ON THE MAXIMAL CROSS NUMBER OF UNIQUE FACTORIZATION ZERO-SUM SEQUENCES OVER A FINITE ABELIAN GROUP

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ABSTRACT. Let $S = (g_1, \cdots, g_l)$ be a sequence of elements from a finite additively written abelian group G. Let

$$k(S) = \sum_{i=1}^{l} \frac{1}{\mathrm{ord}(g_i)}$$

denote the cross number of S. We say a zero-sum sequence S of nonzero elements from G *is a unique factorization zero-sum sequence* if S can be written in the form $S = S_1 \cdots S_r$ uniquely, where all S_i are minimal zero-sum subsequences of S. In this short note we investigate the following invariant of G concerning both cross number and unique factorization. Define

$$K_1(G) = \max\{k(S) | S \text{ is a unique factorization zero-sum sequence over } G \setminus \{0\}\},$$

where the maximum is taken when S runs over all unique factorization zero-sum sequences over $G \setminus \{0\}$. We determine $K_1(G)$ for some special groups including the cyclic groups of prime power order.

1. INTRODUCTION AND MAIN RESULTS

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{Z}$ denote the set of integers. For real numbers $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{Z} | a \leq x \leq b\}$. Let G be an additively written finite abelian group. We denote by $|G|$ the *order* of G. A sequence $S = (g_1, \cdots, g_l)$ of elements (repetition allowed) from G will be called a sequence over G. For convenience, we often write S in the form $S = g_1 \cdot \ldots \cdot g_l$. We call $|S| = l$ the *length* of S. If $g_1 = \cdots = g_l = g$ then we can simply write S in the form $S = g^l$. For every $g \in G$, let $v_g(S)$ denote the number of the times that g occurs in S. Let $T = g_{i_1} \cdots g_{i_t}$ be a subsequence of S. We call $I_T \overset{\text{def}}{=} \{i_1, \cdots, i_t\}$ the *index set* of T. We denote by ST^{-1} the subsequence of S with index set $\{1, \cdots, l\} \setminus I_T$. Let T_1 and T_2 be two subsequences of S. By $T_1 \cap T_2$ we denote the sequence with index set $I_{T_1} \cap I_{T_2}$. We say T_1 and T_2 are disjoint if $I_{T_1} \cap I_{T_2} = \emptyset$, and denote by T_1T_2 the sequence with index set $I_{T_1} \cup I_{T_2}$. We identify two subsequences S_1 and S_2 of S if and only if $I_{S_1} = I_{S_2}$.

Let $\sigma(S) = \sum_{i=1}^{l} g_i \in G$ denote the sum of S. We call the sequence S

- a *zero-sum* sequence if $\sigma(S) = 0$,
- a *zero-sum free* sequence if S contains no nonempty zero-sum subsequence,

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- a *minimal zero-sum* sequence if S is a nonempty zero-sum sequence and S contains no proper zero-sum subsequence.

Every map of abelian groups $\phi : G \to H$ extents to a map from the sequences over G to the sequences over H by $\phi(S) = \phi(g_1) \cdot \ldots \cdot \phi(g_l)$. If ϕ is a homomorphism, then $\phi(S)$ is a zero-sum sequence if and only if $\sigma(S) \in \ker(\phi)$.

Let $D(G)$ be the Davenport constant of G which is the smallest integer d such that every sequence of d elements from G is not zero-sum free. $D(G)$ can also be defined equivalently as the maximal length of a minimal zero-sum sequence over G.

Let

$$k(S) = \sum_{i=1}^{l} \frac{1}{\mathrm{ord}(g_i)}$$

denote the *cross number* of S. Define

$$K(G) = \max\{k(S) | S \text{ is minimal zero-sum over } G\}$$

where the maximum is taken when S runs over all minimal zero-sum sequences over G.

