Noncrossing Linked Partitions and Large $(3,2)$-Motzkin Paths

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Abstract. Noncrossing linked partitions arise in the study of certain transforms in free probability theory. We explore the connection between noncrossing linked partitions and $(3,2)$-Motzkin paths, where a $(3,2)$-Motzkin path can be viewed as a Motzkin path for which there are three types of horizontal steps and two types of down steps. A large $(3,2)$-Motzkin path is a $(3,2)$-Motzkin path for which there are only two types of horizontal steps on the x-axis. We establish a one-to-one correspondence between the set of noncrossing linked partitions of $\{1,\ldots,n+1\}$ and the set of large $(3,2)$-Motzkin paths of length n, which leads to a simple explanation of the well-known relation between the large and the little Schröder numbers.

Keywords: Noncrossing linked partition, Schröder path, large $(3,2)$-Motzkin path, Schröder number

AMS Classifications: 05A15, 05A18.

1 Introduction

The notion of noncrossing linked partitions was introduced by Dykema [5] in the study of the unsymmetrized T-transform in free probability theory. Let $[n]$ denote $\{1,\ldots,n\}$. It has been shown that the generating function of the number of noncrossing linked partitions of $[n+1]$ is given by

$$F(x) = \sum_{n=0}^{\infty} f_{n+1} x^n = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}. \tag{1.1}$$

This implies that the number of noncrossing linked partitions of $[n+1]$ is equal to the n-th large Schröder number S_n, that is, the number of large Schröder paths of length $2n$. To be more specific, a *large Schröder path* of length $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ consisting of up steps $(1,1)$, horizontal steps $(2,0)$ and down steps $(1,-1)$ that does not go below the x-axis. Notice that a large Schröder path is also called a Schröder path. The first few values of S_n are given below

$$1, 2, 6, 22, 90, 394, 1806, \ldots.$$

The sequence of the large Schröder numbers is listed as entry A006318 in OEIS [8]. A bijection from the set of noncrossing linked partitions of $[n+1]$ to the set of large Schröder paths of length $2n$ was established by Chen, Wu, and Yan [2].

In this paper, we aim to construct an explicit correspondence between noncrossing linked partitions and $(3,2)$-Motzkin paths. Recall that a *Motzkin path* of length n is defined as a lattice path from $(0,0)$ to $(n,0)$ consisting of up steps $(1,1)$, horizontal steps $(1,0)$ and down steps $(1,-1)$ that does not go below the x-axis. A $(3,2)$-*Motzkin path* is a Motzkin path for which each horizontal step colored by one of the three colors 1, 2, and 3, and each down step colored by one of the two colors 1 and 2.

It is known that the number of little Schröder paths of length $2n$ equals the number of $(3,2)$-Motzkin paths of length $n-1$, where a *little Schröder path* is defined as a large Schröder path such that there are no horizontal steps on the x-axis. Yan [10] found a bijective proof of this fact. The number of little Schröder paths of length $2n$ is referred to as the little Schröder number s_n. Since the large Schröder numbers and the little Schröder numbers are related by a factor of two, we see that the number of noncrossing linked partitions of $[n+1]$ is twice the number of $(3,2)$-Motzkin paths of length n.

In this paper, we introduce a class of Motzkin paths, called *large* $(3,2)$-*Motzkin paths*, which are defined as $(3,2)$-Motzkin paths such that each horizontal step at the x-axis is colored by one of the two colors 1 and 2. We shall show that noncrossing linked partitions of $[n+1]$ are in one-to-one correspondence with large $(3,2)$-Motzkin paths of length n. By examining the connection between large $(3,2)$-Motzkin paths and ordinary $(3,2)$-Motzkin paths, we immediately get the relation between the large and the little Schröder numbers.

