VERTEX-TRANSITIVE CUBIC GRAPHS OF SQUARE-FREE ORDER

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Abstract. A classification is given of connected vertex-transitive cubic graphs of square-free order. It is shown that such graphs are well-characterised metacirculants (including dihedrants, generalized Petersen graphs, Möbius bands), or the Tutte's 8-cage, or graphs arisen from simple groups $\mathrm{PSL}(2,p)$.

1. Introduction

For a graph $\Gamma = (V,E)$, the number of vertices $|V|$ is called the *order* of Γ. A graph Γ is called *vertex-transitive* if its automorphism group $\mathsf{Aut}\Gamma$ is transitive on V.

In 1967, Turner [22] investigated vertex-transitive graphs of prime order, and enumerated the isomorphism classes of such graphs by using Pó1ya enumeration theorem. Since then, the class of vertex-transitive graphs of square-free order have been studied extensively and numerous interesting results have appeared on classification, isomorphism problem, non-Cayley numbers, etc.. Classification results about vertex-transitive graphs of square-free order usually focus on specific subclasses regarding their symmetry properties, orders, valencies, etc. For instance, see [18, 20] for those graphs of order being a product of two prime, see [4, 5, 9, 10, 15, 17, 19, 24] for those graphs having certain symmetry properties. In a recent paper [23], a classification was given of vertex-transitive cubic graphs of order $2pq$, where p and q are primes.

In this paper, we classify vertex-transitive cubic graphs of square-free order.

A graph is called a *metacirculant* if it has a vertex-transitive metacyclic group of automorphisms. Examples of vertex-transitive cubic graphs of square-free order include a lot of interesting graphs: $\mathsf{K}_{3,3}$, Petersen graph, Tutte's 8-cage (30 vertices), generalized Petersen graphs, Möbius bands, some well-characterised metacirculants, and some graphs arisen from simple groups $\mathrm{PSL}(2,p)$. See Section 2 for definitions and constructions. Among these graphs, some are Cayley graphs. For a group G and a subset $S \subset G$ with $1 \notin S = S^{-1} := \{g^{-1} \mid g \in S\}$, the *Cayley graph* $\mathrm{Cay}(G,S)$ is defined on G such that $\{g,h\}$ is an edge if and only if $gh^{-1} \in S$.

Throughout this paper, for two groups A and B, denote by $A\times B$, $A.B$ and $A{:}B$ the direct product, an extension and a semi-direct product of A by B, respectively; denote respectively by A' and $\mathbf{Z}(A)$ the commutator subgroup and the center of A; for $a \in A$, denote by $o(a)$ the order of a in A; for an positive integer n, denote by $\mathbb{Z}_n$ and D_{2n} the cyclic group of order n and the dihedral group of order $2n$, respectively.

Our classification is stated in the following theorem.

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Theorem 1.1. *Let Γ be a connected vertex-transitive cubic graph of square-free order $2n$. Then one of the following statements holds.*

(1) *Γ is a metacirculant, and one of the following is true:*

(i) *Γ is isomorphic to a generalized Petersen graph $\mathbf{P}(n,r)$ for $1 \le r < \frac{n}{2}$ with $r^2 \equiv 1 \pmod{n}$; $\mathsf{Aut}\Gamma \cong \mathbb{Z}_n{:}\mathbb{Z}_2^2$ has a regular subgroup $\langle a, b \mid a^n = b^2 = 1, bab = a^r\rangle$, and has no regular subgroups isomorphic to $\mathbb{Z}_{2n}$ or D_{2n} unless $r = 1$;*

(ii) *Γ is the Möbius band $\mathbf{M_n}$ of order $2n$; either $\mathsf{Aut}\Gamma \cong \mathbb{Z}_{2n}{:}\mathbb{Z}_2 \cong \mathrm{D}_{4n}$ or $\Gamma \cong \mathsf{K}_{3,3}$;*

(iii) *$\Gamma \cong \mathrm{Cay}(\langle a, b\rangle, S)$ for $S = \{ab, a^k b, b\}$ or $\{ab, a^{1-k}b, b\}$, $\langle a, b\rangle \cong \mathrm{D}_{2n}$, $o(a) = n > 3$ and $o(b) = 2$, where $k \not\equiv -1 \pmod{n}$ and $k^2 \equiv 1 \pmod{n}$; in this case, $\mathsf{Aut}\Gamma \cong \mathrm{D}_{2n}{:}\mathbb{Z}_2$ contains no cyclic regular subgroups;*

(iv) *$\Gamma \cong \mathrm{Cay}(\langle a, b\rangle, \{ab, a^k b, b\})$ for $\langle a, b\rangle \cong \mathrm{D}_{2n}$, $o(a) = n > 3$ and $o(b) = 2$, where $k^2 - k + 1 \equiv 0 \pmod{n}$; in this case, $\mathsf{Aut}\Gamma \cong \mathrm{D}_{2n}{:}\mathbb{Z}_3$ except for Line 1 of Table 1;*

(v) *$\Gamma \cong \mathrm{Cay}(\langle a, b\rangle, \{a^{k'} b, a^k b, b\})$ for $\langle a, b\rangle \cong \mathrm{D}_{2n}$, $o(a) = n > 3$ and $o(b) = 2$, where $(k, k') = 1$, either $(k, n) \ne 1$ and $(k', n) \ne 1$, or $k' \equiv 1 \pmod{n}$, $k^2 \not\equiv 1 \pmod{n}$, $(k-1)^2 \not\equiv 1 \pmod{n}$, $2k \not\equiv 1 \pmod{n}$ and $k^2 - k + 1 \not\equiv 0 \pmod{n}$; in this case, $\mathsf{Aut}\Gamma \cong \langle a, b\rangle$;*

(vi) *$\Gamma \cong \mathrm{Cay}(\langle a, b, c\rangle, \{cab^k, (cab^k)^{-1}, b^l\})$, $\mathbf{Z}(\langle a, b, c\rangle) = \langle c\rangle$, $(\langle a, b, c\rangle)' = \langle a\rangle$, $2 < o(a) < n$, $2 < o(b) = 2l$ and $a^{b^l} = a^{-1}$, where $0 < k < l$ and $(k, l) = 1$; in this case, $\mathsf{Aut}\Gamma \cong \langle a, b, c\rangle$ except for Lines 2-5 of Table 1;*

(vii) *$\Gamma \cong \mathbf{P}(n, r)$ with $1 < r < \frac{n}{2}$ and $r^2 \equiv -1 \pmod{n}$; either $\mathsf{Aut}\Gamma \cong \mathbb{Z}_n{:}\mathbb{Z}_4$, or $\mathsf{Aut}\Gamma = \mathrm{S}_5$ and Γ is isomorphic to the Petersen graph;*

(2) *Γ is isomorphic to the Tutte's 8-cage, $n = 15$ and $\mathsf{Aut}\Gamma = \mathrm{P\Gamma L}(2, 9)$;*

(3) *$\mathsf{Aut}\Gamma = \mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$ for a prime $p \ge 5$, and Γ is isomorphic to one of the graphs constructed in Examples 3.5-3.8;*

(4) *$\mathsf{Aut}\Gamma = \mathrm{PSL}(2, p){:}\mathrm{D}_{2m}$ for a prime $p \ge 5$ and $1 < m = \frac{8n}{p(p^2-1)}$, and Γ is isomorphic to one of the graphs constructed in Construction 4.2.*

