Partitioning complete graphs by heterochromatic trees

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Abstract A *heterochromatic tree* is an edge-colored tree in which any two edges have different colors. The *heterochromatic tree partition number* of an r-edge-colored graph G, denoted by $t_r(G)$, is the minimum positive integer p such that whenever the edges of the graph G are colored with r colors, the vertices of G can be covered by at most p vertex-disjoint heterochromatic trees. In this paper we determine the heterochromatic tree partition number of r-edge-colored complete graphs. We also find at most $t_r(K_n)$ disjoint heterochromatic trees to cover all the vertices in polynomial time for a given r-edge-coloring of K_n.

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1 Introduction

A *monochromatic tree* is an edge-colored tree in which any two edges have the same color, while a *heterochromatic tree* is an edge-colored tree in which any two edges have different colors. The *monochromatic tree partition number* of an r-edge-colored graph G is defined to be the minimum positive integer p such that whenever the edges of G are colored with r colors, the vertices of G can be covered by at most p vertex-disjoint monochromatic trees. The *monochromatic cycle partition number* and the *monochromatic path partition number* are defined analogously.

Erdős et al. [2] proved that the monochromatic cycle partition number of an r-edge-colored complete graph K_n is at most $cr^2 \ln r$ for some constant c. This implies a conjecture from Gyárfás [4] in a stronger form. Recently, the bound was improved by Gyárfás et al. [5]. Almost solving one of the conjectures in [2], Haxell et al. [7] proved that the monochromatic tree partition number of an r-edge-colored complete graph K_n is at most r provided that n is large enough with respect to r. Haxell [6] proved that the monochromatic cycle partition number of an r-edge-colored complete bipartite graph $K_{n,n}$ is also independent of n, which answered a question in [2]. Since the monochromatic path partition number is less than the monochromatic cycle partition number, a natural corollary is that the monochromatic path partition number of K_n or $K_{n,n}$ is also independent of n.

From above, one can see that the monochromatic tree, path, and cycle partition number of r-edge-colored graphs K_n and $K_{n,n}$ are independent of n. This seems to be not true for other graphs such as 2-edge-colored complete multipartite graph. Let $n_1, n_2, \cdots, n_k$ $(k \geq 2)$ be integers such that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ and $n = n_1 + n_2 + \cdots + n_{k-1}, m = n_k$. The authors [9] showed that $t_2'(K_{n_1,n_2,\cdots,n_k}) = \lfloor \frac{m-2}{2^n} \rfloor + 2$, where $t_r'(K_{n_1,n_2,\cdots,n_k})$ denotes the monochromatic tree partition number

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of the r-edge-colored graph $K_{n_1,n_2,\cdots,n_k}$. Given a 2-edge-colored complete multipartite graph, Jin et al. [8] presented a polynomial algorithm to find the minimum number of vertex-disjoint monochromatic trees to cover all the vertices. Other related partition problems can be found in [10, 11, 12].

Analogous to the monochromatic case, the *heterochromatic tree partition number* of an r-edge-colored graph G, denoted by $t_r(G)$, is defined to be the minimum positive integer p such that whenever the edges of the graph G are colored with r colors, the vertices of G can be covered by at most p vertex-disjoint heterochromatic trees. Compared with the monochromatic case, there are few results on the heterochromatic tree partition number. Chen et al. [1] derived the heterochromatic tree partition number of an r-edge-colored complete bipartite graph.

In this paper we consider the heterochromatic tree partition number of an r-edge-colored complete graph K_n. In order to prove our main result, we introduce the following definitions and notation. Throughout this paper, we use r to denote the number of colors. An r-edge-coloring of a graph G means that each color appears at least once in G. Let ϕ be an r-edge-coloring of a graph G. For an edge $e \in E(G)$, denote by $\phi(e)$ the color of e. Denote by $t_r(G, \phi)$ the minimum positive integer p such that under the r-edge-coloring ϕ, the vertices of G can be covered by at most p vertex-disjoint heterochromatic trees. Clearly, $t_r(G) = \max_\phi t_r(G, \phi)$, where ϕ runs over all r-edge-colorings of the graph G. Let F be a spanning forest of G, each component of which is a heterochromatic tree. If F contains exactly $t_r(G, \phi)$ components, then F is called an *optimal heterochromatic tree partition* of the graph G with the edge-coloring ϕ. Note that a tree consisting of a single vertex is also regarded as a heterochromatic tree.

