Rainbow connection of graphs with diameter 2*

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Abstract

A path in an edge-colored graph, where adjacent edges may have the same color, is a *rainbow path* if no two edges of the path are colored the same. The *rainbow connection number* $rc(G)$ of a graph G is the minimum integer k for which there exists a k-edge-coloring of G such that any two distinct vertices of G are connected by a rainbow path. It is known that for a graph G with diameter 2, deciding if $rc(G) = 2$ is NP-complete. In particular, computing $rc(G)$ is NP-hard. So, it is interesting to know the upper bound of $rc(G)$ for such a graph G. In this paper, we show that $rc(G) \leq 5$ if G is a bridgeless graph with diameter 2, and that $rc(G) \leq k+2$ if G is a connected graph with diameter 2 and has k bridges, where $k \geq 1$.

Keywords: Edge-coloring, Rainbow path, Rainbow connection number, Diameter

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1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to book [1] for graph theoretical notation and terminology not described here. A path in an edge-colored graph, where adjacent edges may have the same color, is a *rainbow path* if no two edges of the path are colored the same. An edge-coloring of a graph G is a *rainbow-connected edge-coloring* if any two distinct vertices of G are connected by a rainbow path. Such an edge-coloring is called *rainbow*. The *rainbow connection number* $rc(G)$ of G is the minimum integer k such that G has a rainbow-connected edge-coloring using k colors. It is easy to see that $\text{diam}(G) \leq rc(G)$ for any connected graph G, where $\text{diam}(G)$ is the diameter of G.

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The rainbow connection number was introduced by Chartrand et al. in [5]. It has application in transferring information of high security in multicomputer networks. We refer the readers to [3, 6] for details. Bounds on the rainbow connection numbers of graphs have been studied in terms of other graph parameters, such as radius, dominating number, minimum degree, connectivity, etc. [2, 4, 5, 9, 8, 10]. Chakraborty et al.[3] investigated the hardness and algorithms for the rainbow connection number, and showed the following result.

Theorem 1. [3] *Given a graph G with diameter 2, deciding if $rc(G) = 2$ is NP-complete. In particular, computing $rc(G)$ is NP-hard.*

It is well-known that almost all graphs have diameter 2. So, it is interesting to find a sharp upper bound on $rc(G)$ when G has diameter 2. Clearly, the best lower bound on $rc(G)$ for such a graph G is 2. In this paper, we give sharp upper bounds on the rainbow connection number of a graph with diameter 2: if G is a bridgeless graph with diameter 2, then $rc(G) \leq 5$; if G is a connected graph with diameter 2 and has k bridges, where $k \geq 1$, then $rc(G) \leq k + 2$.

2 Main results

We begin with some notation and terminology. Let G be a graph. The *eccentricity* of a vertex u of G, written as $\epsilon_G(u)$, is $\max\{\, d_G(u, v) : v \in V(G)\,\}$. The *radius* of G, written as $\mathrm{rad}(G)$, is $\min\{\, \epsilon_G(u) : u \in V(G)\,\}$. A vertex u is a *center* of G if $\epsilon_G(u) = \mathrm{rad}(G)$. Let U be a set of vertices of G. The *k-step open neighbourhood* of U in G, denoted by $N_G^k(U)$, is $\{\, v \in V(G) : d_G(U, v) = k\,\}$ for each k, where $0 \leq k \leq \mathrm{diam}(G)$ and $d_G(U, v) = \min\{\, d_G(u, v) : u \in U\,\}$. We write $N_G(U)$ for $N_G^1(U)$ and $N_G(u)$ for $N_G^1(\{u\})$. For any two subsets X and Y of $V(G)$, let $E_G[X, Y]$ denote $\{\, xy : x \in X, y \in Y, xy \in E(G)\,\}$. Let c be a rainbow-connected edge-coloring of G. A path P is a $\{k_1, \cdots, k_r\}$-*rainbow path* if it is a rainbow path and $c(e) \in \{k_1, \cdots, k_r\}$ for each e in $E(P)$. In particular, an edge e is a *k-color edge* if it is colored by k.