Line	Regular subgroup	k	$\mathsf{Aut}\Gamma$	Γ ($\cong$)
1	$\langle a, b\rangle \cong \mathrm{D}_{14}$	3 or 5	$\mathrm{PGL}(2, 7)$	Example 3.6 (2)
2	$\langle a, b\rangle \cong \mathbb{Z}_7{:}\mathbb{Z}_6$	2	$\mathrm{PGL}(2, 7)$	Example 3.8 (2)
3	$\langle a, b\rangle \cong \mathbb{Z}_{\frac{n}{3}}{:}\mathbb{Z}_6, a^b = a^t$ $t^2 - t + 1 \equiv 0 \pmod{n}$	1	$\mathrm{D}_{2n}{:}\mathbb{Z}_3$	Lemma 2.3 (3)
4	$\langle a, b\rangle \cong \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$	$a^{b^k} = a^7$ or a^8	$\mathrm{PGL}(2, 11)$	Example 3.6 (1)
5	$\langle a, b\rangle \cong \mathbb{Z}_{23}{:}\mathbb{Z}_{22}$	$a^{b^k} = a^{17}$ or a^{19}	$\mathrm{PGL}(2, 23)$	Example 3.6 (2)

TABLE 1.

We remark that a characterisation of general cubic metacirculants was given in [16], in which two families of such graphs are proved to be covers of some special graphs but the covers are not yet determined. Part (1) of Theorem 1.1 gives an explicit classification of cubic metacirculants of square-free order.

2. Cubic metacirculants

Let $n \geq 3$ and $1 \leq r < \frac{n}{2}$ be two integers. The *generalized Petersen graph* $\mathbf{P}(n,r)$ is the graph with vertex set and edge set as follows

$$\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} \cup \{\beta_0, \beta_1, \ldots, \beta_{n-1}\},$$
$$\{\{\alpha_i, \alpha_{i+1}\}, \{\alpha_i, \beta_i\}, \{\beta_i, \beta_{i+r}\} \mid 0 \leq i \leq n-1\},$$

reading $i+1$ and $i+r$ modulo n. It was shown in [11] that $\mathbf{P}(n,r)$ is vertex-transitive if and only if either $(n,r) = (10,2)$ or $r^2 \equiv \pm 1 \,(\mathsf{mod}\ n)$. Further, $\mathsf{Aut}\mathbf{P}(n,r)$ has a transitive subgroup isomorphic to $\mathbb{Z}_n{:}\mathbb{Z}_4$ if $r^2 \equiv -1 \,(\mathsf{mod}\ n)$, and has a regular subgroup isomorphic to $\mathbb{Z}_n{:}\mathbb{Z}_2$ if $r^2 \equiv 1 \,(\mathsf{mod}\ n)$. In particular, $\mathsf{Aut}\mathbf{P}(n,1)$ contains two regular subgroups isomorphic to $\mathbb{Z}_{2n}$ and D_{2n}, respectively.

The *Möbius band* $\mathbf{M}_n$ of order $2n$ is the graph with vertex set $\{\alpha_0, \alpha_1, \ldots, \alpha_{2n-1}\}$, and edge set $\{\{\alpha_i, \alpha_{i+1}\}, \{\alpha_i, \alpha_{i+n}\} \mid 0 \leq i \leq 2n-1\}$, reading the subscripts modulo $2n$. For the graph $\mathbf{M}_n$, its automorphism group contains two regular subgroups isomorphic to $\mathbb{Z}_{2n}$ and D_{2n}, respectively.

A graph $\Gamma = (V, E)$ is called a *circulant* or *dihedrant* if $\mathsf{Aut}\Gamma$ contains respectively a cyclic or dihedral subgroup which is regular on the vertex set V.

Let $\Gamma = (V, E)$ be a graph such that $\mathsf{Aut}\Gamma$ has a regular subgroup G. Take $\alpha \in V$. Then each vertex of Γ is uniquely written as α^g for some $g \in G$. Let $\Gamma(\alpha)$ be the set of neighbors of α in Γ. Set $S = \{s \in G \mid \alpha^s \in \Gamma(\alpha)\}$. Then $1 \notin S = S^{-1} := \{s^{-1} \mid s \in S\}$ and $\Gamma \cong \mathrm{Cay}(G, S)$. It is well-known that a Cayley graph $\mathrm{Cay}(G, S)$ is connected whenever S generates the underlying group G, that is, $\langle S \rangle = G$. Moreover, each automorphism $\sigma \in \mathsf{Aut}(G)$ of the group G induces naturally an isomorphism from $\mathrm{Cay}(G, S)$ to $\mathrm{Cay}(G, S^\sigma)$. Set

$$\mathsf{Aut}(G, S) = \{\sigma \in \mathsf{Aut}(G) \mid S^\sigma = S\}.$$

For $g \in G$, by $\bar{g}$ we denote the permutation induced by g on G by right multiplication. Set $\bar{G} = \{\bar{g} \mid g \in G\}$. Then $G \to \bar{G}$, $g \mapsto \bar{g}$ is an isomorphism of groups. By [12, Lemma 2.1], the normalizer $\mathbf{N}_{\mathsf{AutCay}(G,S)}(\bar{G}) = \bar{G}{:}\mathsf{Aut}(G, S)$.

To end this section, let G be a group of square-free order $2n$. Then n is odd.

Lemma 2.1. *For a group G of square-free order $2n$, one of the following holds.*

(1) *$G \cong \mathbb{Z}_{2n}$ or D_{2n};*
(2) *$G' \cong \mathbb{Z}_m$ and $G \cong \mathbb{Z}_m{:}\mathbb{Z}_{\frac{2n}{m}}$ for odd m with $n > m > 2$.*

Proof. Since G has square-free order, G' is cyclic and $G = G'{:}H$, where H is a cyclic Hall subgroup of G. Set $G' = \langle a \rangle$ and $H = \langle b \rangle$. If $G' = 1$, then $G = H \cong \mathbb{Z}_{2n}$.

Let $G' = \langle a \rangle \cong \mathbb{Z}_m$ for $m > 1$. If m is even, then $a^{\frac{m}{2}}$ lies in the center of G, so $G/\langle a^2 \rangle \cong \langle a^{\frac{m}{2}}, b \rangle$ is abelian, hence $G' = \langle a \rangle \leq \langle a^2 \rangle$, which is impossible. Thus m is odd, and so H is of even order $\frac{2n}{m}$. If $n > m$, then part (2) occurs. Assume that $m = n$. Let $C = \mathbf{C}_{\langle a \rangle}(b)$. Then there is a subgroup D of $\langle a \rangle$ with $\langle a \rangle = C \times D$. It is easily shown that D is normal in G. Then $G/D \cong C \times \langle b \rangle$ is abelian, so $G' \leq D$, hence $D = \langle a \rangle$ and $C = 1$. It follows that $a^b = a^{-1}$, hence $G \cong \mathrm{D}_{2n}$. □

Let $\Gamma \cong \mathrm{Cay}(G, S)$, where S be a generating set of G with $|S| = 3$ and $1 \notin S = S^{-1}$. Then S either contains only one involution, or consists of involutions. Since Γ is connected, $\langle S\rangle = G$, we know that $\mathsf{Aut}(G,S)$ is faithful on S. It follows that $\mathsf{Aut}(G,S)$ is isomorphic to a subgroup of the symmetric group S_3 of degree 3.

