is dihedral, and a Sylow 3-subgroup of X is cyclic. Thus, by Lemma 4.1, either X_u is a 2-group, or $X_u \cong \mathrm{A}_4$ or S_4.

Note that TG is a subgroup of X as $T \trianglelefteq X$. Then $|TG : G|$ is a divisor of $|X : G| = |X_u|$. If $X \ne TG$ then $G \le T$, so $|G| = |T : T_u| = |X : X_u|$, yielding $|X_u : T_u| = 2$. Since $|TG| = |T||G|/|T \cap G|$, we have $|TG : G| = |T : (T \cap G)|$, so $|T : (T \cap G)| = |X_u|$ or $|T_u|$ depending on whether or not $X = TG$, respectively.

Assume that $X_u \cong \mathrm{A}_4$ or S_4. Then $T_u \cong \mathrm{A}_4$ or S_4. Consider the action of T on $[T : (T \cap G)]$ induced by right multiplication. Then T has a (faithful) transitive representation of degree 12 or 24. It follows from [12, Table 5.2 A] that $p \le 23$. Checking the subgroups of $\mathrm{PSL}(2,p)$ and $\mathrm{PGL}(2,p)$ in the Atlas [5], we conclude that $p = 5$, 7, 11 or 23, and the triple (X, G, X_u) is described as in Table 4.3.

Now let X_u be a 2-group. Then $|T : (T \cap G)|$ is a power of 2. By [10], $|T : (T \cap G)| = p + 1 = 2^e$ for $e \ge 3$. It follows that $|T_u| = 2^{e-1}$ or 2^e. Thus $T_u \cong \mathrm{D}_{2^e}$ or $\mathrm{D}_{2^{e-1}}$.

Suppose that 32 divides $|T_u|$. Let $v \in \Gamma(u)$. By Lemma 3.1, T_{uv} has index 2 or 4 in both T_u and T_v, then T_{uv} contains a subgroup $C \cong \mathbb{Z}_4$. It is easily shown that C is normal in both T_u and T_v, and so $C \trianglelefteq \langle T_u, T_v \rangle$. Thus $T \ne \langle T_u, T_v \rangle := Q$ as T is simple. Checking the subgroups of T (see [11, 8.27], for example), we conclude that $T_u \cong T_v \cong \mathrm{D}_{2^{e-1}}$, and $Q \cong \mathrm{D}_{2^e} = \mathrm{D}_{p+1}$ which is maximal in T. Let $w \in \Gamma(v)$. Then a similar argument implies that $Q_1 := \langle T_v, T_w \rangle \cong \mathrm{D}_{p+1}$. Note that T_v is normal in both Q and Q_1. Thus $T_v \trianglelefteq \langle Q, Q_1 \rangle$, yielding $Q = Q_1$. By the connectedness of Γ, we conclude that $Q = \langle T_v \mid v \in V \rangle$. Thus, for any $x \in X$, we have $T_v^x = (X_v \cap T)^x = X_{v^x} \cap T = T_{v^x} \leq Q$. Then $Q \trianglelefteq T$, a contradiction. Therefore, $|T_u|$ divides 16, and so $2^e = 8$, 16 or 32. Then $p = 2^e - 1 = 7$ or 31, and $T = \mathrm{PSL}(2,7)$ or $\mathrm{PSL}(2,31)$, respectively.

Suppose that $T = \mathrm{PSL}(2,31)$. Then $T_u \cong \mathrm{D}_{16}$ as $|T : T_u|$ is square-free and $|T_u|$ is not divisible by 32. Checking the subgroups of T, we know that T has no subgroups of order $|T : T_u| = 930$. Thus $X = \mathrm{PGL}(2,31)$ and $X_u \cong \mathrm{D}_{32}$. Note that each Sylow 2-subgroup of X is a maximal subgroup. Then a similar argument as above implies that X has a normal Sylow 2-subgroup, which is impossible.

Therefore, $T = \mathrm{PSL}(2,7)$, so $X = T$ or $\mathrm{PGL}(2,7)$. By Lemma 4.2, checking the subgroups of X implies that $X = \mathrm{PGL}(2,7)$ and $G \cong \mathbb{Z}_7{:}\mathbb{Z}_3$ or $\mathbb{Z}_7{:}\mathbb{Z}_6$.

Case 2. Let $p \in \{2,3\}$. Assume that X_u is a 2-group. Then T has a subgroup of square-free order with index a power of 2. By [10], $T = \mathrm{PSL}(t,s)$ and $\frac{t^s-1}{s-1}$ is a power of 2, where t is a prime and s is a power of some odd prime. Recall that, in this case, $|X|$ is not divisible by r^2 for any odd prime. It follows that $t = 2$ and s is a prime, so $T = \mathrm{PSL}(2,s)$. By Case 1, $X = \mathrm{PGL}(2,7) \cong \mathrm{PSL}(3,2){:}\mathbb{Z}_2$ and $G \cong \mathbb{Z}_7{:}\mathbb{Z}_3$ or $\mathbb{Z}_7{:}\mathbb{Z}_6$.

We next assume that X_u is not a 2-group. Then, by Lemma 4.1, $|X_u|$ is not divisible by 2^5 and 3^7. Thus $|X| = |G||X_u|$ is not divisible by p^8. We check the orders of simple groups. Taking into account the isomorphisms among simple groups (see [12, Proposition 2.9.1]), we know that T is one of $\mathrm{PSL}(2,q)$, $\mathrm{PSL}(3,2)$, $\mathrm{PSL}(3,3)$, $\mathrm{PSL}(3,9)$, $\mathrm{PSL}(4,2)$, $\mathrm{PSL}(4,3)$, $\mathrm{PSU}(3,3)$, $\mathrm{PSU}(3,9)$, $\mathrm{PSp}(4,3)$, $\mathrm{Sz}(8)$ and $\mathrm{G}_2(3)$. However, $\mathrm{PSL}(3,9)$, $\mathrm{PSL}(4,2)$, $\mathrm{PSL}(4,3)$ and the last four groups are excluded as they have orders divisible by 2^6 or 5^2. Recalling that T has a $\{2,3\}$-subgroup of square-free index, $\mathrm{PSU}(3,3)$ is excluded by checking its subgroups in the Atlas [5]. For $T = \mathrm{PSL}(3,2)$ or $\mathrm{PSL}(3,3)$, checking the subgroups of X, the triple (X, G, X_u) is known as in Table 4.3.

To complete the proof, we let $T = \mathrm{PSL}(2,p^f)$ with $f \geq 2$ and $p = 2$ or 3. Then a Sylow p-subgroup of T has order p^f. Suppose that $f \geq 4$. Then p^3 is a divisor of $|T_u|$. Checking the subgroups of T (see [11, 8.27], for example), we know that $T_u \cong \mathbb{Z}_p^f{:}\mathbb{Z}_t$ or $\mathbb{Z}_p^{f-1}{:}\mathbb{Z}_t$, where t divides $p^f - 1$; however, none of the groups in Lemma 4.1 has such a subgroup of index no more than 2, a contradiction. Thus $f \leq 3$. Further, by the Atlas [5], neither $\mathrm{PSL}(2,8)$ nor $\mathrm{PSL}(2,27)$ has subgroups of square-free index. Noting that $\mathrm{PSL}(2,9) \cong \mathrm{A}_6$, we have $T = \mathrm{PSL}(2,4) \cong \mathrm{A}_5$. Then the Lemma follows. □

We now determine the structure of insoluble groups X. Let K be the largest soluble normal subgroup of X. Consider the normal quotient Γ_K. By Lemma 2.9, since X/K is insoluble, Γ is a normal cover of Γ_K. Thus K is the kernel of X acting on $V\Gamma_K$, and K is semiregular on V; in particular, $|K|$ is square-free.

Lemma 4.4. *Assume that X is insoluble. Let K be the largest soluble normal subgroup of X. Then $X = K{:}Y$, where Y is almost simple such that the socle $\mathsf{soc}(Y)$ is normal in X, the greatest common divisor $(|Y|, |K|)$ is a divisor of 6, and $X_u \cong Y_B$ for a K-orbit B and $u \in V$. If further X has a regular subgroup G, then we may* *choose the group Y such that X contains a regular subgroup $K{:}(G\cap Y)$; in this case, Y, $G\cap Y$ and Y_B are known respectively as in the three columns of Table* 3.