For any integer $r \geq 2$, there is a unique positive integer t, such that $\binom{t}{2} + 2 \leq r \leq \binom{t+1}{2} + 1$. Clearly, the integer t is determined completely by r, and here we denote it by $f(r) = t$. This integer $f(r) = t$ will play an important role in expressing the number $t_r(K_n)$. If the color number $r = 1$, clearly a maximum matching (plus a single vertex when n is odd) in K_n is an optimal heterochromatic tree partition, and then $t_r(K_n) = \lceil \frac{n}{2} \rceil$. So, in the rest of this paper we only consider the case $2 \leq r \leq \binom{n}{2}$. The following is the main result of this paper.

Theorem 1.1 *Let $n \geq 3$, $2 \leq r \leq \binom{n}{2}$ and $f(r) = t$. Then $t_r(K_n) = \lceil \frac{n-t}{2} \rceil$.*

As we know, the monochromatic tree partition number of an edge-colored complete graph K_n is bounded by a function independent of n, and from the result mentioned above, the heterochromatic tree partition number does not have this property any more.

2 A canonical r-edge-coloring ϕ_r^*

In this section we present a canonical r-edge-coloring of the graph K_n. Let $S \subseteq V(K_n)$ and $|S| = t$. Take a vertex $u \in V(K_n) - S$. We define the canonical r-edge-coloring ϕ_r^* by

1. assigning distinct colors to the edges of $K_n[S]$;
2. for each color not used in $K_n[S]$, assign it to an edge uv between u and S;
3. finally, color all the remaining edges by the color not used if it exists, or else by the same color used before.

We have the following proposition.

Proposition 2.1 $t_r(K_n, \phi_r^*) = \lceil \frac{n-t}{2} \rceil$.

Proof. First, we present an heterochromatic tree partition with exact $\lceil \frac{n-t}{2} \rceil$ components, which implies that $t_r(K_n, \phi_r^*) \leq \lceil \frac{n-t}{2} \rceil$. Let $X = S \cup \{u\} \cup \{v\}$, where $v \in V(K_n) - S - u$. It is easy to see that $K_n[X]$ contains a heterochromatic spanning tree T, and the vertices not in X induce a monochromatic complete subgraph which can be covered by $\lceil \frac{n-t-2}{2} \rceil$ disjoint heterochromatic trees. So, the union of T and those $\lceil \frac{n-t-2}{2} \rceil$ disjoint heterochromatic trees consist of a heterochromatic tree partition of K_n. This implies that $t_r(K_n, \phi_r^*) \leq \lceil \frac{n-t}{2} \rceil$.

Next, we prove that $t_r(K_n, \phi_r^*) \geq \lceil \frac{n-t}{2} \rceil$. Suppose on the contrary that $t_r(K_n, \phi_r^*) < \lceil \frac{n-t}{2} \rceil$ for some n and r.

Let F be an optimal heterochromatic tree partition of K_n with r-edge-coloring ϕ_r^*. Denote by $T_1, T_2, \cdots, T_k$ the components of F each of which contains a vertex of S. We choose F such that the number of trees covering S is as small as possible. Note that each component of F not containing any vertex of S is an edge or a single vertex, and at most one of the components of F is a single vertex. Since F is an optimal heterochromatic tree partition, from the definition of ϕ_r^*, we have the following facts.

Fact 1. $u \in T_i$ for some $1 \leq i \leq k$.

Fact 2. $|T_j \cap (V(K_n) - S - u)| = 1$ for each T_j.