Let G be abelian. Then G is cyclic, $S=\{x,x^{-1},z\}$ and $\mathsf{Aut}(G,S)\cong\mathbb{Z}_2$, where z is the unique involution in G. Since $\langle S\rangle=G$, either $G=\langle x\rangle$ or $G=\langle x\rangle\times\langle z\rangle$. If $G=\langle x\rangle\times\langle z\rangle$, then $\Gamma\cong\mathbf{P}(n,1)$. Let $G=\langle x\rangle$. Then $z=x^n$. Set $\alpha_i=x^i$. Then α_i and α_j are adjacent whenever $j-i\equiv\pm1\,(\mathsf{mod}\ 2n)$ or $j-i\equiv n\,(\mathsf{mod}\ 2n)$. Thus $\Gamma\cong\mathbf{M}_n$, and the next result follows.

Lemma 2.2. *A connected cubic circulant of order $2n$ is either the ladder graph $\mathbf{P}(n,1)$ or the Möbius band $\mathbf{M}_n$.*

Thus we assume next that G is not abelian. Since G has square-free order, a Sylow 2-subgroup of G has order 2, it follows that all involutions in G are conjugate. The next lemma give a characterisation of connected cubic dihedrants.

Lemma 2.3. *Let G the dihedral group of order $2n$, and let Γ be a connected cubic Cayley graph of G. Set $G=\langle a,b\rangle$ with $o(a)=n$, $o(b)=2$ and $a^b=a^{-1}$. Then $\Gamma\cong\mathrm{Cay}(G,S)$ for one of the following subset S of G.*

(1) *$S=\{a,a^{-1},b\}$; in this case, $\mathsf{Aut}(G,S)\cong\mathbb{Z}_2$ and $\Gamma\cong\mathbf{P}(n,1)$;*
(2) *$n=3$ and $S=\{ab,a^2b,b\}$; in this case, $\Gamma\cong\mathsf{K}_{3,3}$;*
(3) *$S=\{ab,a^kb,b\}$, $k^2-k+1\equiv0\,(\mathsf{mod}\ n)$, $n>3$; in this case, $\mathsf{Aut}(G,S)\cong\mathbb{Z}_3$;*
(4) *$S=\{ab,a^eb,b\}$ or $\{ab,a^{1-e}b,b\}$ for $n>3$ and $e^2\equiv1\,(\mathsf{mod}\ n)$; in this case, $\mathsf{Aut}(G,S)\cong\mathbb{Z}_2$;*
(5) *$S=\{ab,a^kb,b\}$, $n>3$, $k^2\not\equiv1\,(\mathsf{mod}\ n)$, $(k-1)^2\not\equiv1\,(\mathsf{mod}\ n)$, $2k\not\equiv1\,(\mathsf{mod}\ n)$ and $k^2-k+1\not\equiv0\,(\mathsf{mod}\ n)$; in this case, $\mathsf{Aut}(G,S)=1$;*
(6) *$S=\{a^{k'}b,a^kb,b\}$, $n>3$, $(k,k')=1$, $(k,n)\neq1$ and $(k',n)\neq1$; in this case, $\mathsf{Aut}(G,S)=1$.*

Proof. Let $\Gamma=\mathrm{Cay}(G,S)$. Recall that all involutions in G are conjugate. Up to isomorphism of graphs we may choose $b\in S$. If S has only one involution, then $S=\{a^s,a^{-s},b\}$, where $(s,n)=1$. It is easily shown that $\mathsf{Aut}(G,S)\cong\mathbb{Z}_2$. Take $\sigma\in\mathsf{Aut}(G)$ with $(a^s)^\sigma=a$ and $b^\sigma=b$, refer to [14]. Then $\Gamma\cong\mathrm{Cay}(G,S^\sigma)$ and $S^\sigma=\{a,a^{-1},b\}$. Set $\alpha_i=a^i$ and $\beta_i=ba^i$ for $0\le i\le n-1$. Then $\mathrm{Cay}(G,S^\sigma)$ has edges $\{\alpha_i,\alpha_{i+1}\}$, $\{\beta_i,\beta_{i+1}\}$ and $\{\alpha_i,\beta_i\}$. Thus $\Gamma\cong\mathbf{P}(n,1)$.

Assume that $S=\{x,y,b\}$ consists of 3 involutions. Then $S=\{a^ib,a^jb,b\}$ for some positive integers i and j. Let $d=(i,j)$, $i=kd$ and $j=k'd$. Then $G=\langle S\rangle=\langle a^i,a^j,b\rangle=\langle a^i,a^j\rangle{:}\langle b\rangle=\langle a^d\rangle{:}\langle b\rangle$, so $\langle a^d\rangle=\langle a\rangle$, hence $(d,n)=1$. Thus $sd\equiv1\,(\mathsf{mod}\ n)$ for some s coprime to n. Take an automorphism $\sigma\in\mathsf{Aut}(G)$ with $a^\sigma=a^s$ and $b^\sigma=b$, refer to [14]. Then $S^\sigma=\{a^kb,a^{k'}b,b\}$ and $\Gamma\cong\mathrm{Cay}(G,S^\sigma)$.

Suppose that $\mathsf{Aut}(G,S^\sigma)$ has an element τ of order 3. Let $a^\tau=a^t$ for some t coprime to n. Then $t^3\equiv1\,(\mathsf{mod}\ n)$. Noting that $\tau^{-1}\in\mathsf{Aut}(G,S^\sigma)$, without loss of generality, we may set $b^\tau=a^{k'}b$. Since $S^{\sigma\tau}=S^\sigma$, computation shows that $S^\sigma=\{b,a^{k'}b,a^{k'(t+1)}b\}$, $k'(t+1)\equiv k\,(\mathsf{mod}\ n)$, $k'(t^2+t+1)\equiv0\,(\mathsf{mod}\ n)$. By the argument in above paragraph, we know that $(k',n)=1$. Thus we have

(*i*) $S^\sigma=\{b,a^{k'}b,a^{k'(t+1)}b\}$, $(k',n)=1$, $(k,n)=1$, $k'(t+1)\equiv k\,(\mathsf{mod}\ n)$, $(t^2+t+1)\equiv0\,(\mathsf{mod}\ n)$.

Suppose that $\mathsf{Aut}(G, S^\sigma)$ has an involution ε. Let $a^\varepsilon = a^e$ for some e coprime to n. Then $e^2 \equiv 1 \,(\mathsf{mod}\ n)$. Note that ε fixes one involution in S^σ and interchanges the other two. Then one of the following occurs:

(ii) $S^\sigma = \{a^{k'}b, a^{k'e}b, b\}$, $(k', n) = 1$, $(k, n) = 1$ and $k \equiv k'e \,(\mathsf{mod}\ n)$;
(iii) $S^\sigma = \{a^{k'}b, a^{k'(1-e)}b, b\}$, $(k', n) = 1$, $k' - k'e \equiv k \,(\mathsf{mod}\ n)$;
$(iii)'$ $S^\sigma = \{a^{(1-e)k}b, a^k b, b\}$, $(k, n) = 1$, $k \equiv k' + ke \,(\mathsf{mod}\ n)$.

Conversely, it is easily shown that $\mathsf{Aut}(G, S^\sigma) \neq 1$ if S^σ is described as in one of the above items (i)-$(iii)'$. It is easily shown that $\mathsf{Aut}(S^\sigma) \cong \mathrm{S}_3$ if and only if $n = 3$.