Proof. We first show that X is a split extension of K and some $Y\le X$ by induction on the order of K. This is trivial if $K=1$. Let $K\neq 1$, p be the largest prime divisor of $|K|$, and P be the Sylow p-subgroup of K. Then P has order p and is normal in X and, by Lemma 2.9, Γ is a normal cover of Γ_P as X/P is insoluble. Let $u\in V$ and Δ be the P-orbit containing u. Since $|V|$ is square-free, $|\Delta|=p$ is coprime to $|X:X_\Delta|$. Then $X_\Delta=P{:}X_u$ contains a Sylow p-subgroup of X. It follows from Gaschtz' Theorem (see [2, 10.4]) that the extension $X=P.(X/P)$ splits over P, that is, $X=P{:}X_1$ for $X_1\le X$ with $X_1\cap P=1$. Since Γ is a normal cover of Γ_P, the kernel of X acting on $V\Gamma_P$ equals to P. Thus X_1 is faithful and transitive on both $V\Gamma_P$ and $E\Gamma_P$. Further, $K=K\cap PX_1=P(K\cap X_1)$ and $K\cap X_1\trianglelefteq X_1$. Since $|V\Gamma_P|<|V|$, we may assume by induction that $X_1=(K\cap X_1){:}Y$. Then $X=P((K\cap X_1)Y)=KY$, and $K\cap Y\le K\cap X_1$ yielding $K\cap Y\le K\cap X_1\cap Y=1$. Thus $X=K{:}Y$.

Since Γ is a normal cover of Γ_K, we know that Y is faithful and transitive on both $V\Gamma_K$ and $E\Gamma_K$. Let N be a minimal normal subgroup of Y. Then KN is normal in X, so KN is insoluble by the choice of K. Thus N is insoluble, so N is a direct product of isomorphic non-abelain simple groups. Recalling that $|X|$ is not divided by r^2 for a prime $r\ge 5$, it follows that N is simple. Since Γ_K has square-free order, N is not semiregular on $V\Gamma_K$. Thus either N is transitive on $V\Gamma_K$, or Γ_K is not a normal cover of its quotient graph with respect to N. By Lemma 2.9, Y/N is soluble. It follows that N is the unique minimal normal subgroup of Y. Then Y is almost simple. Since $K\trianglelefteq X$, we have $X/\mathbf{C}_X(K)=\mathbf{N}_X(K)/\mathbf{C}_X(K)\lesssim \mathsf{Aut}(K)$. Noting that $\mathsf{Aut}(K)$ is soluble as $|K|$ is square-free, $N=\mathsf{soc}(Y)<\mathbf{C}_X(K)$, yielding $N\trianglelefteq X$.

Let B be the K-orbit containing $u\in V$. Then $K{:}X_u=X_B=X_B\cap(K{:}Y)=K{:}Y_B$, so $X_u\cong Y_B$ is a $\{2,3\}$-group. Noting that $|Y|=|V\Gamma_K||Y_B|$, since $|V|$ is square-free, we have $(|K|,|V\Gamma_K|)=1$. Thus $(|Y|,|K|)=(|Y_B|,|K|)$ is a divisor of 6.

Finally, assume that G is a regular subgroup of X. Let $L\le G$ with $|G|=|K||L|$. Then $R:=K{:}L$ is a regular subgroup of X, and $R=R\cap X=R\cap(K{:}Y)=K{:}(R\cap Y)$. Note L and $R\cap Y$ are Hall subgroups of R. Then L and $R\cap Y$ are conjugate in R, that is, $L=(R\cap Y)^h$ for some $h\in R$. Thus, replacing Y by Y^h, we may assume that $L=R\cap Y$, and so $L=G\cap Y$. It is easily shown that L is regular on the set of all K-orbits on V. Then, identifying Y with a subgroup of $\mathsf{Aut}\Gamma_K$, the quotient graph Γ_K is a Y-edge-transitive Cayley graph of L. Further, since Y is almost simple, the triple (Y,L,Y_B) is known by Lemma 4.3. □

5. Graphs with insoluble automorphism groups

Let G be a group of square-free order, and $\Gamma=(V,E)$ be a connected X-edge-transitive tetravalent Cayley graph of G, where $G\le X\le\mathsf{Aut}\Gamma$ and X is insoluble. Set $X=K{:}Y$ as in Lemma 4.4. Then X has a regular subgroup $K{:}L$ for $L=G\cap Y$.

5.1. **2-arc-transitive graphs.** Assume that Γ is $(X,2)$-arc-transitive. Then, for $u\in V$, the stabilizer X_u is 2-transitive on $\Gamma(u)$. Since $T:=\mathsf{soc}(Y)\trianglelefteq X$, by Lemma 3.1 (i), T_u acts nontrivially on $\Gamma(u)$, and so T_u acts transitively on $\Gamma(u)$. Then, by Lemma 3.1 (ii), T has at most two orbits on V. It follows that Γ is T-edge-transitive.

Since K is semiregular, $|K|$ is a divisor of $|V|$. Then each odd prime divisor of $|K|$ is also a divisor of $|T|$. Recalling that $(|Y|, |K|)$ divides 6, we have $|K| = 1$, 2, 3 or 6.

Lemma 5.1. *Let B be a K-orbit on V and $u \in B$. Then $T_u \trianglelefteq T_B$, and*

(1) $|K| = 1$ *or* 3, $T_B/T_u \cong K$*; or*
(2) $|K| = 2$ *or* 6, $T_B/T_u \cong K$*, and T is transitive on V; or*
(3) $|K| = 2$ *or* 6, $|T_B/T_u| = \frac{|K|}{2}$ *and T has two orbits on V.*

Proof. Let $N = K \times T$. Assume that $|K| = 1$ or 3. Then either T is transitive on V or both N and T have two orbits on V. Thus the K-orbit B lies in one of T-orbits, so T_B is transitive on B. Denote by T_B^B the permutation group induced by T_B on B. Noting that $KT_B = K \times T_B$ and K is regular on B, it follows from [6, Theorem 4.3A] that $T_B^B \cong K^B \cong K$ and T_B^B is regular on B, and so $T_u \trianglelefteq T_B$, hence $K \cong T_B^B \cong T_B/T_u$.

Assume that $|K| = 2$ or 6. Then N is transitive on V. If T is transitive on V, then T_B is transitive on B, so $K \cong T_B^B \cong T_B/T_u$. Suppose that T has two orbits on V. Then T_B has exactly two orbits on B with length $\frac{|K|}{2}$. Let B_1 be the T_B-orbit containing u. Considering the action of $K_{B_1} \times T_B$ on B_1, we get $T_u \trianglelefteq T_B$ and $K_{B_1} \cong T_B^{B_1} \cong T_B/T_u$. Then the lemma follows. □

Lemma 5.2. *If $T = \mathrm{PSL}(3,3)$, then Γ is the point-line incidence graph of the projective plane* $\mathrm{PG}(2,3)$*, which is a* 4*-transitive Cayley graph of* D_{26}.

Proof. Let $T = \mathrm{PSL}(3,3)$. Then $L \cong \mathrm{D}_{26}$, $|K| = 1$ or 3, and $Y_B = T_B \cong \mathbb{Z}_3^2{:}\mathrm{GL}(2,3)$. It is easily shown that T_B has no normal subgroups of index 3. By Lemma 5.1, $K = 1$. Then $X = Y = \mathrm{PSL}(3,3){:}\mathbb{Z}_2$. By [14], the lemma follows. □

Noting that $\mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$ and S_4 has no normal subgroups of index 3, a similar argument as above implies the following lemma.

Lemma 5.3. *If $T = \mathrm{PSL}(2,7)$, then Γ is the point-line non-incidence graph of the projective plane* $\mathrm{PG}(2,2)$*, which is a Cayley graph of* D_{14}.

Lemma 5.4. *If $T = \mathrm{A}_5$, then Γ is isomorphic to one of* K_5, $\mathsf{K}_{5,5} - 5\mathsf{K}_2$ *and the* S_3*-cover of* K_5 *given in Example* 2.14.