If $k = 1$, it is easy to see that F contains exactly $\lceil \frac{n-t}{2} \rceil$ trees. So, assume that $k \geq 2$. Let $S \cap T_i = S_i$ and $v_i = T_i \cap (V(K_n) - S - u)$. From the definition of ϕ_r^*, we have that there exists a heterochromatic tree, denoted by T, covering all the vertices of $T_1 \cup (T_2 - v_2)$. So $F' = (F - T_1 - T_2) \cup \{T\} \cup \{v_2\}$ is an optimal heterochromatic tree partition such that the number of trees covering S is $k-1$, a contradiction, which completes the proof. ∎

3 Proof of Theorem 1.1

Given a complete graph K_n, the heterochromatic tree partition number is closely related to the color number. Before proving our main result, we have the following lemma which gives the relationship between $t_{r+1}(K_n)$ and $t_r(K_n)$.

Lemma 3.1 $t_{r+1}(K_n) \leq t_r(K_n)$.

Proof. Given any $(r+1)$-edge-coloring φ of K_n. Denote by E_i the set of edges colored by the color i. Recoloring the edges of E_{r+1} by the color r, we obtain a r-edge-coloring ψ of K_n. Clearly, $t_{r+1}(K_n, \varphi) \leq t_r(K_n, \psi)$. So, $t_{r+1}(K_n) \leq t_r(K_n)$. ∎

The following lemma gives the relationship between the edge-connectivity and size of a graph. Its proof is omitted.

Lemma 3.2 *Let G be a simple graph of order n. If G contains a cut-edge, then $|E(G)| \leq \binom{n-1}{2} + 1$.*

Proof of Theorem 1.1:

We prove the theorem by induction on r and n. First, we consider the case $r = 2$. Let ϕ be a 2-edge-coloring of K_n. Note that for any 2-edge-coloring of K_n, $n \geq 3$, there is always a heterochromatic tree of order three. Then, we can easily find $1 + \lceil \frac{n-3}{2} \rceil = \lceil \frac{n-1}{2} \rceil$ vertex-disjoint heterochromatic trees which cover all the vertices. So we have $t_r(K_n, \phi) \leq \lceil \frac{n-1}{2} \rceil$. Then, from Proposition 2.1 the result holds for $r = 2$. Obviously, the result holds for $n = 3, 4$.

Assume that the result holds for the color number less than r or the order of a complete graph less than n. Now we consider the r-edge-colored complete graph K_n, $r \geq 3$. Let $f(r) = t$. If $\binom{t}{2} + 3 \leq r \leq \binom{t+1}{2} + 1$, then $f(r-1) = t$. By the induction hypothesis, $t_{r-1}(K_n) = \lceil \frac{n-t}{2} \rceil$. From Lemma 3.1, $t_r(K_n) \leq t_{r-1}(K_n) = \lceil \frac{n-t}{2} \rceil$. And, from Proposition 2.1, $t_r(K_n) \geq t_r(K_n, \phi_r^*) = \lceil \frac{n-t}{2} \rceil$. Then, we have $t_r(K_n) = \lceil \frac{n-t}{2} \rceil$, as desired.

So, we only need to consider the case $r = \binom{t}{2} + 2$. Let ϕ be an r-edge-coloring of K_n. Let G be a heterochromatic subgraph of K_n, such that $\delta(G) \geq 1$ and, for each color i, there is a unique edge colored by the color i in G. Denote by $G_1, G_2, \cdots, G_k$ the components of G, where the order of G_i is n_i, $1 \leq i \leq k$, and $n_1 \geq n_2 \geq \cdots \geq n_k \geq 2$. Choose G such that n_1 is as large as possible. Since the color number $r \geq 3$, we have $n_1 \geq 3$.

Suppose $k = 1$. By $r = \binom{t}{2} + 2$, we have $n_1 \geq t + 1$. If $n_1 \geq t + 2$, then $t_r(K_n, \phi) \leq 1 + \lceil \frac{n-n_1}{2} \rceil \leq \lceil \frac{n-t}{2} \rceil$. So, assume $n_1 = t+1$. By Lemma 3.2, G does not contain any cut-edge. Let $g \in [V(G_1), \overline{V(G_1)}]$, i.e., one end-vertex of g belongs to $V(G_1)$ and the other one belongs to $\overline{V(G_1)}$. From the choice of G, there is an edge $h \in E(G_1)$ with $\phi(h) = \phi(g)$. Since G does not contain any cut-edge, by deleting the edge h and adding the edge g, we can find a heterochromatic graph with r edges, the largest component of which has an order at least $n_1 + 1$, a contradiction to the choice of the graph G.