The following invariant $N_1(G)$ was introduced by Narkiewicz in 1979 [13] which like $D(G)$ and $K(G)$ plays an important role in the study of non-unique factorization problems in algebraic number theory(see [7], [12], [16] and [6]). Let S be a zero-sum sequence over $G \setminus \{0\}$, i.e., S is a zero-sum sequence of non-zero elements from G. Clearly, S can be written in the form $S = S_1 \cdots S_r$ with all S_i are minimal zero-sum subsequences of S, and we call $S = S_1 \cdots S_r$ an *irreducible factorization* of S. We identify two irreducible factorizations $S = S_1 \cdots S_r$ and $S = T_1 \cdots T_m$ if and only if $m = r$, and there is permutation τ on $\{1, \cdots, r\}$ such that $S_i = T_{\tau i}$ holds for every $i \in [1, r]$. We say a zero-sum sequence S over $G \setminus \{0\}$ is *unique factorization* if S has only one irreducible factorization. Narkiewicz constant $N_1(G)$ is the maximal length of a unique factorization sequence over $G \setminus \{0\}$. Unique factorization sequence and therefore $N_1(G)$ can also be formulated in term of the concept of "type" like what Geroldinger and Hater-Koch have done in ([6], Chapter 9).

For $|G| > 1$, define

$$K_1(G) = \max\{k(S) | S \text{ is a unique factorization zero-sum sequence over } G \setminus \{0\}\}$$

where the maximum is taken when S runs over all unique factorization zero-sum sequences over $G \setminus \{0\}$, and let $K_1(G) = 0$ if $|G| = 1$.

The study of cross number has attracted a lot of attention since it was introduced by Krause [8] in 1984. (For example, see [5], [9], [2], [6] and [11]).

Every nontrivial finite abelian group G can be written uniquely in the form $G = \oplus_{i=1}^{r} \oplus_{j=1}^{t_i} C_{{p_i}^{e_{ij}}}$, where $p_1, \cdots, p_r$ are distinct primes. Set

$$K_1^*(G) = \sum_{i=1}^{r} \sum_{j=1}^{t_i} \frac{p_i^{e_{ij}} - 1}{p_i^{e_{ij}} - p_i^{e_{ij}-1}}.$$

and let $K_1^*(G) = 0$ if $|G| = 1$.

It is not difficult to see that

$$K_1(G) \geq K_1^*(G)$$

holds for all finite abelian groups G(See Proposition 2.1 in Section 2). We conjecture that

Conjecture 1.1. *$K_1(G) = K_1^*(G)$ holds for all finite abelian groups G.*

In this paper we shall verify Conjecture 1.1 for some special groups by showing

Theorem 1.2. *Let p be a prime, and let G be a finite abelian group. Then, $K_1(G) = K_1^*(G)$ holds if G is one of the following groups:*

1. *$G = C_{p^m}$ with $m \in \mathbb{N}$.*
2. *$G = C_{pq}$ with q a prime.*
3. *$G = C_2^r$ with $r \in \mathbb{N}$.*
4. *$G = C_3^r$ with $r \in \mathbb{N}$.*
5. *$G = C_p^2$.*

2. An lower bound for $K_1(G)$

Proposition 2.1. *Let G be a finite abelian group. (1) If $G = G_1 \oplus G_2$ for some finite abelian groups G_1 and G_2 then $K_1(G) \geq K_1(G_1) + K_1(G_2)$; (2) $K_1(G) \geq K_1^*(G)$ holds for all finite abelian groups G.*

Proof. If one of G, G_1 and G_2 is trivial then the proposition holds trivially. So, we may assume that none of G, G_1 and G_2 is trivial.