Proof. Let $T = \mathrm{A}_5$. If K is cyclic, then Γ is a circulant and, by [18], Γ is one of K_5 and $\mathsf{K}_{5,5} - 5\mathsf{K}_2$. Thus we assume that $K = \mathrm{S}_3$. Since $X/\mathbf{C}_X(K) = \mathbf{N}_X(K)/\mathbf{C}_X(K) \lesssim \mathsf{Aut}(K) = \mathsf{Inn}(K) \cong \mathrm{S}_3$, we have $Y \leq \mathbf{C}_X(K)$, so $X = K \times Y$. Then, for a K-orbit B and $u \in B$, we have $X_u \cong Y_B \cong \mathrm{A}_4$ or S_4, $X_B = K \times Y_B$, so $|V| = |K||Y : Y_B| = 30$. Recalling that T_u is transitive on $\Gamma(u)$, it follows that $|T_u|$ is divided by 4, so T is not transitive on V. Then T has two orbits on V, so $|T_u| = 4$, hence $T_u \cong \mathbb{Z}_2^2$ as $T_u \trianglelefteq X_u$. Noting that T_u is regular on $\Gamma(u)$, we have $X_u = T_u{:}X_{uv}$, where $v \in \Gamma(u)$

Let $Y = \mathrm{S}_5$. Then $T_B \cong \mathrm{A}_4$, $X_u \cong Y_B \cong \mathrm{S}_4$, and $X_{uv} \cong \mathrm{S}_3$. Write $X_{uv} = \langle g\rangle{:}\langle h\rangle$, where g is of order 3 and h is an involution with $g^h = g^{-1}$. Clearly, g, $h \notin K$. Since $|T_u| = 4$, we have $|Y_u| = 4$ or 8 as $|Y{:}T| = 2$. Consider the action of Y_B on B. Since K is regular on B, it follows from [6, Theorem 4.3A] that Y_B^B is semiregular on B. So $Y_u \trianglelefteq Y_B \cong \mathrm{S}_4$. It follows that $Y_u = T_u \cong \mathbb{Z}_2^2$. Thus g, $h \notin Y$, so $g = y_1y_2$ and $h = z_1z_2$, where $y_1 \in Y$, $y_2 \in K$, $z_1 \in Y$ and $z_2 \in K$. It is easily shown that $\langle y_1, z_1\rangle \cong \mathrm{S}_3$ and $\langle T_u, y_1, z_1\rangle \cong \mathrm{S}_4$. Thus Γ is the S_3-cover of K_5 given in Example 2.14.

Now let $Y = T = \mathrm{A}_5$. Then $X_u \cong Y_B = T_B \cong \mathrm{A}_4$ and $X_{uv} \cong \mathbb{Z}_3$. It is easily shown that $X_{uv} = \langle y_1y_2\rangle$, where $y_1 \in Y$ and $y_2 \in K$ are of order 3. Further, $\mathbf{N}_X(\langle y_1y_2\rangle) = \langle y_1, y_2\rangle{:}\langle z_1z_2\rangle = \langle y_1y_2\rangle{:}\langle y_2, z_1z_2\rangle$, where $z_1 \in Y$ is an involution such that $\langle y_1\rangle{:}\langle z_1\rangle \cong \mathrm{S}_3$. Write $\varGamma \cong \mathsf{Cos}(X, X_u, X_uxX_u)$ for a 2-element $x \in \mathbf{N}_X(\langle y_1y_2\rangle)$ with $\langle x, X_u\rangle = X$. Then $X_uxX_u = X_uy_2^iz_1z_2X_u$. Noting that $X_u^{y_2} = X_u$ and $(X_uy_2^iz_1z_2X_u)^{y_2} = X_uy_2^{i+1}z_1z_2X_u$, it follows that $\varGamma \cong \mathsf{Cos}(X, X_u, X_uz_1z_2X_u)$, that is, $\varGamma$ is unique up to isomorphism. Note that the graph in the above paragraph is $(KT, 2)$-arc-transitive. Thus $\varGamma$ is isomorphic to the S_3-cover of K_5 given in Example 2.14. □

Lemma 5.5. *If* $T = \mathrm{PSL}(2, 23)$, *then* $\varGamma$ *is isomorphic to one of the following graphs:* $\mathrm{P}_{23,11}$, $\mathrm{P}^{(2)}_{23,11}$, *the graphs in Examples* 2.11 *and* 2.14.

Proof. Assume that $T = \mathrm{PSL}(2, 23)$. Then $X = K{\times}T$ and $X_u \cong T_B \cong \mathrm{S}_4$, where $u \in V$ and B is the K-orbit containing u. Noting that S_4 has no quotients isomorphic to $\mathbb{Z}_3$ and $\mathbb{Z}_6$, it follows from Lemma 5.1 that K is one of 1, $\mathbb{Z}_2$ and S_3.

If $K = 1$, then $X = T$ and X_u is a maximal subgroup, so $\mathsf{Aut}\varGamma = X$ and $\varGamma \cong \mathrm{P}_{23,11}$ by [16]. If $K = \mathrm{S}_3$, then $T_u = \mathbb{Z}_2^2$ and a routine argument similar as in Lemma 5.4 implies that $\varGamma$ is the S_3-cover of $\mathrm{P}_{23,11}$ given in Example 2.14.

Let $K = \mathbb{Z}_2$. Then $T_u \cong \mathrm{A}_4$ or S_4 by Lemma 5.1.

Assume first that $T_u \cong \mathrm{S}_4$. Then $X_u = T_u \le T$, and T has two orbits on V, say U and U^z, where $\langle z\rangle = K$. Note that $X_{u^z} = (X_u)^z = X_u$. It follows that all vertex stabilizers are conjugate in T. Recalling that $\varGamma$ is T-edge-transitive, it follows from [8, Lemma 3.4] that $\varGamma$ is the standard double cover of a T-arc-transitive graph Σ of valency 4 and order 253. By [16], $\Sigma \cong \mathrm{P}_{23,11}$, and so $\varGamma \cong \mathrm{P}^{(2)}_{23,11}$.

Assume now that $T_u \cong \mathrm{A}_4$. Then $X_u \cong \mathrm{S}_4$ and $X_{uv} \cong \mathrm{S}_3$. Set $X_u = T_u{:}\langle z_1z_2\rangle$ and $X_{uv} = \langle x\rangle{:}\langle z_1z_2\rangle$, where $z_1 \in T$ and $z_2 \in K$ are involutions, and $x \in T_u$ has order 3. Let $z = z_1z_2$. For $g \in \mathbf{N}_X(X_{uv})$, it is easily shown that g normalizes $\langle x\rangle$. It follows that $\mathbf{N}_X(X_{uv}) \le \mathbf{N}_T(\langle x\rangle){\times}K$. By the Atlas [5], $\mathbf{N}_T(\langle x\rangle) \cong \mathrm{D}_{24}$ and $\mathbf{N}_T(\langle x, z_1\rangle) \cong \mathrm{D}_{12}$. We may write $\mathbf{N}_T(\langle x, z_1\rangle) = \langle x\rangle{:}\langle z_1\rangle{\times}\langle o\rangle$, where o is the involution in the center of $\mathbf{N}_T(\langle x\rangle)$. Note that all involutions of $\langle x, z\rangle$ are conjugate under $\langle x\rangle$. Then an element $h \in \mathbf{N}_T(\langle x\rangle)$ normalizes $\langle x, z\rangle$ if and only if $z^h = z^{x^i}$ for some $0 \le i \le 2$, yielding $x^ih^{-1} \in \mathbf{C}_T(z)$, so $x^ih^{-1} \in \mathbf{C}_T(z_1)$, and hence $x^ih^{-1} \in \mathbf{C}_T(z_1) \cap \mathbf{N}_T(\langle x, z_1\rangle)$. It follows that $h \in \mathbf{N}_X(\langle x, z\rangle) \cap \mathbf{N}_T(\langle x\rangle)$ if and only if $x^ih^{-1} \in \langle z_1, o\rangle$, yielding $h \in \mathbf{N}_T(\langle x, z_1\rangle)$. Therefore, $\mathbf{N}_X(X_{uv}) = \mathbf{N}_T(\langle x, z_1\rangle){\times}K = \langle x\rangle{:}\langle z_1\rangle{\times}\langle o\rangle{\times}\langle z_2\rangle = \langle x, z\rangle{\times}\langle o\rangle{\times}\langle z_2\rangle$. Then, for $g \in \mathbf{N}_X(\langle x, z\rangle) \setminus X_u$, we have $X_ugX_u = HoH$, Hoz_2H or Hz_2H. Note that $\langle H, z_2\rangle = \langle H_1, z_1, z_2\rangle \cong \mathrm{S}_4{\times}\mathbb{Z}_2$. Thus, writing $\varGamma$ as a coset graph, $\varGamma$ is one of the graphs in Example 2.11. □

We next determine the 2-arc-transitive graphs associated with $\mathrm{PSL}(2, 11)$.

Lemma 5.6. *Let* $\varGamma = (V, E)$ *be a connected tetravalent* $(\mathrm{PSL}(2, 11), 2)$*-arc-transitive graph of order* 55. *Then* $\varGamma \cong \mathrm{P}_{11,5}$.