Let us give a brief review of some terminology. Let m_n denote the n-th $(3,2)$-Motzkin number, that is, the number of $(3,2)$-Motzkin paths with n steps. An *irreducible large* $(3,2)$-*Motzkin path* is defined as a large $(3,2)$-Motzkin path that does not touch the x-axis except for the origin and the destination. Bear in mind that a horizontal step on the x-axis is considered as an irreducible large $(3, 2)$-Motzkin path. The *length* of a path is defined to be the number of steps in the path. Denote the set of large $(3, 2)$-Motzkin paths by L and the set of large $(3, 2)$-Motzkin paths of length n by L_n. Let l_n be the number of paths in L_n.

By the decomposition of a large $(3, 2)$-Motzkin path into irreducible segments, we see that the generating function

$$L(x) = \sum_{n=0}^{\infty} l_n x^n$$

satisfies the functional equation

$$L(x) = 1 + 2xL(x) + 2x^2 M(x)L(x), \tag{1.2}$$

where

$$M(x) = \sum_{n=0}^{\infty} m_n x^n = \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x^2} \tag{1.3}$$

is the generating function of the $(3, 2)$-Motzkin numbers. A similar decomposition has been used by Cheon, Lee, and Shapiro [3] to derive generating function identities for the Catalan numbers and the Fine numbers. From (1.2) and (1.3) it follows that $L(x) = F(x)$. This yields

$$l_n = f_{n+1}. \tag{1.4}$$

Using the connection between the large $(3, 2)$-Motzkin paths and ordinary $(3, 2)$-Motzkin paths, we are led to a simple explanation of the following relation:

$$l_n = 2m_{n-1}. \tag{1.5}$$

Since the little Schröder number s_n is equal to the $(3, 2)$-Motzkin number m_{n-1} (Chen, Li, Shapiro, and Yan [1] and Yan [10]), we find that relation (1.5) is equivalent to the well-known relation

$$S_n = 2s_n. \tag{1.6}$$

Combinatorial interpretations of (1.6) have been given by Shapiro and Sulanke [9], Deutsch [4], Gu, Li, and Mansour [6], and Huq [7].

2 Noncrossing Linked Partitions

In this section, we give a bijection from the set of large $(3, 2)$-Motzkin paths of length n to the set of noncrossing linked partitions of $[n + 1]$.

A *linked partition* of $[n]$ is a collection of nonempty subsets $B_1, \ldots, B_k$ of $[n]$, called *blocks*, such that the union of $B_1, \ldots, B_k$ is $[n]$ and any two distinct blocks are nearly disjoint. Two blocks B_i and B_j are said to be *nearly disjoint* if for any $k \in B_i \cap B_j$, one of the following conditions holds:

(a) $k = \min(B_i), |B_i| > 1$ and $k \neq \min(B_j)$, or

(b) $k = \min(B_j), |B_j| > 1$ and $k \neq \min(B_i)$.

We say that $\pi = \{B_1, \ldots, B_k\}$ is a *noncrossing linked partition* if in addition, for any two distinct blocks A and B in π, there does not exist $a, b \in A$ and $c, d \in B$ such that $a < c < b < d$. Let $NCL(n)$ denote the set of noncrossing linked partitions of $[n]$.

In this paper, we adopt the *linear representation* of linked partitions, introduced by Chen, Wu, and Yan [2]. For a linked partition π of $[n]$, first we draw n vertices $1, \ldots, n$ on a horizontal line in increasing order. For each block $B = \{i_1, \ldots, i_k\}$, we write the elements $i_1, \ldots, i_k$ in increasing order, and we use $\min(B)$ to denote the minimum element i_1 of B. If $k \geq 2$, then we draw an arc joining i_1 and any other vertex in B. We shall use a pair (i, j) to denote an arc between i and j, where we assume that $i < j$. It can be seen that a linked partition is noncrossing if and only if it does not contain any crossing arcs in its linear representation. For example, the linear representation of a noncrossing linked partition $\pi = \{1, 4, 9\}\{2, 3\}\{5, 6\}\{6, 7\}\{8\}$ is illustrated in Figure 2.1, where 6 belongs to both blocks $\{5, 6\}$ and $\{6, 7\}$.