(1). Let $S_1 = a_1 \cdots a_u$ be a unique factorization zero-sum sequence over G_1 with $k(S_1) = K_1(G_1)$, and Let $S_2 = b_1 \cdots b_v$ be a unique factorization zero-sum sequence over G_2 with $k(S_2) = K_1(G_2)$. Let $\mathbf{0}_{G_1}$ denote the identity element of G_1, and let $\mathbf{0}_{G_2}$ denote the identity element of G_2. Let

$$S_1' = (a_1, \mathbf{0}_{G_2})(a_2, \mathbf{0}_{G_2}) \cdots (a_u, \mathbf{0}_{G_2})$$

and let

$$S_2' = (\mathbf{0}_{G_1}, b_1)(\mathbf{0}_{G_1}, b_2) \cdots (\mathbf{0}_{G_1}, b_v).$$

Then S_1' and S_2' are both sequences over $G = G_1 \oplus G_2$ with $|S_1'| = |S_1|, |S_2'| = |S_2|, k(S_1') = k(S_1)$ and $k(S_2') = k(S_2)$. Let $S = S_1'S_2'$. Clearly, S is a unique factorization zero-sum sequence over G. Therefore, $K_1(G) \geq k(S) = k(S_1') + k(S_2') = k(S_1) + k(S_2) = K_1(G_1) + K_1(G_2)$.

(2). By (1), it suffices to prove $K_1(G) \geq K_1^*(G)$ for every cyclic group G of prime power order. Let $G = C_{p^m}$ with p a prime, and let g be a generating element of G. Let

$$S = g^{p-1} \cdot ((1-p)g) \cdot (pg)^{p-1} \cdot ((1-p)pg) \cdots (p^{m-2}g)^{p-1} \cdot ((1-p)p^{m-2}g) \cdot (p^{m-1}g)^p,$$

i.e., S is the sequence with $v_{p^ig}(S) = p-1$ and $v_{(1-p)p^ig}(S) = 1$ for every $i \in [0, m-2]$, and $v_{p^{m-1}g}(S) = p$. Clearly, S is a unique factorization zero-sum sequence. Thus we have $K_1(C_{p^m}) \geq k(S) = 1 + \frac{1}{p} + \cdots + \frac{1}{p^{m-1}} = \frac{p^m-1}{p^m-p^{m-1}} = K_1(G)$.

$\square$

3. Proof of Theorem 1.2

To prove Theorem 1.2 we need some preliminaries begin with a result due to Olson [15].

Let p be a prime, and let G be a finite abelian p-group. For $g \in G$, define $\alpha(g) = p^n$ where n is the largest integer such that $g \in p^nG = \{p^nx | x \in G\}$ ($\alpha(0) = \infty$). Let $S = g_1 \cdot \ldots \cdot g_l$ be a sequence over G. Define

$$\alpha(S) = \sum_{i=1}^{l} \alpha(g_i).$$

Lemma 3.1. *([15]) Let p be a prime, and let $G = C_{p^{e_1}} \oplus \cdots \oplus C_{p^{e_r}}$. Let $S = g_1 \cdots g_k$ be a sequence over G. If $\alpha(S) = \sum_{i=1}^{r} \alpha(g_i) \geq 1 + \sum_{i=1}^{r}(p^{e_i} - 1)$, then S is not zero-sum free.*

Lemma 3.2. *([3]) Let S be a zero-sum sequence of nonzero elements from a finite abelian group G. Then, the following statements are equivalent.*
(1) S is unique factorization.
(2) For any two zero-sum subsequences S_1 and S_2 of S we have that the intersection $S_1 \cap S_2$ is also zero-sum.

Let G be a finite abelian group. It is well known that either $|G| = 1$ or G can be written uniquely in the form $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 | \cdots | n_r$. Narkiewicz [13] conjectured that $N_1(G) = n_1 + \cdots + n_r$ hold all finite abelian groups. This conjecture has been verified only for some very special groups. Here we list some of them we need in the proof of Theorem 1.2.

Lemma 3.3. *([14], [1], [4]) Let p be a prime. Then $N_1(G) = n_1 + \cdots + n_r$ holds if G is one of the following*

1. *$G = C_n$ with $n \in \mathbb{N}$;*
2. *$G = C_2^r$;*
3. *$G = C_3^r$;*
4. *$G = C_p^2$.*

Lemma 3.4. *Let p be a prime, and let r be a positive integer. Then, $N_1(C_p^r) = rp$ if and only if $K_1(C_p^r) = r$.*

Proof. Let $G = C_p^r$. Since every nonzero element of G has order p, the result follows from the definition of $N_1(G)$ and $K_1(G)$. □

Proof of Theorem 1.2. 1. By Proposition 2.1, it suffices to prove the upper bound.