Proof. Let $X = \mathrm{PSL}(2, 11)$. Then $X_u \cong \mathrm{A}_4$ and $X_{uv} \cong \mathbb{Z}_3$ for $u \in V$ and $v \in \varGamma(u)$. Write $\varGamma$ as a coset graph $\mathsf{Cos}(X, X_u, X_uxX_u)$, where $x \in \mathbf{N}_X(X_{uv})$ with $\langle x, X_u\rangle = X$ and $x^2 \in X_{uv}$. By the Atlas [5], $\mathbf{N}_X(X_{uv}) \cong \mathrm{D}_{12} = \mathbb{Z}_3{:}\mathbb{Z}_2^2$. Then $X_uxX_u = X_uyX_u$ for some involution $y \in \mathbf{N}_X(X_{uv})$. Checking the subgroups of $\mathrm{PSL}(2, 11)$, we know that $X_u = \mathbf{N}_X(P)$ for a Sylow 2-subgroup P of X. It follows that the subgroups isomorphic to A_4 are all conjugate in X. Then there are two non-conjugate maximal subgroups M_1 and M_2 of X such that $X_u \le M_i \cong \mathrm{A}_5$, $i = 1, 2$. Note that $\mathbf{N}_{M_i}(X_{uv}) \cong \mathrm{S}_3$. Then $\mathbf{N}_X(X_{uv}) = \mathbf{N}_{M_i}(X_{uv}){\times}\langle o\rangle$, where o is the involution in the center of $\mathbf{N}_X(X_{uv})$. It is easily shown that $\langle X_u, \mathbf{N}_{M_i}(X_{uv})\rangle = M_i$, $i = 1, 2$. Thus $\mathbf{N}_{M_1}(X_{uv}) \cap \mathbf{N}_{M_2}(X_{uv})$ contains no involutions; otherwise, $\mathbf{N}_{M_1}(X_{uv}) = \mathbf{N}_{M_2}(X_{uv})$, so $M_1 = M_2$, a contradiction. Then $\mathbf{N}_{M_1}(X_{uv}) \cup \mathbf{N}_{M_2}(X_{uv})$ contains exactly 6 of the 7 involutions in $\mathbf{N}_X(X_{uv})$, and so we have $X_u x X_u = X_u y X_u = X_u o X_u$. Thus $\mathit{\Gamma} = \mathsf{Cos}(X, X_u, X_u o X_u)$ is unique. Since $\mathrm{P}_{11,5}$ is $(\mathrm{PSL}(2, 11), 2)$-arc-transitive, $\mathit{\Gamma} \cong \mathrm{P}_{11,5}$. □

Lemma 5.7. *If* $T = \mathrm{PSL}(2, 11)$ *and* $K \ne 1$, *then* K *is isomorphic to* $\mathbb{Z}_2$, $\mathbb{Z}_3$ *or* $\mathbb{Z}_6$.

Proof. Assume $T = \mathsf{soc}(Y) = \mathrm{PSL}(2, 11)$ and $K \ne 1$. Then $|K| = 2$, 3 or 6.

Suppose that $K \cong \mathrm{S}_3$. Recall that $X = K{:}Y$ has a regular subgroup $K{:}(G \cap Y)$. Then $|G \cap Y|$ is odd. By Lemmas 4.4 and 4.3, $T_B \cong \mathrm{A}_4$ and $G \cap Y \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5$. Thus, by Lemma 5.1, $T_u \cong \mathbb{Z}_2^2$ and $T = \mathrm{PSL}(2, 11)$ has two orbits on V. It is easily shown that $\mathit{\Gamma}$ is $(KT, 2)$-arc-transitive. Without loss of generality, we assume $X = K{\times}T$. Then $X_{uv} = \langle xy\rangle \cong \mathbb{Z}_3$ and $X_u = T_u{:}X_{uv}$ for $v \in \mathit{\Gamma}(u)$, where $x \in T$ and $y \in K$ are of order 3 such that $T_u{:}\langle x\rangle \cong \mathrm{A}_4$. Computation shows that $\mathbf{N}_X(X_{uv}) = \langle o\rangle{\times}(\langle x\rangle{\times}\langle y\rangle){:}\langle z_1 z_2\rangle$, where o is the involution in the center of $\mathbf{N}_T(\langle x\rangle)$, $z_1 \in T$ and $z_2 \in K$ are involutions with $x^{z_1} = x^{-1}$ and $y^{z_2} = y^{-1}$. For an arbitrary element $g = o^i x^s y^t (z_1 z_2)^j \in \mathbf{N}_X(\langle xy\rangle)$, set $W = \langle g, X_u\rangle$. Then $W \le \langle T_u, x, o^i z_1^j\rangle{\times}\langle y, z_2^j\rangle$. If $j \equiv 0\,(\mathsf{mod}\ 2)$, then $W \ne X$. Assume that $j \equiv 1\,(\mathsf{mod}\ 2)$. Then $W \le \langle T_u, x, o^i z_1\rangle{\times}K$. Checking the subgroups of $T = \mathrm{PSL}(2, 11)$, we conclude that $\mathbf{N}_T(T_u) = T_u{:}\langle x\rangle$. Let M_1 and M_2 be two non-conjugate maximal subgroups of T with $M_i \cong \mathrm{A}_5$ and $T_u \le M_i$, $i = 1, 2$. Then $\mathbf{N}_{M_i}(T_u) \cong \mathrm{A}_4$ for $i = 1, 2$. Thus $M_1 \cap M_2 = T_u{:}\langle x\rangle$. Noting that $\mathbf{N}_{M_1}(\langle x\rangle) \cong \mathrm{S}_3 \cong \mathbf{N}_{M_2}(\langle x\rangle)$, a similar argument as in the proof of Lemma 5.6 implies that $\mathbf{N}_{M_1}(\langle x\rangle) \cup \mathbf{N}_{M_2}(\langle x\rangle)$ contains 6 of the 7 involutions in $\mathbf{N}_T(\langle x\rangle) = \langle o\rangle{\times}\langle x, z_1\rangle$. Since A_5 has no elements of order 6, we have $o \notin M_i$ for $i = 1, 2$. Thus $o^i z_1 \in \mathbf{N}_{M_1}(\langle x\rangle) \cup \mathbf{N}_{M_2}(\langle x\rangle)$. Then $\langle T_u, x, o^i z_1\rangle \cong \mathrm{A}_5$, and so $W \ne X$. Thus there is no $g \in \mathbf{N}_X(X_{uv})$ with $\langle g, X_u\rangle = X$, a contradiction.

Therefore, $K \not\cong \mathrm{S}_3$, so K is isomorphic to one of $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_6$. □

Lemma 5.8. *Assume that* $T = \mathrm{PSL}(2, 11)$. *Then* $\mathit{\Gamma}$ *is isomorphic to one of* $\mathrm{P}_{11,5}$, $\mathrm{P}_{11,5}^{(2)}$, *the graph in Example* 2.12 *and its standard double cover.*

Proof. Let K be the largest soluble normal subroup of X. Then, by Lemma 5.7, either $K = 1$, or K is isomorphic to one of $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_6$.

Case 1. Let $K = 1$. By Lemma 4.3, $G \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5$ or $\mathbb{Z}_{11}{:}\mathbb{Z}_{10}$. If $|G| = 55$, then $T_u \cong \mathrm{A}_4$ and $\mathit{\Gamma}$ is $(T, 2)$-arc-transitive, so $\mathit{\Gamma} \cong \mathrm{P}_{11,5}$ by Lemma 5.6. Thus we assume next that $X = \mathrm{PGL}(2, 11)$, $G \cong \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$ and $X_u \cong \mathrm{A}_4$. Then $G \cap T \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5$, $X_u = T_u \cong \mathrm{A}_4$ and $X_{uv} \cong \mathbb{Z}_3$ for $v \in \mathit{\Gamma}(u)$. Let M be a maximal subgroup of X with $X_u \le M \cong \mathrm{S}_4$. Then $M = X \cap M = GX_u \cap M = (G \cap M)X_u = X_u{:}(G \cap M)$. Let $G \cap M = \langle z\rangle$. Then z is an involution. Replacing v by v^h for $h \in X_u$ if necessary, we assume that z normalizes X_{uv}. Then $X_{uv}{:}\langle z\rangle \cong \mathrm{S}_3$. By the Atlas [5], we conclude that $\mathbf{N}_X(X_{uv}) = (X_{uv}{\times}\langle y\rangle){:}\langle z\rangle \cong \mathrm{D}_{24}$, where $y \in X$ has order 4.