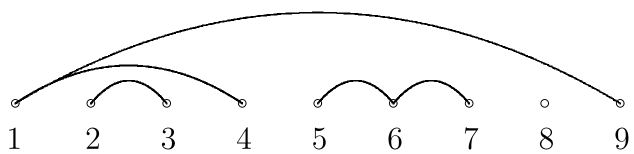

Figure 2.1: The linear representation of $\pi = \{1, 4, 9\}\{2, 3\}\{5, 6\}\{6, 7\}\{8\}$.

Below is the main result of this paper.

Theorem 2.1 *There is a bijection from the set of large* $(3, 2)$*-Motzkin paths of length* n *to the set of noncrossing linked partitions of* $[n + 1]$*.*

Proof. To establish the correspondence, we define a map φ from L_n to $NCL(n + 1)$ in terms of a recursive procedure. Let P be a large $(3, 2)$-Motzkin path in L_n, which is represented as a sequence on $\{u, d_1, d_2, h_1, h_2, h_3\}$, where u is an up step, d_i is an down step with color i for $i = 1, 2$, and h_j is a horizontal step with color j for $j = 1, 2, 3$. We proceed to construct a noncrossing linked partition $\pi = \varphi(P)$.

If $P = \emptyset$, then set $\varphi(P) = \{1\}$. If P is nonempty, then it can be decomposed into a sequence of irreducible large $(3, 2)$-Motzkin paths, say, $P = P_1 P_2 \cdots P_k$. Note that a horizontal step on the x-axis is an irreducible large $(3, 2)$-Motzkin path. For each segment

P_i, let p_i denote the length of P_i. We wish to construct a noncrossing linked partition $\varphi(P_i)$ on the set $\{1, \ldots, p_i + 1\}$. We can then recover a noncrossing linked partition π by piecing together the noncrossing linked partitions $\varphi(P_1), \varphi(P_2), \ldots, \varphi(P_k)$ and relabeling the elements from left to right with $1, \ldots, n+1$.

Case 1: P_i contains only one step. If $P_i = h_1$, then set $\varphi(P_i) = \{1, 2\}$; if $P_i = h_2$, then set $\varphi(P_i) = \{1\}\{2\}$. Figure 2.2 is an illustration of this case.

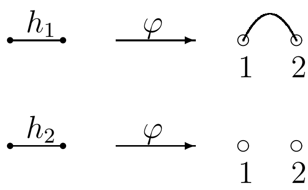

Figure 2.2: Case 1.

Case 2: P_i contains at least two steps. In this case, we may write P_i in the form $uQ_1h_3Q_2h_3 \cdots h_3Q_rd$, where $r \geq 1$, $d = d_1$ or d_2, and $Q_j \in L$ is a large $(3,2)$-Motzkin path that is allowed to be empty. Then $\varphi(P_i)$ can be generated by the following operations on the linear representations of $\varphi(Q_1), \varphi(Q_2), \ldots, \varphi(Q_r)$.

For the case $d = d_1$, arrange the linear representations of $\varphi(Q_1), \varphi(Q_2), \ldots, \varphi(Q_r)$ from left to right, and relabel the vertices also from left to right by $1, \ldots, p_i - 1$. For $j = 1, \ldots, r-1$, add an arc connecting the minimal vertex of $\varphi(Q_j)$ and the minimal vertex of $\varphi(Q_{j+1})$. Then add two vertices p_i and $p_i + 1$ to the right of $\varphi(Q_r)$. Finally, add an arc connecting the minimal vertex of $\varphi(Q_r)$ and the vertex p_i and add an arc connecting 1 and the vertex $p_i + 1$. See Figure 2.3.