Proposition 2. *If G is a bridgeless graph with diameter 2, then either G is 2-connected, or G has only one cut-vertex. Furthermore, the vertex is the center of G, and G has radius 1.*

Proof. Let G be a bridgeless graph with diameter 2. Suppose that G is not 2-connected, that is, G has a cut-vertex, say v. Moreover, G has only one cut-vertex, since $\mathrm{diam}(G) = 2$. If some vertex other than v is not adjacent to v, then its distance from the vertices in the other components of $G - v$ is at least 3, a contradiction. Therefore v is the center of G, and G has radius 1. □

Lemma 3. *Let G be a bridgeless graph with diameter 2. If G has a cut vertex, then $rc(G) \leq 3$.*

Proof. Let u be a cut-vertex of G. By Proposition 2, the vertex u is the only cut-vertex of G and is also adjacent to all other vertices. Let F be a spanning forest of $G - u$, and let (X, Y) be one of the bipartition defined by F. Note that F has no isolated vertices, because G has no bridges. We provide a 3-edge-coloring c of G as follows: $c(e) = 1$, if $e \in E[u, X]$; $c(e) = 2$, if $e \in E[u, Y]$; $c(e) = 3$, if $e \in E[X, Y]$. By the construct, paths joining any vertex of X to any vertex of Y through u are rainbow. Rainbow paths $\langle x, u, y, x' \rangle$ join any two vertices $x, x' \in X$, where y is a neighbor of x' in F, and similarly there are rainbow paths of length 3 joining any two vertices in Y. $\square$

Let $X_1, X_2, \cdots X_k$ be pairwise disjoint vertex subsets of G. Notation $X_1 \sim X_2 \sim \cdots \sim X_k$ means that there exists some desired rainbow path $\langle x_1, x_2, \cdots, x_k \rangle$, where $x_i \in X_i$ for each $i \in \{1, \ldots, k\}$.

Lemma 4. *If G is a 2-connected graph with diameter 2, then $rc(G) \leq 5$.*

Proof. Pick a vertex v in $V(G)$ arbitrarily. Let

$$B = \{\, u \in N_G^2(v) : \textit{there exists a vertex } w \textit{ in } N_G^2(v) \textit{ such that } uw \in E(G)\}.$$

We consider the following two cases distinguishing either $B \neq \emptyset$ or $B = \emptyset$.

Case 1. $B \neq \emptyset$.

In this case, the subgraph $G[B]$ of G induced by B has no isolated vertices. Let F be a spanning forest F of $G[B]$, and let (B_1, B_2) be one of the bipartition defined by F. Now we divide $N_G(v)$ as follows. Set $X = \emptyset$ and $Y = \emptyset$. For each u in $N_G(v)$, if $u \in N_G(B_1)$, then we put u into X. If $u \in N_G(B_2)$, then we put u into Y. If $u \in N_G(B_1)$ and $u \in N_G(B_2)$, then we put u into X. By the argument above, we know that for each x in X (y in Y), there exists a vertex y in Y (x in X) such that x and y are connected by a path P with length 3 satisfying $(V(P) - \{x, y\}) \subseteq B$.

We have the following claim for each u in $N_G(v) - (X \cup Y)$.

Claim 1. For each u in $N_G(v) - (X \cup Y)$, either u has a neighbor w in X, or u has a neighbor w in Y.

Proof of Claim 1. Let u be a vertex in $N_G(v) - (X \cup Y)$. Note that B_1 is nonempty. If $z \in B_1$, then u and z are nonadjacent since $u \notin X \cup Y$. Moreover, $\mathrm{diam}(G) = 2$ implies that u and z have a common neighbor w. We see that $w \notin N_G^2(v)$, otherwise, $w \in B$ and $u \in X \cup Y$, a contradiction. Similarly, we have that $w \notin N_G(v) - (X \cup Y)$. Thus w must be contained in $X \cup Y$.