So, assume $k \geq 2$. If $n_1 \geq t + 2$, then $t_r(K_n, \phi) \leq 1 + \lceil \frac{n-n_1}{2} \rceil \leq \lceil \frac{n-t}{2} \rceil$, as desired. Thus, assume $n_1 \leq t + 1$. We have the following claim.

Claim: G_1 contains a cut-edge, and then $|E(G_1)| \leq \binom{n_1-1}{2} + 1$.

If G_1 does not contain any cut-edge, then it is easy to find a heterochromatic graph with r edges, the largest component of which has order at least $n_1 + 1$, a contradiction. From Lemma 3.2, $|E(G_1)| \leq \binom{n_1-1}{2} + 1$ follows clearly.

Now we consider the graph $K_n - V(G_1)$, a complete graph of order $n - n_1$. When restricting the r-edge-coloring ϕ on the graph $K_n - V(G_1)$, we have that $K_n - V(G_1)$ is edge-colored by r_0 colors, where $r_0 \geq r - (\binom{n_1-1}{2} + 1)$. If $r_0 \geq 2$, let $f(r_0) = t_0$. It follows that either $r_0 = 1$, or $t_0 \geq t - n_1 + 1$. We distinguish the following cases.

Case 1. $r_0 = 1$.

Then $K_n - V(G_1)$ is monochromatic, and then it follows that $k = 2$ and $n_2 = 2$. Let $G_2 = uv$. From the choice of G, we have $|E(G_1)| = r - 1 = \binom{t}{2} + 1$. By $n_1 \leq t + 1$, we have $n_1 = t + 1$. From Claim 1, let e be a cut-edge in G_1. Since $|E(G_1)| = \binom{t}{2} + 1$ and $n_1 = t + 1$, we have $G_1 - e \cong K_t \cup K_1$. Let $w \in V(G_1)$. From the choice of G, we have $\phi(uw) \neq \phi(uv)$, and there is a cut-edge in G_1 colored by the same color $\phi(uw)$.

If $n_1 \geq 4$, from $G_1 - e \cong K_t \cup K_1$, we have that e is the unique cut-edge in G_1. By $G_1 - e \cong K_t \cup K_1$, we can take a vertex w which is not single in $G_1 - e$. Then $\phi(uw) = \phi(e)$. By deleting the edge e and adding the edge uw, we can find a heterochromatic graph with r edges, the largest component of which has an order at least n_1+1, a contradiction to the choice of G.

So, assume $n_1 = 3$. Then $r = 3$ and $G_1 \cong P_3$. Let $G_1 = xyz$. Then either $\phi(yu) = \phi(xy)$ or $\phi(yu) = \phi(yz)$. Without loss of generality, assume $\phi(yu) = \phi(xy)$. Then $\phi(yu) \neq \phi(yz)$. Again, the graph $zyuv$ is heterochromatic and of size r, a contradiction to the choice of G.

Case 2. $t_0 \geq t - n_1 + 1$.

Since $r_0 \geq 2$, we have $t_0 \geq 1$. If $t_0 \geq t - n_1 + 2$, then by the induction hypothesis, the graph $K_n - V(G_1)$ can be covered by at most $\lceil \frac{n-n_1-t_0}{2} \rceil$ vertex-disjoint heterochromatic trees. Thus, $t_r(K_n, \phi) \leq 1 + \lceil \frac{n-n_1-t_0}{2} \rceil \leq \lceil \frac{n-t}{2} \rceil$, as desired.