By the above argument, $\mathsf{Aut}(G, S^\sigma) = 1$ if neither $(k, n) = 1$ nor $(k', n) \neq 1$, and then part (6) follows. Thus, without loss of generality, we assume next that $(k', n) = 1$. Then, by [14], there is $\delta \in \mathsf{Aut}(G)$ with $(a^{k'})^\delta = a$ and $b^\delta = b$. Since $\mathrm{Cay}(G, S^\sigma) \cong \mathrm{Cay}(G, S^{\sigma\delta})$, replacing S^σ by $S^{\sigma\delta}$, we may assume that $S^\sigma = \{ab, a^k b, b\}$, that is, take $k' = 1$. If $n = 3$, then the part (2) of the lemma follows. Let $n > 3$. If item (i) holds, then part (3) follows. If item (ii) or (iii) holds, then part (4) follows. Assume that $(iii)'$ holds then $1 = k' \equiv k(1-e) \,(\mathsf{mod}\ n)$, so $(1-e, n) = 1$. Hence $e \equiv -1 \,(\mathsf{mod}\ n)$ as $e^2 \equiv 1 \,(\mathsf{mod}\ n)$. Thus $2k \equiv 1 \,(\mathsf{mod}\ n)$. Noting that $(k, n) = 1$, we may take an automorphism of G with $a^k \mapsto a$ and $b \mapsto b$. Then $\Gamma \cong \mathrm{Cay}(G, \{ab, a^k b, b\}) \cong \mathrm{Cay}(G, \{a^2 b, ab, b\})$, which is a graph given in part (4). For $S^\sigma = \{ab, a^k b, b\}$, by the above argument, $\mathsf{Aut}(G, S^\sigma) = 1$ if and only if $n > 3$, $k^2 \not\equiv 1 \,(\mathsf{mod}\ n)$, $(k-1)^2 \not\equiv 1 \,(\mathsf{mod}\ n)$, $2k \not\equiv 1 \,(\mathsf{mod}\ n)$ and $k^2 - k + 1 \not\equiv 0 \,(\mathsf{mod}\ n)$. Then part (5) follows. □

Corollary 2.4. *Let $n > 3$ and $G = \langle a\rangle{:}\langle b\rangle \cong \mathrm{D}_{2n}$ be of square-free order, and let $S = \{ab, a^e b, b\}$ or $\{ab, a^{1-e}b, b\}$ be as in Lemma 2.3 (4). Then $\bar{G}{:}\mathsf{Aut}(G, S)$ has a cyclic regular subgroup if and only if $e \equiv -1 \,(\mathsf{mod}\ n)$.*

Proof. Let $\Gamma = \mathrm{Cay}(G, S)$. Then $\mathsf{Aut}(G, S) = \langle\sigma\rangle \cong \mathbb{Z}_2$, where $\sigma \in \mathsf{Aut}(G)$ with $a^\sigma = a^e$ and either $b^\sigma = b$ for $S = \{ab, a^e b, b\}$ or $b^\sigma = a^{1-e}b$ for $S = \{ab, a^{1-e}b, b\}$. Let $g \in S$ with $g^\sigma = g$. It is easily shown that each regular subgroup of $\bar{G}{:}\mathsf{Aut}(G, S)$ can be written as $R := \langle \bar{a}, \sigma^j \bar{g}\rangle$ for $j = 0$ or 1. Clearly, R is cyclic if and only if $j = 1$ and $\bar{a}^{-e} = \bar{a}^{\sigma\bar{g}} = (\sigma\bar{g})^{-1}\bar{a}\sigma\bar{g} = \bar{a}$, that is, $e \equiv -1 \,(\mathsf{mod}\ n)$. □

Now assume that G satisfies Lemma 2.1 (2). Then G can not be generated by three involutions. Thus, for a connected cubic graph $\mathrm{Cay}(G, S)$, the subset S contains only one involution of G. Since G is not abelian, this involution is not contained in the center of G. Let $H < G$ with $G = G'{:}H$, and let $C = C_H(G')$. Then C is the center of G and of odd order, and $G = C \times (G'{:}\langle b\rangle)$ for a cyclic subgroup $\langle b\rangle$ of H of even order. Set $C = \langle c\rangle$ and $G' = \langle a\rangle$. Then $o(c)o(b) > 2$, and so $2 < o(a) < n$.

Lemma 2.5. *Let $G = \langle c\rangle \times (\langle a\rangle{:}\langle b\rangle)$ be a group of square-free order $2n$, where $\mathbf{Z}(G) = \langle c\rangle$ and $G' = \langle a\rangle \cong \mathbb{Z}_m$ with $2 < m < n$. Let Γ be a connected cubic Cayley graph of G. Then $o(b) = 2l$, $a^{b^l} = a^{-1}$ and $\Gamma \cong \mathrm{Cay}(G, S_k)$ for $S_k = \{cab^k, (cab^k)^{-1}, b^l\}$, where $l \geq 1$, $0 \leq k \leq l$ and $(k, l) = 1$. Moreover, $\mathsf{Aut}(G, S_k) \neq 1$ if and only if $l = 1$; in this case, either Γ is a dihedrant, or $\Gamma \cong \mathbf{P}(n, r)$ with $1 < r < \frac{n}{2}$ and $r^2 \equiv 1 \,(\mathsf{mod}\ n)$.*

Proof. Let $\Gamma \cong \mathrm{Cay}(G, S)$. By the above argument, $o(b)$ is even. Set $o(b) = 2l$. Recall that all involutions in G are conjugate. Up to isomorphism of graphs, we may choose $b^l \in S$ and set $S = \{xyz, (xyz)^{-1}, b^l\}$, where $x \in \langle c\rangle$, $y \in \langle a\rangle$ and $z \in \langle b\rangle$. Since $\langle S\rangle = G$, we have $\langle x\rangle = \langle c\rangle$, $\langle y\rangle = \langle a\rangle$ and $\langle z, b^l\rangle = \langle b\rangle$. Take $\sigma \in \mathsf{Aut}(G)$

with $x^\sigma = c$, $y^\sigma = a$ and $b^\sigma = b$, refer to [14]. Then $S_k := S^\sigma = \{cab^k, (cab^k)^{-1}, b^l\}$ for some $0 \le k < 2l$ coprime to l, and so $\Gamma \cong \mathrm{Cay}(G, S_k)$. Setting $a^b = a^r$, by [14], we may take $\rho \in \mathsf{Aut}(G)$ with $c^\rho = c^{-1}$, $a^\rho = a^{-r^{2l-k}}$ and $b^\rho = b$. Then $S_k^\rho = \{cab^{2l-k}, (cab^{2l-k})^{-1}, b^l\} = S_{2l-k}$, so $\mathrm{Cay}(G, S_k) \cong \mathrm{Cay}(G, S_{2l-k})$. Thus, up to isomorphism of graphs, we may choose $k < l$ or $k = l = 1$.