We proceed by induction on m. $m = 1$, let $S = g_1 \cdots g_k$ be a zero-sum sequence over G with $k(S) = \frac{k}{p} > 1$. Since $N_1(C_p) = p$ we know that S is not unique factorization. Therefore we obtain $K_1(C_p) = 1$.

Let now $m \geq 2$. Let S be a unique factorization zero-sum sequence over $G^* = C_{p^m} \setminus \{0\}$. We need to show that $k(S) \leq 1 + \frac{1}{p} + \cdots + \frac{1}{p^{m-1}}$.

Assume to the contrary that $k(S) > 1+\frac{1}{p}+\cdots+\frac{1}{p^{m-1}}$. We shall derive a contradiction. Write S in the form

$$S = g_{11}\cdots g_{1r_1}g_{21}\cdots g_{2r_2}\cdots g_{m1}\cdots g_{mr_m} = \prod_{i=1}^{m}\prod_{j=1}^{r_i} g_{ij}$$

with $g_{ij}\in C_{p^m}$ and $\operatorname{ord}(g_{ij}) = p^i$. Then

$$k(S) = \sum_{i=1}^{m}\sum_{j=1}^{r_i}\frac{1}{\operatorname{ord}(g_{ij})} = \frac{r_1}{p}+\cdots+\frac{r_m}{p^m}.$$

Therefore, $\frac{r_1}{p}+\cdots+\frac{r_m}{p^m} > 1+\frac{1}{p}+\cdots+\frac{1}{p^{m-1}}$. Multiple the two sides of the above inequality with p we obtain

$$r_1+\frac{r_2}{p}+\cdots+\frac{r_m}{p^{m-1}} > p+1+\frac{1}{p}+\cdots+\frac{1}{p^{m-2}}.$$

Let ϕ be the canonical epimorphism from C_{p^m} to C_{p^m}/C_p. Let $T = g_{11}\cdots g_{1r_1}$ and let $S' = ST^{-1}$. Then $\phi(S') = \phi(ST^{-1}) = \prod_{i=2}^{m}\prod_{j=1}^{r_i}\phi(g_{ij})$ and

$$k(\phi(S')) = \frac{r_2}{p}+\cdots+\frac{r_m}{p^{m-1}} > p-r_1+1+\frac{1}{p}+\cdots+\frac{1}{p^{m-2}}.$$

By multiple the two sides of the above inequality with p^{m-1} we obtain that $r_2p^{m-2}+r_3p^{m-3}+\cdots+r_m \geq p^{m-1}(p-r_1+1)+p^{m-2}+\cdots+p+1$. Therefore,

$$\alpha(\phi(S')) = r_2p^{m-2}+r_3p^{m-3}+\cdots+r_m \geq p^{m-1}(p-r_1+1)+p^{m-2}+\cdots+p+1.$$

Let $t\geq 0$ be maximal such that there are disjoint subsequences $S_1,\ldots,S_t$ of S' with $\sigma(S_i)\in\ker\phi\setminus\{0\}$. By the maximality of t we infer that $\phi(S_i)$ is minimal zero-sum for each $i\in[1,t]$. It follows from Lemma 3.1 that