Write $\mathit{\Gamma} = \mathsf{Cos}(X, X_u, X_u x X_u)$ for $x \in \mathbf{N}_X(X_{uv})$ with $\langle x, X_u\rangle = X$ and $x^2 \in X_{uv}$. It implies that $x = hy^i z$ for $i \in \{1, 2, 3\}$ and $h \in X_{uv}$, so $X = \langle x, X_u\rangle = \langle y^i z, X_u\rangle$. In particular, $y^i z \notin T$ as $X_u \le T$. It is easy to know that $y^2 \in T$, $y \notin T$, $z \notin T$,

$\langle y,z\rangle \cong \mathrm{D}_8$ and $T\cap\langle y,z\rangle \cong \mathbb{Z}_2^2$. It follows that $yz, y^3z\in T$. Thus $i=2$, and so $X_uxX_u = X_uy^2zX_u$. Hence $\mathit{\Gamma} = \mathsf{Cos}(X, X_u, X_uy^2zX_u)$.

Identify $V\mathit{\Gamma}$ with $U\cup U'$, where $U=\{X_ug \mid g\in G\cap T\}$ and $U'=\{X_uzg \mid g\in G\cap T\}$ are in fact the bipartition subsets of $\mathit{\Gamma}$. Then X_ug and X_uzg_1 are adjacent whenever $zg_1g^{-1}\in X_uy^2zX_u = zX_uy^2X_u$, i.e., $g_1g^{-1}\in X_uy^2X_u$. Noting that $T_u=X_u$, it follows that $\mathit{\Gamma}$ is the standard double cover of $\Sigma := \mathsf{Cos}(T, T_u, T_uy^2T_u)$. Clearly, Σ is $(T,2)$-arc-transitive and of order 55. By Lemma 5.6, $\Sigma\cong \mathrm{P}_{11,5}$, so $\mathit{\Gamma}\cong \mathrm{P}_{11,5}^{(2)}$.

Case 2. Let $K\cong\mathbb{Z}_2$. Then $X=Y{\times}K$. By Lemmas 4.4 and 4.3, $(Y,Y_B) = (\mathrm{PSL}(2,11),\mathrm{A}_4)$ or $(\mathrm{PGL}(2,11),\mathrm{S}_4)$. Then $T_B\cong\mathrm{A}_4$. By Lemma 5.1, $T_u\cong\mathrm{A}_4$ and T has two orbits on V. Moreover, $\mathit{\Gamma}$ is $(K{\times}T,2)$-arc-transitive. Recalling that $\mathit{\Gamma}$ is T-edge-transitive, by [8, Lemma 3.4], $\mathit{\Gamma}$ is the standard double cover of a T-edge-transitive graph Σ of order 55. It is easily shown that Σ is $(\mathrm{PSL}(2,11),2)$-arc-transitive. By Lemma 5.6, $\Sigma\cong\mathrm{P}_{11,5}$, and so $\mathit{\Gamma}\cong\mathrm{P}_{11,5}^{(2)}$.

Case 3. Let $K=\mathbb{Z}_3$. By Lemma 5.1, $T_B\cong\mathrm{A}_4$ and $T_u\cong\mathbb{Z}_2^2$, where $u\in V$ and B is the K-orbit containing u. Since T_u is normal in X_u and transitive on $\mathit{\Gamma}(u)$, we have $X_u=T_u{:}X_{uv}$ for $v\in\mathit{\Gamma}(u)$. Write $\mathit{\Gamma}\cong\mathsf{Cos}(X,X_u,X_ugX_u)$ for a 2-element $g\in\mathbf{N}_X(X_{uv})$ with $\langle g,X_u\rangle = X$ and $g^2\in X_{uv}$.

Assume first that T is transitive on V. Then $|V|=|T:T_u|$ is odd, and $(Y,X_u) = (\mathrm{PSL}(2,11),\mathrm{A}_4)$ or $(\mathrm{PGL}(2,11),\mathrm{S}_4)$ by Lemmas 4.4 and 4.3.

Suppose that $(Y,X_u) = (\mathrm{PSL}(2,11),\mathrm{A}_4)$. Then $X_{uv}\cong\mathbb{Z}_3$. Write $X_{uv}=\langle xy\rangle$, where $x\in T$ and $y\in K$ are of order 3. Then $\mathbf{N}_X(X_{uv}) = \langle o\rangle\times\langle x\rangle\times\langle y\rangle$, and so $\mathit{\Gamma}\cong\mathsf{Cos}(X,X_u,X_uoX_u)$ is unique up to isomorphism. Noting that the graph in Example 2.12 is $(K{\times}T,2)$-arc-transitive, $\mathit{\Gamma}$ is isomorphic to the graph in Example 2.12.

Suppose that $(Y,X_u) = (\mathrm{PGL}(2,11),\mathrm{S}_4)$. Then $X_{uv}\cong\mathrm{S}_3$ and, since T is transitive, $K{:}Y = X = (K{\times}T)X_u = (K{\times}T)X_{uv}$. Since $|X:(K{\times}T)|=2$, we conclude that the Sylow 3-subgroup of X_{uv} is contained in $K{\times}T$. Then we may set $X_{uv}=\langle xy\rangle{:}\langle z\rangle$, where $x\in T$, $y\in K$ and z is an involution. Then $X=(K{\times}T){:}\langle z\rangle$. If $y=1$ then $x\in T\cap X_u = T_u\cong\mathbb{Z}_2^2$, a contradiction. If $x=1$ then $X_u=T_u{:}X_{uv} = \langle T_u,y,z\rangle = (T_u{\times}\langle y\rangle)\langle z\rangle\not\cong\mathrm{S}_4$, again a contradiction. Thus both x and y have order 3. Since $X_{uv}\cong\mathrm{S}_3$, we have $(xy)^z=(xy)^{-1}$, so $x^z=x^{-1}$ and $y^z=y^{-1}$. Then a routine argument implies that $\mathit{\Gamma}$ is isomorphic to the graph in Example 2.12.

Assume now that T has two orbits on V. Then $(X,X_u) = (K{:}\mathrm{PGL}(2,11),\mathrm{A}_4)$. It follows that $X_{uv}=\langle xy\rangle$, where $x\in T$ and $y\in K$ are of order 3 such that $T_u{:}\langle x\rangle\cong\mathrm{A}_4$.

Let $z\in\mathrm{PGL}(2,11)\setminus T$ be an involution with $x^z=x^{-1}$ and $T_u{:}\langle x,z\rangle\cong\mathrm{S}_4$. Let o be the involution in the center of $\mathbf{N}_T(\langle x\rangle)\cong\mathrm{D}_{12}$. If $yz=zy$, $\mathbf{N}_X(\langle xy\rangle) = \langle o\rangle\times\langle x,y\rangle$, and $\langle g,X_v\rangle\le K\times T$, a contradiction. Thus $y^z=y^{-1}$, and $\mathbf{N}_X(\langle xy\rangle) = \langle o\rangle\times\langle x,y\rangle{:}\langle z\rangle = \langle xy\rangle{:}\langle x,z\rangle\times\langle o\rangle$. Then we may take $g=zo$, xzo or x^2zo. Noting that $X_u^x=X_u$ and $(X_ux^izoX_u)^x = X_ux^{i-2}zoX_u$, it follows that $\mathit{\Gamma}\cong\mathsf{Cos}(X,X_u,X_uzoX_u)$.

Write $V\mathit{\Gamma}=\{X_ug\mid g\in KT\}\cup\{X_uzg\mid g\in KT\}$. Then X_uzg_2 and X_ug_1 are adjacent in $\mathit{\Gamma}$ if and only if $zg_2g_1^{-1}\in X_uzoX_u = zX_uoX_u$, that is, $g_2g^{-1}\in X_uoX_u$. Noting that $X_u=T_u{:}X_{uv}\le K{\times}T$, it follows that $\mathit{\Gamma}$ is the standard double cover of $\Sigma:=\mathsf{Cos}(KT,X_u,X_uoX_u)$. By the argument in the third paragraph of this case, Σ is isomorphic to the graph in Example 2.12.

Case 4. $K=\mathbb{Z}_6$. Then $(X,X_u) = (K{\times}\mathrm{PSL}(2,11),\mathrm{A}_4)$ or $(K{:}\mathrm{PGL}(2,11),\mathrm{S}_4)$. In this case, $T=\mathrm{PSL}(2,11)$ has two orbits on V, and $\mathit{\Gamma}$ is $(KT,2)$-arc-transitive. It is easily shown that $\mathrm{A}_4\cong(KT)_u\le Q{\times}T$, where $Q\cong\mathbb{Z}_3$ is the Sylow 3-subgroup of K.