Since Γ is connected, $G = \langle S_k\rangle = \langle c\rangle \times \langle ab^k, b^l\rangle$, we have $\langle ab^k, b^l\rangle = \langle a, b\rangle$. Since $\langle a\rangle$ is normal in $\langle a, b\rangle$, we may set $a^{b^l} = a^e$ for some integer e. Since $o(a) = m$, we have $e^2 \equiv 1 \,(\mathsf{mod}\ m)$, and so $H := \langle a, b\rangle = \langle ab^k, b^l\rangle = \langle a^e b^k, ab^k, b^l\rangle = \langle a^{e-1}, ab^k, b^l\rangle = \langle a^{e-1}\rangle\langle ab^k, b^l\rangle$. Let $K = \langle a^{e-1}\rangle$. Since $(ab^k)^{b^l} = a^e b^k = a^{e-1}ab^k$, we have $K(ab^k)^{b^l} = Ka^{e-1}ab^k = Kab^k$. Thus the quotient group H/K is abelian, so $\langle a\rangle = H' \le K = \langle a^{e-1}\rangle$. Then $\langle a\rangle = \langle a^{e-1}\rangle$, and so $(e-1, m) = 1$. Hence $e \equiv -1 \,(\mathsf{mod}\ m)$ as $e^2 \equiv 1 \,(\mathsf{mod}\ m)$, and so $a^{b^l} = a^{-1}$.

Now we show that $\mathsf{Aut}(G, S_k) \ne 1$ if and only if $l = 1$. Suppose that $\mathsf{Aut}(G, S_k) \ne 1$. Then, since S_k contains only one involution, we conclude that $\mathsf{Aut}(G, S_k) = \langle\tau\rangle \cong \mathbb{Z}_2$, $b^l = (b^l)^\tau$ and $(cab^k)^\tau = (cab^k)^{-1}$. Then $c^\tau = c^{-1}$ and $(ab^k)^\tau = (ab^k)^{-1} = b^{-k}a^{-1} = (a^{-1})^{b^k}b^{-k} = a^s b^{-k}$ for some s. By [14], we set $a^\tau = a^i$ and $b^\tau = a^j b$ for some i and j. Then, noting $a^b \in \langle a\rangle$, computation shows that $(ab^k)^\tau = a^\tau(b^\tau)^k = a^{i+t}b^k$ for some t. Thus $a^{i+t}b^k = a^s b^{-k}$, yielding $k \equiv -k \,(\mathsf{mod}\ 2l)$, and so $l = 1$ as $(l, k) = 1$.

Conversely, suppose that $l = 1$. Then $o(c) = \frac{2n}{o(a)o(b)} = \frac{n}{m} > 1$, $k = 0$ or 1, and $S_k = \{ca, c^{-1}a^{-1}, b\}$ or $\{cab, c^{-1}ab, b\}$. Assume first that $S_k = \{cab, c^{-1}ab, b\}$. Take $\tau \in \mathsf{Aut}(G)$ with $c^\tau = c^{-1}$, $a^\tau = a$ and $b^\tau = b$. Then $1 \ne \tau \in \mathsf{Aut}(G, S_k)$, and $\mathsf{Aut}\mathrm{Cay}(G, S_k)$ has a regular subgroup $\langle \bar{c}\bar{a}, \bar{b}\tau\rangle \cong \mathrm{D}_{2n}$, so Γ is a dihedrant. Now let $S_k = \{ca, c^{-1}a^{-1}, b\}$. By [14], take $\tau \in \mathsf{Aut}(G)$ with $c^\tau = c^{-1}$, $a^\tau = a^{-1}$ and $b^\tau = b$. Then $1 \ne \tau \in \mathsf{Aut}(G, S_k)$. Since $\langle ca\rangle$ is normal in G, we set $(ca)^b = (ca)^t$ for some $1 < t < n$. Then $t^2 \equiv 1 \,(\mathsf{mod}\ n)$ as $o(b) = 2$ and $o(ca) = n$. Let $r = t$ or $n - t$ such that $r < \frac{n}{2}$. For $0 \le i \le n-1$, we label $\alpha_i = (ca)^i$ and $\beta_i = b(ca)^i$ if $r = t$, or $\alpha_i = (ca)^{-i}$ and $\beta_i = b(ca)^{-i}$ if $r = n - t$. Then $\mathrm{Cay}(G, S_k)$ has edges $\{\alpha_i, \alpha_{i+1}\}$, $\{\alpha_i, \beta_i\}$ and $\{\beta_i, \beta_{i+r}\}$. Thus $\Gamma \cong \mathrm{Cay}(G, S_k) \cong \mathbf{P}(n, r)$. $\square$

3. Cubic coset graphs

In a graph, an *arc* is an ordered pair of adjacent vertices, and a 2-*arc* is a directed path of length 2. A graph Γ is called *arc-transitive* or 2-*arc-transitive* if $\mathsf{Aut}\Gamma$ is transitive on the arcs or the 2-arcs of Γ, respectively. For a graph Γ and $G \le \mathsf{Aut}\Gamma$, we say Γ to be G-*vertex-transitive* or G-*arc-transitive* if G acts transitively on the vertices or the arcs of Γ, respectively.

Let $\Gamma = (V, E)$ be a G-vertex-transitive graph. Then, for $\alpha \in V$, the stabilizer G_α is a core-free subgroup in G, that is, $\cap_{g\in G} G_\alpha^g = 1$. Set $H = G_\alpha$ and $D = \{x \mid \alpha^x \in \Gamma(\alpha)\}$, where $\Gamma(\alpha)$ is the set of neighbors of α in Γ. Then D is a union of several double cosets HxH. Since Γ is undirected, we have $D = D^{-1} := \{x^{-1} \mid x \in D\}$. Moreover, Γ is isomorphic the *coset graph* $\mathsf{Cos}(G, H, D)$ defined over $\{Hx \mid x \in G\}$ with edge set $\{\{Hg_1, Hg_2\} \mid g_2 g_1^{-1} \in D\}$.

The following statements for coset graphs are well-known.

(a) Γ is connected if and only if $\langle H, D\rangle = G$.
(b) Γ is G-arc-transitive if and only if $D = HgH$ for $g \in G$ with $g^2 \in H$; moreover, g can be chosen as a 2-element with $g \in \mathbf{N}_G(H \cap H^g)$ and $g^2 \in H \cap H^g$.

The next lemma gives a characterisation of the prime divisors of $|G_\alpha|$.

Lemma 3.1 ([7])**.** *If Γ is connected and of valency k, then each prime divisor of $|G_{\alpha\beta}|$ is less than k, where $\{\alpha,\beta\}$ is an edge of Γ.*

Now assume that Γ is cubic and connected. If G is regular on V, then Γ is a Cayley graph of G. If G is transitive on the arcs of Γ, then $\Gamma \cong \mathsf{Cos}(G, G_\alpha, G_\alpha g G_\alpha)$ where g is a 2-element with $\langle g, G_\alpha\rangle = G$, $\alpha^g \in \Gamma(\alpha)$, $g \in \mathbf{N}_G(G_{\alpha\alpha^g})$ and $g^2 \in G_{\alpha\alpha^g}$; moreover, the well-known result of Tutte determines G_α, refer to [2].

Theorem 3.2. *If Γ is G-arc-transitive, then $G_\alpha \cong \mathbb{Z}_3$, S_3, D_{12}, S_4 or $\mathrm{S}_4{\times}\mathrm{S}_2$.*

Suppose that G is not regular on V and not transitive on the arcs of Γ. Then G_α fixes one of neighbors, say γ, and transitive on the other two neighbors, say β_1 and β_2, of α. Thus G_α is a non-trivial 2-group by Lemma 3.1. Moreover, Γ is an arc-disjoint union of two G-arc-transitive graphs, one of valency 2 and the other of valency 1. Then $\Gamma \cong \mathsf{Cos}(G, G_\alpha\{x,y\}G_\alpha)$, where x and y are 2-elements such that $\alpha = \beta_1^x$, $x \in \mathbf{N}_G(G_{\alpha\beta_1})$, $x^2 \in G_{\alpha\beta_1}$, $\alpha^y = \gamma$, $y \in \mathbf{N}_G(G_\alpha)$, $y^2 \in G_\alpha$ and $\langle x, y, G_\alpha\rangle = G$. Thus, if a characteristic subgroup $M \le G_{\alpha\beta_1}$ is normal in $\langle y, G_\alpha\rangle$ then $M = 1$; if G has an abelian Sylow 2-subgroup, then $\langle y, G_\alpha\rangle$ is an abelian 2-group, and so $G_{\alpha\beta_1}$ is normal in G, hence $G_{\alpha\beta_1} = 1$. Then the next lemma follows.