Suppose $t_0 = t-n_1+1$. Then we have $r = \binom{t}{2}+2 \leq |E(G_1)|+r_0 \leq \binom{n_1-1}{2}+1+\binom{t_0+1}{2}+1 = \binom{n_1-1}{2}+1+\binom{t-n_1+1+1}{2}+1$. This implies that $\binom{t}{2} \leq \binom{n_1-1}{2}+\binom{t-(n_1-1)+1}{2}$, i.e., $(n_1-1)(t-(n_1-1)) \leq t-(n_1-1)$. By $n_1 \geq 3$ and $n_1 \leq t+1$, we have $n_1 = t+1$, and then $t_0 = 0$, a contradiction to the fact $t_0 \geq 1$. The proof is now complete. ∎

4 Algorithmic aspect

From the result in previous section, we know that given an r-edge-coloring ϕ of K_n, $t_r(K_n, \phi) \leq \lceil \frac{n-t}{2} \rceil$. A natural question is how to find an optimal heterochromatic tree partition of K_n. For general graphs, the decision version of this problem is defined formally as follows:

Heterochromatic Tree Partition Problem

Instance: An r-edge-coloring ϕ of a graph G, and a positive integer k.

Question: Are there k or less vertex-disjoint heterochromatic trees which cover all the vertices of G ?

For general graphs, the authors [10] showed that the problem above is NP-complete. Here, we present some positive results for the complete graphs. We show that given an r-edge-coloring ϕ of K_n, we can find at most $\lceil \frac{n-t}{2} \rceil$ vertex-disjoint heterochromatic trees to cover all the vertices in polynomial time. Our main technique comes from the proof in Section 3. A heterochromatic connected subgraph H is called *maximal* if the following hold:

1. For any $u, v \in V(H)$ with $uv \notin E(H)$, there is an edge $e \in E(H)$ such that $\phi(e) = \phi(uv)$.

2. For any $u \in V(H)$ and $v \notin V(H)$, there is a cut-edge $e \in E(H)$ such that $\phi(e) = \phi(uv)$.

The following proposition is obvious and the detailed proof is omitted.

Proposition 4.1 *If H is a maximal heterochromatic connected subgraph of an edge-colored complete graph K_n and $V(H) \subset V(K_n)$, then $|E(H)| \leq \binom{|V(H)|-1}{2} + 1$.*

Let ϕ be an r-edge-coloring of K_n and H be a maximal heterochromatic connected subgraph with $V(H) \subset V(K_n)$. From the definition, one can easily see that the graph $K_n - V(H)$ is edge-colored by $r_0 \geq r - |E(H)|$ colors. Let $f(r) = t$ and $f(r_0) = t_0$ if $r_0 \geq 2$. Also, if $r_0 \geq 2$, then $t_0 \geq t - |V(H)| + 1$.

If $|V(H)| \geq t+2$, Then we can find at most $\lceil \frac{n-t}{2} \rceil$ vertex-disjoint heterochromatic trees to cover all the vertices.

If $r_0 = 1$, as showed in proof of Theorem 1.1, we have that $|V(H)| \geq t+2$. Then we can find at most $\lceil \frac{n-t}{2} \rceil$ vertex-disjoint heterochromatic trees to cover all the vertices.

If $r_0 \geq 2$, as showed in proof of Theorem 1.1, we have that $|V(H)| \geq t+2$ or $t_0 \geq t-|V(H)|+2$. If $t_0 \geq t-|V(H)|+2$, by Theorem 1.1, $K_n - V(H)$ can be covered by at most $\lceil \frac{n-t-2}{2} \rceil$ vertex-disjoint heterochromatic trees.

So, from the analysis above, in order to find at most $\lceil \frac{n-t}{2} \rceil$ vertex-disjoint heterochromatic trees to cover all the vertices, we only need to find maximal heterochromatic connected subgraphs one by one. Given an r-edge-coloring of K_n, the following procedure produces a maximal heterochromatic connected subgraph.

Procedure

0. Initial state: Let H be an edge of K_n.

1. For any $u, v \in V(H)$, $uv \notin E(H)$, if there is no any edge $e \in E(H)$ such that $\phi(e) = \phi(uv)$, then $H \leftarrow H + uv$.