By [8, Lemma 3.4], Γ is the standard double cover of a $(QT, 2)$-arc-transitive graph Σ. By the argument in the third paragraph of Case 3, Σ is isomorphic to the graph in Example 2.12. This completes the proof. □

5.2. **Graphs associated with** $\mathrm{PSL}(2,7)$**.** Now we consider graphs associated with the simple group $\mathrm{PSL}(2,7)$. If Γ is $(X,2)$-arc-transitive, then Γ is known. Thus we assume that $X = K{:}Y$, $Y = \mathrm{PGL}(2,7)$ and $X_u \cong Y_B \cong \mathrm{D}_{16}$ or D_8, where B is the K-orbit containing $u \in V$. In particular, $|V| = 21|K|$ or $42|K|$, and $(|K|,|Y|) \le 2$.

Lemma 5.9. *If $X = \mathrm{PGL}(2,7)$, then either $\Gamma \cong \mathrm{P}_{7,3}$, or Γ is bipartite and is isomorphic one of $\mathrm{P}^{(2)}_{7,3}$ and the graphs in Example* 2.8.

Proof. Assume that $X = \mathrm{PGL}(2,7)$. If $X_u = \mathrm{D}_{16}$, then X_u is maximal in X, so $\Gamma \cong \mathrm{P}_{7,3}$ by [16]. Thus we assume further that $X_u \cong \mathrm{D}_8$ in the following.

Suppose that $X_u \not\le T = \mathrm{PSL}(2,7)$. Then $|T_u| = 4$. Note that X has a factorization $X = GX_u$ with $G \cap X_u = 1$, where $G \cong \mathbb{Z}_7{:}\mathbb{Z}_6$. Let P be a Sylow 2-subgroup of X with $X_u < P$. Then $P \cong \mathrm{D}_{16}$, and P contains exactly two subgroups isomorphic to D_8: one is X_u and the other one, say Q, is a Sylow 2-subgroup of T. It is easily shown that $X_u \cap Q \cong \mathbb{Z}_4$. Then $X_u = \langle h\rangle{:}\langle z_1\rangle$, where $h \in T$ is of order 4 and $z_1 \in X \setminus T$ is an involution. Noting that $P = X \cap P = (G \cap P)X_u$, we find that $G \cap P = \langle z_2\rangle$ and $P = X_u{:}\langle z_2\rangle$ for an involution z_2. Since T has no subgroups isomorphic to $\mathbb{Z}_7{:}\mathbb{Z}_6$, we have $z_2 \in X \setminus T$. Clearly, $z_2 \notin X_u$. Thus P contains another subgroup $\langle h, z_2\rangle$ which is isomorphic to D_8 and not contained in T, a contradiction. Therefore, $X_u = T_u$. In particular, T has two orbits on V, and so Γ is a bipartite graph.

Suppose that Γ is X-half-transitive. Write $\Gamma \cong \mathsf{Cos}(X, T_u, T_u\{x, x^{-1}\}T_u)$, where $x \in X$ with $\langle T_u, x\rangle = X$. Then $|T_u : (T_u \cap T_u^x)| = 2$, so $T_u \cap T_u^x$ is normal in both T^u and T_u^x, hence $T_u \cap T_u^x \trianglelefteq M := \langle T_u, T_u^x\rangle$. Checking the subgroups of T, since $T_u \ne T_u^x$, we have $M \cong \mathrm{S}_4$. Then there is an element $y \in M$ of order 3 such that $T_u^x = T_u^y$, so $xy^{-1} \in \mathbf{N}_X(T_u) \cong \mathrm{D}_{16}$. Write $\mathbf{N}_X(T_u) = T_u{:}\langle z\rangle$ for an involution $z \notin T$. Then $xy^{-1} = hz^i$ for $h \in T_u$ and $i = 0$ or 1, so $x = hz^iy$. Since $\langle T_u, x\rangle = X = \mathrm{PGL}(2,7)$, we have $i = 1$. Thus $T_u\{x, x^{-1}\}T_u = T_u\{zy, (zy)^{-1}\}T_u$, so Γ is isomorphic to the graph in Example 2.15 (1). Then $\Gamma \cong \mathrm{P}^{(2)}_{7,3}$ by Lemma 2.16.

Suppose that Γ is X-arc-transitive. Then $|T_u : T_{uv}| = 4$, so $T_{uv} \cong \mathbb{Z}_2$. By the information given in the Atlas [5], we have $\mathbf{N}_X(T_{uv}) \cong \mathrm{D}_{16}$ and $\mathbf{N}_T(T_{uv}) \cong \mathrm{D}_8$. Write $\mathbf{N}_X(T_{uv}) = \mathbf{N}_T(T_{uv}){:}\langle z\rangle$ for an involution $z \notin T$. Set $T_{uv} = \langle o\rangle$. If o lies in the center of T_u, then $\mathbf{N}_T(T_{uv}) = T_u$, so $|T_{uv}| = |T_u \cap T_v| \ge 4$ by noting that $|\mathbf{N}_{T_v}(T_{uv})| \ge 4$, a contradiction. Thus $\mathbf{N}_{T_u}(T_{uv}) \cong \mathbb{Z}_2^2$. Let y be an element of order 4 in $\mathbf{N}_T(T_{uv})$. Then $y^2 = o$, $\mathbf{N}_T(T_{uv}) = \mathbf{N}_{T_u}(T_{uv})\langle y\rangle$, and so $\mathbf{N}_X(T_{uv}) = (\mathbf{N}_{T_u}(T_{uv})\langle y\rangle){:}\langle z\rangle$. Thus $T_u\mathbf{N}_X(T_{uv})T_u = T_u \cup (T_uyT_u) \cup (T_uzT_u) \cup (T_uyzT_u)$. Since Γ is connected, we conclude that Γ is isomorphic to one of the graphs in Example 2.8. □

Lemma 5.10. *If $K \cong \mathbb{Z}_2$ then $\Gamma \cong \mathrm{P}^{(2)}_{7,3}$ and $X \cong \mathrm{PGL}(2,7){\times}\mathbb{Z}_2$.*

Proof. Assume that $K = \mathbb{Z}_2$. Then $X = Y{\times}K$, $\Gamma_K \cong \mathrm{P}_{7,3}$ and $|V| = 21|K| = 42$. Suppose that T is transitive on V. Then $|T_u| = 4$, so $T_u \cong \mathbb{Z}_4$ as T_u is normal in $X_u \cong \mathrm{D}_{16}$. Since T is not regular, either $T_{uv} \cong \mathbb{Z}_2$ or T_u is transitive on $\Gamma(u)$, where $v \in \Gamma(u)$. The latter case yields that Γ is $T{\times}K$-arc-transitive, so Γ_K is a T-arc-transitive graph of order 21, which contradicts Lemma 4.2. By Lemma 3.1, the former case implies that $T_{vw} \cong \mathbb{Z}_2$ for $w \in \mathit{\Gamma}(v)$, so $T_{uv} = T_{vw}$ as $T_v \cong \mathbb{Z}_4$ has a unique subgroup of order 2. By the connectedness of $\mathit{\Gamma}$ we conclude that T_{uv} fixes every vertex of $\mathit{\Gamma}$, a contradiction. Therefore, T is intransitive on V. Noting that T_u is a 2-group, $T_u \cong \mathrm{D}_8$ and T has exactly two orbits. Then $\mathit{\Gamma}$ is bipartite with two parts being T-orbits on V. Let $\tilde{Y}$ be the maximal subgroup of X preserving the bipartition of $\mathit{\Gamma}$. Then $\mathrm{PGL}(2,7) \cong \tilde{Y} = Y$ or $T{:}\langle z_1 z_2\rangle$, where $z_1 \in Y \setminus T$ and $z_2 \in K$ are involutions. It is easily shown that $X = \tilde{Y}{\times}K$. Note that $\tilde{Y}$, viewed as a subgroup of $\mathsf{Aut}\mathit{\Gamma}_K$, is transitive on the arcs of $\mathit{\Gamma}_K \cong \mathrm{P}_{7,3}$. It follows that $\mathit{\Gamma}$ is X-arc-transitive, so $\tilde{Y}_u$ is transitive on $\mathit{\Gamma}(u)$, hence $\mathit{\Gamma}$ is $\tilde{Y}$-edge-transitive. By [8, Lemma 3.4], $\mathit{\Gamma}$ is isomorphic to the standard double cover of a $\tilde{Y}$-arc-transitive graph Σ of order 21. Then, by Lemma 5.9, $\Sigma \cong \mathrm{P}_{7,3}$, and this lemma follows. □

Lemma 5.11. *If $|K| > 3$ is odd then $Y_B \cong X_u \cong \mathrm{D}_8$ for a K-orbit B containing u.*

Proof. Assume that $|K| > 3$ is odd. Set $Y = T{:}\langle z\rangle$, where $T = \mathrm{PSL}(2,7)$ and $z \in Y \setminus T$ is an involution. Then $X = (T{\times}K){:}\langle z\rangle$. Since $\mathit{\Gamma}$ is X-edge-transitive, we may write $\mathit{\Gamma} = \mathsf{Cos}(X, X_u, X_u\{x, x^{-1}\}X_u)$, where $x \in X \setminus X_u$ with $\langle x, X_u\rangle = X$. Write $x = yz^ic$ for $c \in K$, $y \in T$ and $i = 0$ or 1.