By the claim above, for each u in $N_G(v) - (X \cup Y)$, either we can put u into X such that $u \in N_G(Y)$, or we can put u into Y such that $u \in N_G(X)$. Now X and Y form a partition of $N_G(v)$.

For $N_G^2(v) - B$, let

$$A = \{\, u \in N_G^2(v) : u \in N_G(X) \cap N_G(Y) \,\};$$
$$D_1 = \{\, u \in N_G^2(v) : u \in N_G(X) - N_G(Y) \,\};$$
$$D_2 = \{\, u \in N_G^2(v) : u \in N_G(Y) - N_G(X) \,\}.$$

We see that at least one of D_1 and D_2 is empty. Otherwise, there exist $u \in D_1$ and $v \in D_2$ such that $d_G(u, v) \geq 3$, a contradiction. Without loss of generality, suppose $D_2 = \emptyset$.

First, we provide a 5-edge-coloring $c : E(G) - E_G[D_1, X] \to \{1, 2, \ldots, 5\}$ defined by

$$c(e) = \begin{cases} 1, & if\ e \in E_G[v, X]; \\ 2, & if\ e \in E_G[v, Y]; \\ 3, & if\ e \in E_G[X, Y] \cup E_G[Y, A] \cup E_G[B_1, B_2]; \\ 4, & if\ e \in E_G[X, A] \cup E_G[X, B_1]; \\ 5, & if\ e \in E_G[Y, B_2],\ or\ otherwise. \end{cases}$$

Next, we color the edges in $E_G[X, D_1]$ as follows. For each u in D_1, color one edge incident with u by 5 (solid lines) and the other edges incident with u by 4 (dotted lines). See Figure 1.

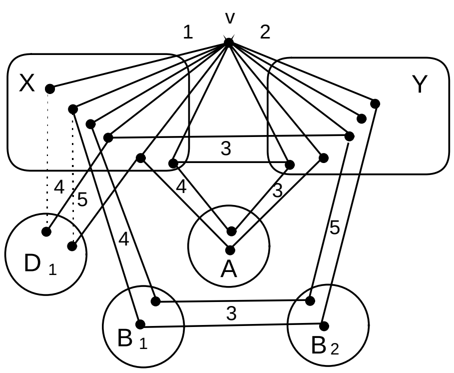

Figure 1.

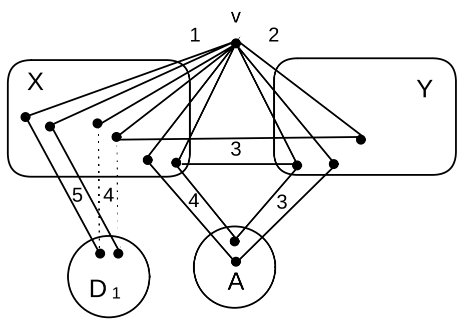

Figure 2.

We have the following claim for the coloring above.

Claim 2. (i) *For each x in X, there exists a vertex y in Y such that x and y are connected by a $\{3, 4, 5\}$-rainbow path in $G - v$.*

(ii) *For each y in Y, there exists a vertex x in X such that x and y are connected by a $\{3, 4, 5\}$-rainbow path in $G - v$.*

(iii) *For any two vertices u and u' in D_1, there exists a rainbow path connecting u and u'.*

(iv) *For each u in D_1 and each u' in X, there exists a rainbow path connecting u and u'.*

Proof of Claim 2. First, we show that (i) and (ii) hold. We only prove part (i), since part (ii) can be proved by a similar argument. By the procedure of constructing X and Y, we know that for any $x \in X$, either there exists a vertex $y \in Y$ such that $xy \in E(G)$, or there exists a vertex $y \in Y$ such that x and y are connected by a path P with length 3 satisfying

$(V(P) - \{x, y\}) \subseteq B$. Clearly, this path is a $\{3, 4, 5\}$-rainbow path.