Lemma 3.3. *Assume that $\{\beta_1,\beta_2\}$ and $\{\gamma\}$ are the two G_α-orbits on $\Gamma(\alpha)$. Then G_α and $G_{\alpha\beta_1}$ do not contain a common non-trivial characteristic subgroup. If further G has an abelian Sylow 2-subgroup, then $G_\alpha \cong \mathbb{Z}_2$.*

Some of the generalized Petersen graphs can be constructed as coset graphs.

Lemma 3.4. *Let Γ be a connected G-vertex-transitive cubic graph with $\mathbb{Z}_n{:}\mathbb{Z}_4 \cong G \le \mathsf{Aut}\Gamma$, where n is odd and square-free. Then either G is a regular subgroup of $\mathsf{Aut}\Gamma$, or $\Gamma \cong \mathbf{P}(n,r)$ for $1 < r < \frac{n}{2}$ with $r^2 \equiv -1 \,(\mathsf{mod}\ n)$.*

Proof. Let $\langle a\rangle$ be the normal subgroup of G of order n. Then $\langle a\rangle$ is a semiregular subgroup of G. Since $\langle a\rangle$ has odd order and Γ has valency 3, we conclude that $\langle a\rangle$ is intransitive on $V\Gamma$. Thus Γ has order $2n$ or $4n$. If Γ has order $4n$, then G is a regular subgroup of $\mathsf{Aut}\Gamma$. Hence, we assume Γ has order $2n$. Let $b \in G$ be of order 4. Then $G = \langle a\rangle{:}\langle b\rangle$ and $a^b = a^r$ as $\langle a\rangle$ normal in G, where $1 \le r < n$ with $r^4 \equiv 1 \,(\mathsf{mod}\ n)$.

Note all involutions of G are conjugate and contained in $\langle a, b^2\rangle$. Then $H := G_\alpha = \langle b^2\rangle$ for some $\alpha \in V\Gamma$. Write $\Gamma \cong \mathsf{Cos}(G, H, H\{x,y\}H)$, where x is an involution and $y \in \mathbf{N}_G(H)$ with $y^2 \in H$. Let $\mathbf{C}_{\langle a\rangle}(b^2) = \langle a_1\rangle$. Since $o(a) = n$ is square-free, we may write $\langle a\rangle = \langle a_1\rangle{\times}\langle a_2\rangle$. Then $a_2 \ne 1$; otherwise, $\mathbf{C}_{\langle a\rangle}(b^2) = \langle a\rangle$, yielding $H = \langle b^2\rangle$ is normal in G, a contradiction. It is easily shown that $a_2^{b^2} = a_2^{-1}$, yielding $a_2^{r^2} = a_2^{-1}$, and hence $r^2 \equiv -1 \,(\mathsf{mod}\ o(a_2))$. Note that $\mathbf{N}_G(H) = \langle a_1\rangle{:}\langle b\rangle$ and all involutions of G are contained in $\langle a_2, b^2\rangle$. Since $HbH = Hb^{-1}H$ and $\langle x, y, H\rangle = G$, we may choose $x = a_2^t b^2$ and $y = a_1^i b$ with $y^2 \in H$. Then $y^2 = a_1^i b^2(b^{-1}a_1^i b) = a_1^{i+ri}b^2$, yielding $y^2 = b^2$. In particular, y has order 4. Thus, since Γ is connected, $G = \langle x, y, H\rangle = \langle a_2^t b^2, y, y^2\rangle = \langle a_2^t, y\rangle = \langle a_2^t\rangle{:}\langle y\rangle$. It follows that $\langle a\rangle = \langle a_2^t\rangle$, and so $n = o(a) = o(a_2) = o(a_2^t)$, $a_1 = 1$ and $r^2 \equiv -1 \,(\mathsf{mod}\ n)$. Thus $y = b$, and it is easily shown that $\mathbf{N}_G(H) = \langle b\rangle$. Write $a_2^t = a^s$. Then $x = a^s b^2$ and $G = \langle a^s\rangle{:}\langle b\rangle$.

Since $H\{x,y\}H = H\{a^s, b\}H$, we have $\Gamma \cong \mathsf{Cos}(G, H, H\{a^s, b\}H)$. Since $HbH = Hb^3H$ and $a^{b^3} = a^{n-r}$, replacing b by b^3 if necessary, we assume that $r < \frac{n}{2}$.

Now label $\alpha_i = Ha^{si}$ and $\beta_i = Hba^{si}$, where $0 \le i \le n-1$, which gives rise to all vertices of Γ. Then, $\{\alpha_i, \alpha_{i+1}\}$ and $\{\alpha_i, \beta_i\}$ are edges. Moreover, $\beta_i = Hba^{si}$ and $\beta_j = Hba^{sj}$ are adjacent whenever $(a^s)^{(j-i)(-r)} = ba^{sj-si}b^{-1} = ba^{sj}(ba^{si})^{-1}$ equals to a^s or a^{-s}, i.e., $(j-i)(-r) \equiv \pm 1 \pmod{n}$. Thus $\{\beta_i, \beta_j\}$ is an edge if and only if $j \equiv i \pm r \pmod{n}$. Therefore, $\Gamma \cong \mathsf{Cos}(G, H, H\{a^s, b\}H) \cong \mathbf{P}(n, r)$. □

We next describe some graphs associated with simple groups $\mathrm{PSL}(2,p)$ with p prime. As usual, for two integers d, n, by $d \,\|\, n$ we mean d divides n, and $(d, \frac{n}{d}) = 1$.

Example 3.5. Let $T = \mathrm{PSL}(2,p)$, where p is a prime.

(1) Assume that $p \equiv \pm 3 \pmod{8}$. Then $4 \,\|\, (p-\varepsilon)$, where $\varepsilon = 1$ or -1. Take a subgroup $H \cong \mathrm{S}_3$ of T, and let $K \cong \mathbb{Z}_2$ be a Sylow 2-subgroup of H. Then $\mathbf{N}_T(K) = \mathrm{D}_{p-\varepsilon}$, and let $g \in \mathbf{N}_T(K) \setminus K$ be an involution such that $\langle H, g\rangle = T$.

(2) Assume that $p \equiv \pm 7 \pmod{16}$. Then $8 \,\|\, (p-\varepsilon)$, where $\varepsilon = 1$ or -1. Take a subgroup $H \cong \mathrm{D}_{12}$ of T, and let $K \cong \mathbb{Z}_2^2$ be a Sylow 2-subgroup of H. Then $\mathbf{N}_T(K) = \mathrm{S}_4$, and let $g \in \mathbf{N}_T(K) \setminus K$ be an involution such that $\langle H, g\rangle = T$.