Suppose that $X_u \cong \mathrm{D}_{16}$. Then X_u is a Sylow 2-subgroup of X. Since $|K|$ is odd, we may assume that $z \in X_u \le Y$. Then $Y_B = X_u = T_u{:}\langle z\rangle$. Since $Y_B \cong \mathrm{D}_{16}$, the quotient $\mathit{\Gamma}_K \cong \mathrm{P}_{7,3}$ is Y-arc-transitive, and so $\mathit{\Gamma}$ is X-arc-transitive. Thus we may choose x such that $x^2 \in X_u$. So $x^2 = yz^icyz^ic = yc^{z^i}y^{z^i}c = yy^{z^i}c^{z^i}c \in X_u$. Since $yy^{z^i} \in T$ and $c^{z^i}c \in K$ has odd order, $c^{z^i}c = 1$. Since $X = \langle x, X_u\rangle = \langle yz^ic, X_u\rangle \le \langle yz^i, c, X_u\rangle \le K{:}\langle yz^i, X_u\rangle$, we have $c \ne 1$ and $Y = \langle yz^i, X_u\rangle$. It implies that $i = 1$, $x = yzc$, $yy^z \in T_u$ and $cc^z = 1$, so $c^z = c^{-1}$. Recalling $Y = \langle yz, X_u\rangle$, we have $T{:}\langle z\rangle = Y = \langle yz, T_u{:}\langle z\rangle\rangle = \langle y, T_u, z\rangle = \langle y, y^z, T_u, z\rangle = \langle y, y^z, T_u\rangle{:}\langle z\rangle = \langle y, T_u\rangle{:}\langle z\rangle$. Then $T = \langle y, T_u\rangle$, and so $\Sigma := \mathsf{Cos}(T, T_u, T_u\{y, y^{-1}\}T_u)$ is a connected T-edge-transitive graph of order 21. Since $\mathit{\Gamma}$ is X-arc-transitive, $|X_u : (X_u \cap X_u^x)| = 4$, so $|X_u \cap X_u^x| = 4$. Noting that $X_u \cap X_u^x \le Y$ and $T_u, T_u^y \le T$, we have $4 = |X_u \cap X_u^x| = |X_u \cap X_u^{yzc}| = |X_u \cap X_u^{yc^{-1}z}| = |X_u \cap X_u^{c^{-1}y}| = |(T_u{:}\langle z\rangle) \cap (T_u^y{:}\langle c^2z^y\rangle)| = |T_u \cap T_u^y|$. Thus $|T_u : (T_u \cap T_u^y)| = 2$, and Σ has valency 4, which contradicts Lemma 4.2. Then the lemma follows. □

Lemma 5.12. *Assume that $|K| > 3$. Then $\mathit{\Gamma}$ is isomorphic to the graph in Example 2.15, $X \cong \mathrm{PGL}(2,7){\times}\mathbb{Z}_l$ or $\mathrm{PGL}(2,7){\times}\mathrm{D}_{2l}$, where l is odd and square-free.*

Proof. Assume first $|K| = l$ is odd. Then $Y_B \cong X_u \cong \mathrm{D}_8$, $|V| = 42l$ and Y contains a Sylow 2-subgroup of X. Thus, without loss of generality, we assume that $X_u < Y$, and so $X_u = Y_B$. Let $z \in Y \setminus T$ be an involution such that $\langle X_u, z\rangle \cong \mathrm{D}_{16}$. Then $Y = T{:}\langle z\rangle$ and $X = (T{\times}K){:}\langle z\rangle = \langle z\rangle(T{\times}K)$. Write $\mathit{\Gamma} = \mathsf{Cos}(X, X_u, X_u\{x, x^{-1}\}X_u)$, where $x \in X \setminus X_u$ with $\langle x, X_u\rangle = X$. Write $x = z^igc$ for $c \in K$, $g \in T$ and $i = 0$ or 1.

Since T centralizes K, we have $\langle c, c^z\rangle = \langle c^y \mid y \in Y\rangle$; in particular, Y normalizes $\langle c, c^z\rangle$. Noting that $\mathit{\Gamma}$ is connected, $K{:}Y = X = \langle z^igc, X_u\rangle \le \langle z^ig, c, X_u\rangle \le \langle c, c^z, z^ig, X_u\rangle = \langle c, c^z\rangle{:}\langle z^ig, X_u\rangle$. It follows that $K = \langle c, c^z\rangle$ and $Y = \langle z^ig, X_u\rangle$.

Note that the quotient $\mathit{\Gamma}_K$ is Y-edge-transitive and of order 42. By Lemma 5.9, $\mathit{\Gamma}_K$ is bipartite, so $\mathit{\Gamma}$ is also bipartite. It is easily shown that $T{\times}K$ is the maximal subgroup preserving the bipartition of $\mathit{\Gamma}$. Thus $X_u \le T{\times}K$, and so $X_u = T_u$ as $|K| = l$ is odd. Since $Y = \langle z^ig, X_u\rangle$ and $g \in T$, we have $i = 1$, so $Y = \langle zg, T_u\rangle$.

Suppose that Γ_K is Y-arc-transitive. Then Γ is X-arc-transitive, so we may choose x with $x^2 \in T_u$. Since $x^2 = zgczgc = g^z c^z gc = g^z g c^z c$, we have $c^z c = 1$, so $c^z = c^{-1}$, hence zc has order 2. Then $X = \langle zgc, T_u\rangle \le \langle g, T_u, cz\rangle \le \langle T, cz\rangle \cong \mathrm{PGL}(2,7)$, a contradiction. Therefore, by Lemma 5.9, $\Gamma_K \cong \mathrm{P}_{7,3}^{(2)}$ is Y-half-transitive.

Note that $T{:}(\langle c, z\rangle) = (T\times K){:}\langle z\rangle = X = \langle zgc, T_u\rangle \le \langle g, T_u, zc\rangle \le T\langle zc\rangle \le X$. Thus $\langle c, z\rangle \cong X/T \cong (T\langle zc\rangle)/T$ is cyclic. Then $c^z = c$ and $K = \langle c\rangle$. Thus $X = Y\times K$. Since Γ_K is Y-half-transitive, Γ is X-half-transitive, so $2 = |X_u : (X_u \cap X_u^x)| = |T_u : (T_u \cap T_u^{zg})|$. Recalling that $\langle T_u, z\rangle = \langle X_u, z\rangle \cong \mathrm{D}_{16}$, we know that z normalizes T_u, so $|T_u \cap T^g| = |T_u \cap T_u^{zg}| = 4$. It follows that $T_u \cap T_u^g$ is normal in $M := \langle T_u, T_u^g\rangle$, so $M \cong \mathrm{S}_4$ by checking the subgroups of T. Then $T_u^g = T_u^y$ for an element $y \in M$ of order 3. Thus $gy^{-1} \in \mathbf{N}_T(T_u) = T_u$, so $g = hy$ for $h \in T_u$. Then $T_u\{x, x^{-1}\}T_u = T_u\{zhyc, (zhyc)^{-1}\}T_u = T_u\{h^z zyc, (h^z zyc)^{-1}\}T_u = T_u\{zyc, (zyc)^{-1}\}T_u$. Noting that $\mathrm{S}_4 \cong M = \langle T_u, y\rangle$, it follows that Γ is the graph in Example 2.15.

Now let $|K| = 2l$. Then $\Gamma_K \cong \mathrm{P}_{7,3}$ and $X_u \cong Y_B \cong \mathrm{D}_{16}$. In this case, Γ is X-arc-transitive as Γ_K is Y-arc-transitive. Since K is of square-free order, K has a unique $2'$-Hall subgroup $K_{2'}$, which is characteristic in K. It implies that $K_{2'} \trianglelefteq X$, and $X/K_{2'} \cong \mathrm{PGL}(2,7)\times\mathbb{Z}_2$. Then the quotient graph $\Gamma_{K_{2'}}$ is $X/K_{2'}$-arc-transitive. By Lemma 5.10, $\Gamma_{K_{2'}} \cong \mathrm{P}_{7,3}^{(2)}$. It is easily shown that $X/K_{2'}$ contains a subgroup $Z/K_{2'} \cong \mathrm{PGL}(2,7)$ acting transitively on the edges of $\Gamma_{K_{2'}}$. Then Γ is Z-edge-transitive. By the argument for the odd $|K|$ case, we conclude that $K_{2'} \cong \mathbb{Z}_l$, $Z = \tilde{Y}\times K_{2'}$ with $\tilde{Y} \cong \mathrm{PGL}(2,7)$, and Γ is isomorphic to the graph in Example 2.15.