2. For each $u \in V(H), v \notin V(H)$, if there is no any edge $e \in E(H)$ such that $\phi(e) = \phi(uv)$, then $H \leftarrow H + uv$ and Goto 1.

3. For each $u \in V(H), v \notin V(H)$, if there is a non-cutedge $e \in E(H)$ such that $\phi(e) = \phi(uv)$, then $H \leftarrow H + uv - e$ and Goto 1.

4. Stop!

Theorem 4.2 *The procedure above can find a maximal heterochromatic connected subgraph in polynomial time.*

Proof. Let the current graph H have k vertices. Then Step 1 can be checked in at most $\binom{k}{2}$ times. And Steps 2 and 3 can be checked in at most $k(n-k)$ times and $k(n-k)+k$ times, respectively. So the procedure can find a maximal heterochromatic connected subgraph in $O(n^3)$. ∎

Repeating the procedure at most $O(n)$ times, we can find at most $\lceil \frac{n-t}{2} \rceil$ vertex-disjoint heterochromatic trees to cover all the vertices, and then we have the following result.

Theorem 4.3 *For any r-edge-colored complete graph K_n and $k \geq \lceil \frac{n-t}{2} \rceil$, the heterochromatic tree partition problem can be solved in polynomial time.*

5 Further discussion

There are several possible directions for further investigation. In our construction of the canonical r-edge-coloring ϕ_r^* in Section 2, one of the color classes contains lots of edges while each of the other color classes contains only one edge. One could therefore consider the problem for r-edge-colorings such that each color classes contains a bounded number of edges. In our construction of ϕ_r^*, some vertices have large color degree (i.e., the number of colors used on the incident edges) while the others have color degree only one. This is also a possible direction for generalization. In Section 4, we show that the heterochromatic tree partition problem can be solved in polynomial time for any r-edge-colored complete graph K_n and $k \geq \lceil \frac{n-t}{2} \rceil$. However, we do not know the complexity for the case $k < \lceil \frac{n-t}{2} \rceil$.

References

[1] Chen, H., Jin, Z., Li, X. and Tu, J. Heterochromatic tree partition numbers for complete bipartite graphs. *Discrete Math.* 308:3871-3878 (2008)

[2] Erdős, P., Gyárfás, A. and Pyber, L. Vertex coverings by monochromatic cycles and trees. *J. Combin. Theory Ser.B.* 51:90-95 (1991)

[3] Gyárfás, A. Vertex coverings by monochromatic paths and cycles. *J. Graph Theory* 7:131-135 (1983)

[4] Gyárfás, A. Covering complete graphs by monochromatic paths, in Irregularities of Partitions, Algorithms and Combinatorics, Vol.8, Springer-Verlag, 1989

[5] Gyárfás, A., Ruszinkó, M. , Sárközy, G. N. and Szemerédi, E. An improved bound for the monochromatic cycle partition number. *J. Combin. Theory Ser.B.* 96:855-873 (2006)

[6] Haxell, P. E. Partitioning complete bipartite graphs by monochromatic cycles. *J. Combin. Theory Ser.B.* 69:210-218 (1997)

[7] Haxell, P. E., Kohayakawa, Y. Partitioning by monochromatic trees. *J. Combin. Theory Ser.B.* 68:218-222 (1996)

[8] Jin, Z., Kano, M., Li, X. and Wei, B. Partitioning 2-edge-colored complete multipartite graphs into monochromatic cycles, paths and trees. *J. Combin. Optim.* 11:445-454 (2006)

[9] Kaneko, A., Kano, M. and Suzuki, K. Partitioning complete multipartite graphs by monochromatic trees. *J. Graph Theory* 48:133-141 (2005)

[10] Li, X., Zhang, X. On the minimum monochromatic or multicolored subgraph partition problems. *Theoret. Comput. Sci.* 385:1-10, (2007)

[11] Luczak, T., Rödl, V. and Szemerédi, E. Partitioning 2-edge-colored complete graphs into 2 monochromatic cycles. *Combin. Probab. Comput.* 7:423-436 (1998)

[12] Rado, R. Monochromatic paths in graphs. *Ann. Discrete Math.* 3:191-194 (1978)