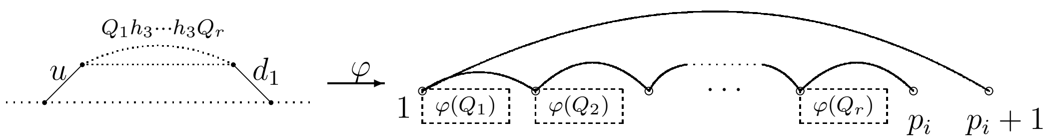

Figure 2.3: The case for $d = d_1$.

For the case $d = d_2$, the construction of $\varphi(P_i)$ is similar to the case $d = d_1$, except that we do not add the arc connecting the vertex 1 and the minimal vertex of $\varphi(Q_2)$. See Figure 2.4. If $r = 1$, namely $P_i = uQ_1d_2$, then p_i is an isolated vertex in $\varphi(P_i)$.

Finally, we join the last vertex of $\varphi(P_i)$ and the first vertex of $\varphi(P_{i+1})$, for $i = 1, \ldots, k-1$. Now $\pi = \varphi(P)$ can be obtained by relabeling the vertices from left to right with

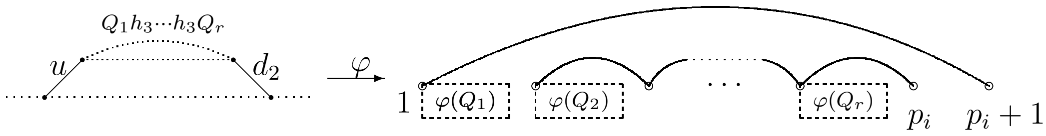

Figure 2.4: The case for $d = d_2$.

$\{1, \ldots, n+1\}$. It can be seen that π is a noncrossing linked partition of $[n+1]$. Figure 2.5 is an illustration of the operation of piecing together noncrossing linked partitions that correspond to irreducible large $(3, 2)$-Motzkin paths, where we use a dotted arc to represent a boundary arc. More precisely, a boundary arc of a partition is an arc that is not covered by any other arc.

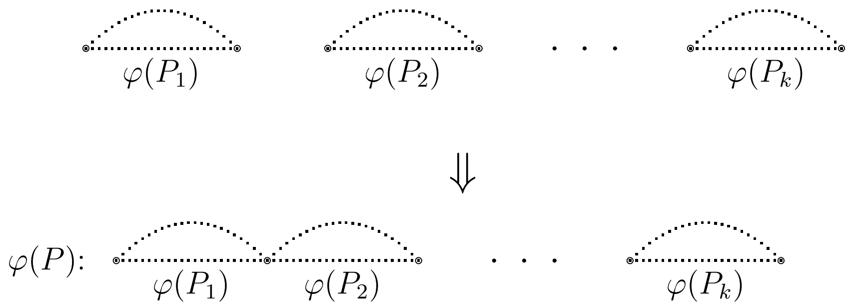

Figure 2.5: The operation of piecing together noncrossing linked partitions.

To show that φ is a bijection, we aim to construct the inverse map φ^{-1} from noncrossing linked partitions in $NCL(n+1)$ to large $(3, 2)$-Motzkin paths in L_n. Let π be a noncrossing linked partition in $NCL(n+1)$. As the inverse step of decomposing a large $(3, 2)$-Motzkin path into irreducible segments, we can decompose a noncrossing linked partition also into irreducible segments. We say that a noncrossing linked partition π of $[n+1]$ is irreducible if it has a boundary arc or it is $\{1\}\{2\}$ for $n = 1$. It is easy to decompose π into irreducible segments. In the linear representation of π, if there is a boundary arc from 1 to j, for $j \geq 2$, then the partition of $[j]$ consisting of the arcs of the linear representation of π forms an irreducible noncrossing linked partition. Removing the vertices $1, \ldots, j-1$, we obtain a noncrossing linked partition. If 1 is an isolated vertex, then we may form an irreducible partition $\{1\}\{2\}$. Removing the vertex 1, we obtain a noncrossing linked partition. In either case, we can iterate this process to decompose π into irreducible segments.