Next, we show that (iii) holds. The vertices u and y have a common neighbor w in X since $\text{diam}(G) = 2$. Furthermore, without loss of generality, suppose that uw is a 5-color edge. Therefore $\langle u, w, y, v, w', u' \rangle$ is a rainbow path connecting u and u', where u' is adjacent to w' by a 4-color edge $u'w'$.

Finally, we show that (iv) holds. Pick a vertex y in Y. The vertices u and y have a common adjacency vertex w in X since $\text{diam}(G) = 2$. Therefore $\langle u, w, y, v, u' \rangle$ is a rainbow path connecting u and u'. The proof of Claim 2 is complete.

It is easy to see that the edge-coloring above is rainbow in this case by Figure 1 and Table 1.

	v	X	Y	A	B_1	B_2	D_1
v	$\cdots$	$v \sim X$	$v \sim Y$	$v \sim X \sim A$	$v \sim X \sim B_1$	$v \sim X \sim B_1 \sim B_2$	$v \sim X \sim D_1$
X	$\cdots$	Claim 2 and $Y \sim v \sim X$	$X \sim v \sim Y$	$X \sim v \sim Y \sim A$	$X \sim v \sim Y \sim B_2 \sim B_1$	$X \sim v \sim Y \sim B_2$	Claim 2
Y	$\cdots$	$\cdots$	Claim 2 and $X \sim v \sim Y$	$Y \sim v \sim X \sim A$	$Y \sim v \sim X \sim B_1$	$Y \sim v \sim X \sim B_1 \sim B_2$	$Y \sim v \sim X \sim D_1$
A	$\cdots$	$\cdots$	$\cdots$	$A \sim X \sim v \sim Y \sim A$	$A \sim Y \sim v \sim X \sim B_1$	$A \sim X \sim v \sim Y \sim B_2$	$A \sim Y \sim v \sim X \sim D_1$
B_1	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$B_1 \sim X \sim v \sim Y \sim B_2 \sim B_1$	$B_1 \sim X \sim v \sim Y \sim B_2$	$B_1 \sim B_2 \sim Y \sim v \sim X \sim D_1$
B_2	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$B_2 \sim B_1 \sim X \sim v \sim Y \sim B_2$	$B_2 \sim Y \sim v \sim X \sim D_1$
D_1	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	Claim 2

Table 1. Rainbow paths in G

Case 2. $B = \emptyset$.

In this case, clearly, $N_G(u) \subseteq N_G(v)$ for each u in $N_G^2(v)$. To show a rainbow coloring of G, we need to construct a new graph H. The vertex set of H is $N_G(v)$, and the edge set is $\{\, xy : x, y \in N_G(v),\ x \textit{ and } y \textit{ are connected by a path } P \textit{ of length at most } 2 \textit{ in } G - v,\ \textit{and } V(P) \cap N_G(v) = \{x, y\} \,\}$.

Claim 3. The graph H is connected.

Proof of Claim 3. Let x and y be any two distinct vertices of H. Since G is 2-connected, the vertices x and y are connected by a path in $G - v$. Let $\langle v_0, v_1, \ldots, v_k \rangle$ is a shortest path joining x and y in $G - v$, where $x = v_0$ and $v_k = y$.

If $k = 1$, then by the definition of H, the vertices x and y are adjacent in H. Otherwise, $k \geq 2$. Since $\text{diam}(G) = 2$, the vertex v_i is adjacent to v, or v_i and v have a common neighbor u_i if $d_G(v, v_i) = 2$. For each integer i with $0 \leq i \leq k - 1$, if $d_G(v, v_i) = 1$ and $d_G(v, v_{i+1}) = 1$, then v_i and v_{i+1} are contained in $V(H)$, and they are adjacent in H. If

$d_G(v, v_i) = 1$ and $d_G(v, v_{i+1}) = 2$, then v_i and u_{i+1} are contained in $V(H)$, and they are adjacent in H. If $d_G(v, v_i) = 2$ and $d_G(v, v_{i+1}) = 1$, then u_i and v_{i+1} are contained in $V(H)$, and they are adjacent in H. If $d_G(v, v_i) = 2$ and $d_G(v, v_{i+1}) = 2$, then u_i and u_{i+1} should be contained in B, which contradicts the fact that $B = \emptyset$. Therefore there exists a path connecting x and y in H. The proof of Claim 3 is complete.