$$\alpha(\phi(S_i)) \leq p^{m-1}$$

for each $i\in[1,t]$. We assert that

$$t+r_1 \geq p+1.$$

Assume to the contrary that $t+r_1\leq p$. Then, $\alpha(\phi(S'(S_1\cdots S_t)^{-1})) = \alpha(\phi(S')) - \sum_{i=1}^{t}\alpha(\phi(S_i)) \geq p^{m-1}(p-r_1+1)+p^{m-2}+\cdots+p+1-(p-r_1)p^{m-1} \geq p^{m-1}+p^{m-2}+\cdots+p+1$. Let $S'' = S'(S_1\cdots S_t)^{-1}$. We just proved that $\alpha(\phi(S''))\geq p^{m-1}+p^{m-2}+\cdots+p+1$. Let r''_j be the number of elements x(counted with multiple) of $\phi(S'')$ with $\operatorname{ord}(x) = p^j$ for every $j\in[1,m-1]$. It follows that $r''_1p^{m-2}+\cdots+r''_{m-2}p+r''_{m-1} = \alpha(\phi(S''))\geq p^{m-1}+p^{m-2}+\cdots+p+1$. Therefore,

$$K(\phi(S'')) = \frac{r''_1}{p}+\cdots+\frac{r''_{m-2}}{p^{m-2}}+\frac{r''_{m-1}}{p^{m-1}} \geq 1+\frac{1}{p}+\cdots+\frac{1}{p^{m-2}}+\frac{1}{p^{m-1}}.$$

By the induction hypothesis, we have $K_1(\phi(C_{p^m})) = K_1(C_{p^{m-1}}) = 1+\frac{1}{p}+\cdots+\frac{1}{p^{m-2}}$. Therefore, $\phi(S'')$ is not unique factorization. By Lemma 3.2 there exist two subsequences T_1,T_2 of S'' such that both $\phi(T_1)$ and $\phi(T_2)$ are minimal zero-sum sequences but $\phi(T_1\cap T_2)$

is not zero-sum over $\phi(G) = C_{p^{m-1}}$. Hence, $T_1 \cap T_2$ is not zero-sum over C_{p^m}. Since S is unique factorization, again by Lemma 3.2 we obtain that either $\sigma(T_1) \in \ker\phi \setminus \{0\}$, or $\sigma(T_2) \in \ker\phi \setminus \{0\}$, a contradiction of the maximality of t. This proves that $t + r_1 \geq p+1$.

Since $\sigma(\phi(S(TS_1 \cdots S_t)^{-1})) = 0$, $S(TS_1 \cdots S_t)^{-1} = R_1 \cdots R_\ell$ with $\phi(R_i)$ is minimal zero-sum for each $i \in [1, \ell]$. By the maximality of t, $\sigma(R_i) = 0$ for each $i \in [1, \ell]$. It follows that both $S(TS_1 \cdots S_t)^{-1}$ and $TS_1 \cdots S_t$ are a zero-sum sequences. Now $T\sigma(S_1) \cdots \sigma(S_t)$ is a zero-sum sequence over $C_p \setminus \{0\}$ and $|T\sigma(S_1) \cdots \sigma(S_t)| = r_1 + t \geq p+1$. By $N_1(C_p) = p$ we obtain that $T\sigma(S_1) \cdots \sigma(S_t)$ is not unique factorization, and so is S, a contradiction. □

2. From 1 we may assume that $p \neq q$. It suffices to prove the upper bound. Let S be a unique factorization zero-sum sequence over $C_{pq} \setminus \{0\}$. We need to show that $k(S) \leq 2$. Assume to the contrary that,

$$k(S) > 2.$$

Write S in the form

$$S = g_{11} \cdots g_{1m} g_{21} \cdots g_{2n} g_{31} \cdots g_{3k}$$

with

$$\operatorname{ord}(g_{ij}) = \begin{cases} p & \text{if } \ i = 1 \\ q & \text{if } \ i = 2 \\ pq & \text{if } \ i = 3. \end{cases}$$

Then

$$k(S) = \frac{m}{p} + \frac{n}{q} + \frac{k}{pq} > 2.$$

Therefore,

$$mq + np + k \geq 2pq + 1. \tag{3.1}$$

Let $T = g_{11} \cdots g_{1m}$, and let ϕ be the canonical epimorphism from C_{pq} to C_{pq}/C_p. Then

$$\phi(ST^{-1}) = \phi(g_{21}) \cdots \phi(g_{2n}) \phi(g_{31}) \cdots \phi(g_{3k})$$

and $k(\phi(ST^{-1})) = \frac{n+k}{q}$. Since $\sigma(S) = 0$ we have $\sigma(\phi(ST^{-1})) = 0$.