It is routine to verify that for any irreducible noncrossing linked partition, one can reverse every step of the map φ to obtain an irreducible large $(3, 2)$-Motzkin path. Thus the map φ is a bijection. This completes the proof. ∎

For example, the decomposition of $\pi = \{1,3,5\}\{2\}\{4\}\{5,6\}\{7\}\{8\} \in NCL(8)$ is shown in Figure 2.6.

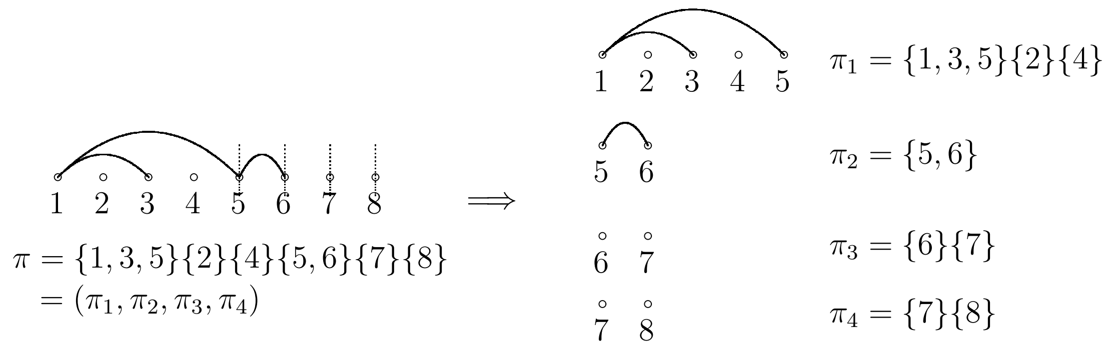

Figure 2.6: The decomposition of $\pi = \{1,3,5\}\{2\}\{4\}\{5,6\}\{7\}\{8\}$.

An example of the above bijection is given in Figure 2.7.

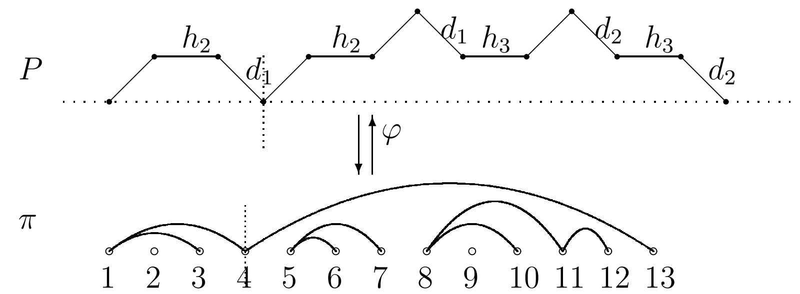

Figure 2.7: Bijection $\varphi \colon L_{12} \to NCL(13)$.

The above bijection implies that the large Schröder number S_n equals the number l_n of large $(3,2)$-Motzkin paths of length n. On the other hand, there is a one-to-one correspondence between $(3,2)$-Motzkin paths of length $n-1$ and little Schröder paths of length $2n$. Therefore, the relation $S_n = 2s_n$ can be rewritten as

$$l_n = 2m_{n-1}, \tag{2.7}$$

that is, the number of large $(3,2)$-Motzkin paths of length n is twice the number of ordinary $(3,2)$-Motzkin paths of length $n-1$. Here we give a combinatorial interpretation of this fact. Let P be a $(3,2)$-Motzkin path of length $n-1$. If P does not have any horizontal step h_3 on the x-axis, then we can get two large $(3,2)$-Motzkin paths by adding a horizontal step h_1 or h_2 at the end of P. Otherwise, we remove the first horizontal step

h_3 on the x-axis in P, and elevate the path after this h_3 horizontal step by adding an up step at the beginning and a down step at the end so that the resulting path is a large $(3, 2)$-Motzkin path of length n. In this case, there are also two choices for the last down step. It is easy to see that the above construction is reversible. Hence we obtain (2.7).

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