Recalling that $|V| = 42l$ is square-free, we conclude that $\tilde{Y}$ is a Hall subgroup of Z, so $\tilde{Y}$ is characteristic in Z. Thus $\tilde{Y}$ is normal in X as Z has index 2 in X. Let z be an involution in K. Then $\langle \tilde{Y}, z\rangle = \tilde{Y}\times\langle z\rangle$. Thus $X = Z{:}\langle z\rangle = \tilde{Y}\times K$.

Suppose that K is not a dihedral group. Then $K = N\times M$, where $N \ne 1$ is cyclic and of odd order. By Lemma 2.9, Γ is normal cover of $\Sigma := \Gamma_M$. Identify $\bar{X} := \tilde{Y}\times N$ with a subgroup of $\mathsf{Aut}\Sigma$. Then Σ is $\bar{X}$-edge-transitive as Γ is X-edge-transitive. Since $|N|$ is odd, for $\alpha \in V\Sigma$, $\bar{X}_\alpha \cong \mathrm{D}_8$ by Lemma 5.11; in particular, $|V\Sigma| = 42|N|$. Thus $|V| = 42|N||M| = 42|K| = 84l$, a contradiction. Then $K \cong \mathrm{D}_{2l}$ is dihedral. □

6. Proofs of Theorem 1.1 and Corollary 1.2

Let G be a finite group of square-free order, and let Γ be a connected edge-transitive Cayley graph of G of valency 4. If $\mathsf{Aut}\Gamma$ is soluble then, by Lemmas 3.3 and 3.7, one of Theorem 1.1 (1-3) occurs. Thus we assume next that $X := \mathsf{Aut}\Gamma$ is insoluble. Then Γ and X are known and listed in either Table 1 by Lemmas 4.3, 4.4, 5.2, 5.3, 5.5, 5.8 and 5.9, or Table 2 by Lemmas 5.4, 5.9, 5.10 and 5.12.

Proof of Theorem 1.1. It suffices to determine G up to isomorphism. If X is almost simple, then all possible G are known by checking the subgroups of X in the Atlas [5]. Thus we assume that X is not almost simple. By Lemma 4.4 and checking the automorphism group X listed in Tables 1 and 2, we know that $X = K{:}Y$ has a regular subgroup $R := L\times K$, where $L \le T\cap G$, $T = \mathsf{soc}(Y)$ and K is the largest soluble normal subgroup of X. Thus $G \le \mathbf{N}_X(G) \le \mathbf{N}_X(L) = K{:}\mathbf{N}_Y(L) = \mathbf{N}_X(R)$, and further Y, K, L and $\mathbf{N}_Y(L)$ are known as in the following table:

Y	K	L	$\mathbf{N}_Y(L)$
$\mathrm{PSL}(2,23)$	$\mathbb{Z}_2, \mathrm{S}_3$	$\mathbb{Z}_{23}{:}\mathbb{Z}_{11}$	L
$\mathrm{PGL}(2,11)$	$\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6$	$\mathbb{Z}_{11}{:}\mathbb{Z}_5$	$\mathbb{Z}_{11}{:}\mathbb{Z}_{10}$
S_5	$\mathbb{Z}_2, \mathrm{S}_3$	$\mathbb{Z}_5$	$\mathbb{Z}_5{:}\mathbb{Z}_4$
$\mathrm{PGL}(2,7)$	$\mathbb{Z}_2, \mathrm{D}_{2l}$	$\mathbb{Z}_7{:}\mathbb{Z}_3$	$\mathbb{Z}_7{:}\mathbb{Z}_6$

Let $K_{2'}$ be the $2'$-Hall subgroup of K. Then $L{\times}K_{2'}$ is the unique $2'$-Hall subgroup of R. It follows that $L{\times}K_{2'} \trianglelefteq \mathbf{N}_X(R)$. Let Q be a Sylow 2-subgroup of K. Then $\mathbf{N}_X(R)$ has a Sylow 2-subgroup $Q{:}P$, where P is a Sylow 2-subgroup of $\mathbf{N}_Y(L)$. Noting that $|Q| \leq 2$, we have $QP = Q{\times}P$, and so QP is abelain. Considering the subgroup $(L{\times}K_{2'})G$, we conclude that $L{\times}K_{2'} \leq G$ and $G \trianglelefteq \mathbf{N}_X(R)$. In particular, $G = (L{\times}K_{2'}){:}\langle z\rangle$ for an involution $z \in Q{\times}P$. Checking all possible involutions z, we conclude that G is listed in Tables 1 and 2 up to isomorphism.

Proof of Corollary 1.2. It is easy to check that $\mathbf{N}_X(G) = G{:}\mathbb{Z}_3$ or G while Γ is a graph listed in Lines 1 to 7 of Table 1, so Γ is not normal-edge-transitive. For Lines 1 to 3 of Table 2, we have $\mathbf{N}_X(G) \cong G{:}\mathbb{Z}_4$, so Γ is normal-edge-transitive. We next deal with the rest of the graphs in Tables 1 and 2.

Suppose that X is not almost simple. Then, by the argument in Proof of Theorem 1.1, $X = K{:}Y$ has a regular subgroup $R = L{\times}K$, where $L \leq T \cap G$, $T = \mathsf{soc}(Y)$ and K is the largest soluble normal subgroup of X. Recalling that $\mathbf{N}_X(G) \leq \mathbf{N}_X(L) = K{:}\mathbf{N}_Y(L) = \mathbf{N}_X(R)$ and $G \trianglelefteq \mathbf{N}_X(R)$, we have $\mathbf{N}_X(G) = \mathbf{N}_X(R) = K{:}\mathbf{N}_Y(L)$. Then Γ is normal-edge-transitive with respect to G whenever Γ is a normal-edge-transitive Cayley graph of R. Noting that K is a normal Hall subgroup of R, it follows that $\mathbf{N}_X(R)/K = \mathbf{N}_{X/K}(R/K) \cong \mathbf{N}_Y(L)$. Note that the quotient graph Γ_K has automorphism group isomorphic to Y. Thus, it suffices to determine whether or not Γ_K is a normal-edge-transitive Cayley graph of L.

Therefore, the above argument allows us to assume that X is almost simple, that is, Γ is described as either Line 8 of Table 1 or Line 4 of Table 2.

Suppose that Γ is described as Line 8 of Table 1. Then $X = \mathrm{PGL}(2,11)$ and, for $u \in V\Gamma$, the stabilizer $X_u \cong \mathrm{S}_4$ and $X = GX_u$. Hence $\mathbb{Z}_{11}{:}\mathbb{Z}_{10} \cong \mathbf{N}_X(G) = G{:}(\mathbf{N}_X(G) \cap X_u)$. Thus $\mathbf{N}_X(G) \cap X_u = \langle o\rangle \cong \mathbb{Z}_2$. It is easily shown that $o \notin T = \mathsf{soc}(X)$. Noting that $T_u \cong \mathrm{A}_4$, it follows that o induces an odd permutation on $\Gamma(u)$; in particular, o fixes at least one vertex in $\Gamma(u)$. Thus Γ is not normal-edge-transitive.

Finally, let Γ be described as Line 4 of Table 2. Then $X = \mathrm{PGL}(2,7)$, $\mathbb{Z}_7{:}\mathbb{Z}_3 \cong G < T = \mathrm{PSL}(2,7)$ and $\mathbf{N}_X(G) \cong \mathbb{Z}_7{:}\mathbb{Z}_6$. Write $\Gamma = \mathsf{Cay}(G,S)$ with $S = \{x, x^{-1}, y, y^{-1}\}$. Then $\mathsf{Aut}(G,S) \cong \mathbb{Z}_2$. Let u be the vertex corresponding to the identity of G. Then $X_u \cong \mathrm{D}_{16}$ and $T_u \cong \mathrm{D}_8$. By Lemma 4.2, Γ is not T-edge-transitive, it follows from Lemma 3.2 that T_u has two orbits $\{x, x^{-1}\}$ and $\{y, y^{-1}\}$ on S. Noting that $\mathbf{N}_X(G) \not\leq T$, we have $X_u = T_u{:}\mathsf{Aut}(G,S)$. Then, since Γ is arc-transitive, there is $\sigma \in \mathsf{Aut}(R,S)$ such that $x^\sigma = y$ or y^{-1}. Thus Γ is normal-edge-transitive.

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