Let $t \geq 0$ be maximal such that there are disjoint subsequences $S_1, \ldots, S_t$ of ST^{-1} with $\sigma(S_i) \in \ker\phi \setminus \{0\}$. By the maximality of t we infer that $\phi(S_i)$ is minimal zero-sum over $\phi(C_{pq}) \cong C_q$. It follows from $D(C_q) = q$ that

$$|S_i| = |\phi(S_i)| \leq q$$

for each $i \in [1, t]$. In a similar way to the proof of 1 we get that $T\sigma(S_1) \cdots \sigma(S_t)$ is a zero-sum sequence over $C_p \setminus \{0\}$. If $m + t \geq p + 1 > p = N_1(C_p)$ then $T\sigma(S_1) \cdots \sigma(S_t)$ is not unique factorization, and so is S, a contradiction. Therefore,

$$m + t \leq p.$$

If $n \geq q + 1$ then switch p and q and repeat the procedure above we can derive a contradiction. Therefore,

$$n \leq q.$$

By equation (3.1) we have that $np+k-(p-m)q \geq pq+1$. This together with $n \leq q$ gives that $k-(p-m)q > 0$. Therefore, $np+(k-(p-m)q)p > np+k-(p-m)q \geq pq+1$. Hence,

$$n+k-(p-m)q \geq q+1.$$

Now $|S(TS_1\cdots S_t)^{-1}| \geq |S|-m-tq = n+k-tq \geq n+k-(p-m)q \geq q+1 > q = N_1(C_q)$. So, $\phi(S(TS_1\cdots S_t)^{-1})$ is not unique factorization. Now in a similar way to the proof of 1 we can derive a contradiction.

3-5. The result follows from Lemma 3.3 and Lemma 3.4. □

4. Concluding Remarks

For general case we have the following

Proposition 4.1. *Let G be a nontrivial finite abelian group, and p be the smallest prime divisor of $|G|$. Then $K_1(G) < \ln|G| + \frac{1}{p}\log_2|G|$.*

Proof. Let S be a unique factorization sequence over $G \setminus \{0\}$. Let $S = S_1\cdots S_t$ be an irreducible factorization of S, where $t \in \mathbb{N}$, and all $S_1,\ldots,S_t$ are minimal zero-sum subsequences of S. Then we have $|S_i| \geq 2$ for every $i \in [1,t]$. By a result due to Narkiewicz (see [14], Proposition 6; or [1], Lemma 2), $\Pi_{i=1}^{t}|S_i| \leq |G|$. Therefore,

$$t \leq \log_2|G|.$$

For every $i \in [1,t]$ we choose an element $g_i \in \text{supp}(S_i)$. It follows from S is unique factorization that the sequence $T = g_1^{-1}S_1\cdots g_t^{-1}S_t$ is zero-sum free. Now by a result due to Geroldinger and Schneider [9], $k(T) \leq \ln|G|$. Therefore,

$$k(S) = k(T) + \sum_{i=1}^{t}\frac{1}{\text{ord}(g_i)} \leq \ln|G| + t\frac{1}{p} \leq \ln|G| + \frac{\log_2|G|}{p}.$$

□

Let G be a finite abelian group. It is easy to see that

$$K(G) \leq K_1(G)$$

holds for all nontrivial finite abelian group G. Unlike the Davenport constant $D(G)$, we even don't known the exact value of $K(G)$ for most of cyclic groups. Also, very little is known about the Narkiewicz constant $N_1(G)$. So, we can't go too far in the determining of $K_1(G)$ since it is essentially involving the determining of $K(G)$ and $N_1(G)$.

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