(3) Assume that $p \equiv \pm 15 \pmod{32}$. Then $16 \,\|\, (p-\varepsilon)$, where $\varepsilon = 1$ or -1. Take a subgroup $H \cong \mathrm{S}_4$ of T, and let $K \cong \mathrm{D}_8$ be a Sylow 2-subgroup of H. Then $\mathbf{N}_T(K) = \mathrm{D}_{16}$, and let $g \in \mathbf{N}_T(K) \setminus K$ be an involution such that $\langle H, g\rangle = T$.

In each of these three cases, the coset graph $\Gamma = \mathsf{Cos}(T, H, HgH)$ is a connected 2-arc-transitive cubic graph, and the order of Γ is even and indivisible by 4.

Example 3.6. Let $T = \mathrm{PSL}(2,p)$, and let $G = \mathrm{PGL}(2,p)$, where p is a prime.

(1) Assume that $p \equiv \pm 3 \pmod{8}$. Then $4 \,\|\, (p-\varepsilon)$, where $\varepsilon = 1$ or -1. Take a subgroup $H \cong \mathrm{D}_{12}$ of T, and let $K \cong \mathbb{Z}_2^2$ be a Sylow 2-subgroup of H. Then $\mathbf{N}_G(K) = \mathrm{S}_4$. Let $g \in \mathbf{N}_G(K) \setminus K$ be an involution such that $\langle H, g\rangle = G$.

(2) Assume that $p \equiv \pm 7 \pmod{16}$. Then $8 \,\|\, (p-\varepsilon)$, where $\varepsilon = 1$ or -1. Take a subgroup $H \cong \mathrm{S}_4$ of T, and let $K \cong \mathrm{D}_8$ be a Sylow 2-subgroup of H. Then $\mathbf{N}_G(K) = \mathrm{D}_{16}$, and let $g \in \mathbf{N}_G(K) \setminus K$ be an involution such that $\langle H, g\rangle = G$.

If g is described as in (1) or (2), then the coset graph $\Gamma = \mathsf{Cos}(G, H, HgH)$ is bipartite, connected, cubic and 2-arc-transitive.

The final two examples give several families of cubic graphs associated with $\mathrm{PSL}(2,p)$, which are not arc-transitive.

Example 3.7. Let $T = \mathrm{PSL}(2,p)$, where p is a prime.

(1) Assume that $p \equiv \pm 3 \pmod{8}$. Then $4 \,\|\, (p-\varepsilon)$, where $\varepsilon = 1$ or -1. Let $\mathbb{Z}_2 \cong H < T$. Then $\mathbf{N}_T(H) = \mathrm{D}_{p-\varepsilon}$. Let $x \in \mathbf{N}_T(H) \setminus H$ and $y \in T \setminus \mathbf{N}_T(H)$ be two involutions. Then $\langle H, x, y\rangle = T$.

(2) Assume that $p \equiv \pm 7 \pmod{16}$. Then $8 \,\|\, (p-\varepsilon)$, where $\varepsilon = 1$ or -1. Let $\mathbb{Z}_2^2 \cong H < T$, and let $K \cong \mathbb{Z}_2$ be a subgroup of H. Then $\mathbf{N}_T(H) = \mathrm{S}_4$ and $\mathbf{N}_T(K) = \mathrm{D}_{p-\varepsilon}$. Let $x \in \mathbf{N}_T(H) \setminus H$ and $y \in \mathbf{N}_T(K) \setminus \mathbf{N}_{\mathbf{N}_T(H)}(K)$ be involutions such that $\langle H, x, y\rangle = T$.

(3) Assume that $p \equiv \pm 15 \pmod{32}$. Then $16 \,\|\, (p-\varepsilon)$, where $\varepsilon = 1$ or -1. Let $\mathrm{D}_8 \cong H < T$ and $K \cong \mathbb{Z}_2^2$ be a subgroup of H. Then $\mathbf{N}_T(H) = \mathrm{D}_{16}$ and $\mathbf{N}_T(K) = \mathrm{S}_4$. Let $x \in \mathbf{N}_T(H) \setminus H$ and $y \in \mathbf{N}_T(K) \setminus H$ be involutions such that $\langle H, x, y\rangle = T$.

Take x and y as in (1), (2) or (3). Then the coset graph $\Gamma = \mathsf{Cos}(T, H, H\{x, y\}H)$ is a connected cubic graph, and Γ has even indivisible by 4.

Example 3.8. Let $T = \mathrm{PSL}(2, p)$, and let $G = \mathrm{PGL}(2, p)$, where p is a prime.

(1) Assume that $p \equiv \pm 3 \pmod 8$. Then $4 \,\|\, (p - \varepsilon)$, where $\varepsilon = 1$ or -1. Let $\mathbb{Z}_2^2 \cong H < T$ and $K \cong \mathbb{Z}_2$ be a subgroup of H. Then $\mathbf{N}_G(K) = \mathrm{D}_{2((p-\varepsilon))}$ and $\mathbf{N}_G(H) = \mathrm{S}_4$. Let $x \in \mathbf{N}_G(H) \setminus H$ and $y \in \mathbf{N}_G(K) \setminus \mathbf{N}_{\mathbf{N}_G(K)}(H)$ be two involutions such that $\langle H, x, y\rangle = G$.

(2) Assume that $p \equiv \pm 7 \pmod{16}$. Then $8 \,\|\, (p - \varepsilon)$, where $\varepsilon = 1$ or -1. Let $\mathrm{D}_8 \cong H < T$ and let $K \cong \mathbb{Z}_2^2$ be a subgroup of H. Then $\mathbf{N}_G(H) = \mathrm{D}_{16}$ and $T > \mathbf{N}_G(K) = \mathrm{S}_4$. Let $x \in \mathbf{N}_G(H) \setminus H$ and $y \in \mathbf{N}_G(K) \setminus H$ be an involution such that $\langle H, x, y\rangle = G$.

For each of (1) and (2), the coset graph $\Gamma = \mathsf{Cos}(G, H, H\{x, y\}H)$ is bipartite, connected and cubic, and the order of Γ is even and indivisible by 4.

4. Normal quotients

Let $\Gamma = (V, E)$ be a connected G-vertex-transitive graph, where $G \leq \mathsf{Aut}\Gamma$.

For a normal subgroup $N \lhd G$, the *normal quotient* Γ_N of Γ, induced by N, is the graph whose vertices are the N-orbits on V such that B and C are adjacent if and only if there exists an edge $\{\beta, \gamma\} \in E$ with $\beta \in B$ and $\gamma \in C$. Clearly, the valency of Γ_N is at most the number of N_α-orbits on $\Gamma(\alpha)$. Let K be the kernel of G acting on the N-orbits. Then G/K can be viewed as a subgroup of $\mathsf{Aut}\Gamma_N$. If the valency of Γ_N equals the valency of Γ, then Γ is a *cover* of Γ_N and, in this case, $K = N$ is semiregular on V.

From now on, we assume that Γ is connected and cubic. Suppose that G is neither regular on V nor transitive on the arcs of Γ. Then G_α is a non-trivial 2-group, where $\alpha \in V$. Set $\Gamma(\alpha) = \{\beta_1, \beta_2, \gamma\}$ such that G_α is transitive on $\{\beta_1, \beta_2\}$ and fixes γ.

Let $N \lhd G$ have at least 3 orbits on V, and V_N be the set of N-orbits. Then the quotient graph Γ_N has valency 2 or 3. If Γ_N has valency 3, then Γ is a cover of Γ_N.