Let T be a spanning tree of H, and let (X, Y) be the bipartition defined by T. Now divide $N_G^2(v)$ as follows. For $N_G^2(v)$,

$$let\ A = \{\, u \in N_G^2(v) :\in N_G(X) \cap N_G(Y) \,\};$$

For $N_G^2(v) - A$,

$$let\ D_1 = \{\, u \in N_G^2(v) : u \in N_G(X) - N_G(Y) \,\},$$
$$D_2 = \{\, u \in N_G^2(v) : u \in N_G(Y) - N_G(X) \,\}.$$

We see that at least one of D_1 and D_2 is empty. Otherwise, there exist $u \in D_1$ and $v \in D_2$ such that $d_G(u, v) \geq 3$, a contradiction. Without loss of generality, suppose $D_2 = \emptyset$. Therefore A and D_1 form a partition of $N_G^2(v)$. See Figure 2.

First, we provide a 4-edge-coloring $c : E(G) - E_G[D_1, X] \to \{1, 2, \ldots, 4\}$ defined by

$$c(e) = \begin{cases} 1, & if\ e \in E_G[v, X]; \\ 2, & if\ e \in E_G[v, Y]; \\ 3, & if\ e \in E_G[X, Y] \cup E_G[Y, A]; \\ 4, & if\ e \in E_G[X, A],\ or\ otherwise. \end{cases}$$

Next, we color the edges in $E_G[X, D_1]$ as follows. For each u in D_1, color one edge incident with u by 5 (solid lines), the other edges incident with u by 4 (dotted lines). See Figure 2.

It is easy to check that the edge-coloring above is rainbow in this case by Figure 2 and Table 2.

	v	X	Y	A	D_1
v	$\cdots$	$v \sim X$	$v \sim Y$	$v \sim X \sim A$	$v \sim X \sim D_1$
X	$\cdots$	Claim 2 and $Y \sim v \sim X$	$X \sim v \sim Y$	$X \sim v \sim Y \sim A$	Claim 2
Y	$\cdots$	$\cdots$	Claim 2 and $X \sim v \sim Y$	$Y \sim v \sim X \sim A$	$Y \sim v \sim X \sim D_1$
A	$\cdots$	$\cdots$	$\cdots$	$A \sim X \sim v \sim Y \sim A$	$A \sim Y \sim v \sim X \sim D_1$
D_1	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$D_1 \sim A \sim Y \sim v \sim X \sim D_1$

Table 2. Rainbow paths in G

By this both possibilities have been exhausted and the proof is thus complete. □

Combining Proposition 2 with Lemmas 3 and 4, we have the following theorem.

Theorem 5. *If G is a bridgeless graph with diameter 2, then $rc(G) \leq 5$.*

Remark 1. Recently, Dong and Li [7] gave a class of graphs with diameter 2 that achieve equality of this bound.

For graphs containing bridges, the following proposition holds.

Proposition 6. *If G is a connected graph with diameter 2 and has k bridges, where $k \geq 1$, then $rc(G) \leq k + 2$.*

Proof. Since $\mathrm{diam}(G) = 2$, all bridges have a common endpoint u. Moreover, the vertex u is adjacent to all other vertices. For all bridges, we color them with different colors. The remaining edges can be colored similar to Lemma 1 with two new colors and one old color. It is easy to check that the coloring above is a rainbow-coloring of G with $k+2$ colors. □

Tight examples: The upper bound in Proposition 6 is tight. The graph $(kK_1 \cup rK_2) \vee v$ has a rainbow connection number achieving this upper bound, where $k \geq 1, r \geq 2$.

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