Lemma 4.1. *Let K be the kernel of G acting on V_N. If Γ is not a cover of Γ_N, then Γ_N is an l-cycle and either*

(1) *each N-orbit is a matching, $K = N$ is semiregular, $G/N \cong \mathrm{D}_{2l}$, and G has a regular subgroup $N.\mathbb{Z}_l$; or*

(2) *$G_\alpha = K_\alpha$ is a 2-group, l is even, and $G/K \cong \mathrm{D}_l$ acting on V_N regularly.*

Proof. Suppose that Γ_N has valency 2. Then Γ_N is an l-cycle for some integer l. Noting that $(\gamma^N)^{G_\alpha} = \gamma^N$ and $(\beta_1^N)^g = \beta_2^N$ for some $g \in G_\alpha$, either $\alpha^N = \gamma^N$ and $\beta_1^N \neq \beta_2^N$, or $\alpha^N \neq \gamma^N$ and $\beta_1^N = \beta_2^N$.

We assume first that $\alpha^N = \gamma^N$ and $\beta_1^N \neq \beta_2^N$. Then α^N induces a matching, and G/K is transitive on the arcs of Γ_N, and so $G/K \cong \mathrm{D}_{2l}$. Noting that K_α fixes $\Gamma(\alpha) = \{\beta_1, \beta_2, \gamma\}$ point-wise, it implies that $K_\alpha = 1$, hence $N = K$ is a semiregular subgroup of G. Then G contains a subgroup $N.\mathbb{Z}_l$ which is regular on V.

Now let $\alpha^N \neq \gamma^N$ and $\beta_1^N = \beta_2^N$. Then the induced subgraphs $[\alpha^N \cup \beta_1^N]$ and $[\alpha^N \cup \gamma^N]$ are regular and have valency 2 and 1, respectively. Thus there is no an element in G which maps $\{\alpha^N, \beta_1^N\}$ to $\{\alpha^N, \gamma^N\}$. Therefore, G/K is transitive on

V_N but not on the edges of Γ_N. Noting that $\mathsf{Aut}\Gamma_N \cong \mathrm{D}_{2l}$, it follows that l is even, $G/K \cong \mathrm{D}_l$ and G/K acting on V_N regularly. Moreover, $K_\alpha = G_\alpha$. $\square$

This leads us to define a special type of cover for some cubic graphs.

Construction 4.2. Assume that $X = \mathrm{PGL}(2,p)$, $T = \mathrm{PSL}(2,p)$ and $p \equiv \pm 3 \,(\mathsf{mod}\ 8)$. Then $4 \,\|\, (p-\varepsilon)$, where $\varepsilon = 1$ or -1. Let $\mathbb{Z}_2^2 \cong H < T$ and $K \cong \mathbb{Z}_2$ be a subgroup of H. Then $\mathbf{N}_X(K) = \mathrm{D}_{2((p-\varepsilon))}$ and $\mathbf{N}_X(H) = \mathrm{S}_4$. Let $x \in \mathbf{N}_X(H) \setminus T$ and $y \in \mathbf{N}_X(K) \setminus T$ be such that $x^2 \in H$, $y^2 \in K$ and $\langle H, x, y\rangle = X$. Let $M = \langle c\rangle \cong \mathbb{Z}_m$ with odd m coprime to $|T|$, and let $G = (T{\times}M)\langle x\rangle$ such that $c^x = c^{-1}$ (and so $c^y = c^{-1}$). Then $G = T{:}\mathrm{D}_{2m}$, and $\Sigma = \mathsf{Cos}(G, H, H\{c^ix, c^jy\}H)$ is a cubic graph.

It is easily shown that Σ is connected if and only if $(i-j, m) = 1$. Moreover, $\Sigma_M \cong \mathsf{Cos}(X, H, H\{x, y\}H)$ and Σ_T is a cycle of length $2m$.

5. Soluble automorphism groups

Let $\Gamma = (V, E)$ be a connected cubic G-vertex-transitive graph of square-free order $2n$, where $G \le \mathsf{Aut}\Gamma$. In this section, we consider the case where G is soluble.

If G is regular on V, then Γ is a Cayley graph of G, and Γ is known by Lemmas 2.1-2.5 and Corollary 2.4. Thus, in the following, we assume that G is not regular on V, that is, $G_\alpha \ne 1$ for $\alpha \in V$. Then Lemma 4.1 is available.

As usual, for a prime divisor p of $|G|$, let $\mathbf{O}_p(G)$ be the largest normal p-subgroup of G. Since the order $|G : G_\alpha|$ of Γ is square-free and G_α is a $\{2,3\}$-group, either $|\mathbf{O}_p(G)| \le p$, or $|\mathbf{O}_p(G)| \ge p^2$ and $p \in \{2, 3\}$.

Lemma 5.1. *If $\mathbf{O}_2(G) \ne 1$, then $G \cong \mathbb{Z}_{2n}{:}\mathbb{Z}_2 \cong \mathrm{D}_{4n}$, and $\Gamma = \mathbf{M}_n$ or $\mathbf{P}(n, 1)$.*

Proof. Let $N = \mathbf{O}_2(G) \ne 1$. Then each N-orbit has length 2, and the quotient graph Γ_N is of odd order n. It follows from Lemma 4.1 that $G_\alpha \cong \mathbb{Z}_2$, $N \cong \mathbb{Z}_2$ and G contains a regular subgroup $N.\mathbb{Z}_n \cong \mathbb{Z}_{2n}$, and so $G \cong \mathbb{Z}_{2n}{:}\mathbb{Z}_2$. Thus G contains a normal regular subgroup $R \cong \mathbb{Z}_{2n}$. Write $\Gamma = \mathrm{Cay}(R, S)$. Then $S = \{a, a^{-1}, b\}$, where b is the unique involution in R, and $o(a) = n$ or $2n$. Thus, $\Gamma = \mathbf{M}_n$ or $\mathbf{P}(n, 1)$.

Let α be the vertex corresponding the identity of R. Then $G_\alpha \le \mathsf{Aut}(R)$. Set $G_\alpha = \langle\sigma\rangle$. Then $a^\sigma = a^{-1}$ as $S^\sigma = S$, and thus $G = R{:}\langle\sigma\rangle \cong \mathrm{D}_{4n}$. $\square$

Lemma 5.2. *If $\mathbf{O}_3(G)$ has order divisible by 9, then $\Gamma = \mathsf{K}_{3,3}$ and $\mathsf{Aut}\Gamma = \mathrm{S}_3 \wr \mathrm{S}_2$.*

Proof. Let $N = \mathbf{O}_3(G)$. Assume that $|N| > 3$. Then N is not semiregular on V, and N_α is a non-trivial 3-group. It follows that N_α is transitive on $\Gamma(\alpha)$. For $\beta \in \Gamma(\alpha)$, the orbit β^{N_α} has size 3. It follows that the induced subgraph of Γ with vertex set $\alpha^N \cup \beta^N$ is isomorphic to $\mathsf{K}_{3,3}$. So $\Gamma \cong \mathsf{K}_{3,3}$, and clearly, $\mathsf{Aut}\Gamma = \mathrm{S}_3 \wr \mathrm{S}_2$ $\square$

Let F be the Fitting subgroup of G, the largest nilpotent normal subgroup of G. Then $F \ne 1$ and $\mathbf{C}_G(F) \le F$ as G is soluble, and $F = \langle \mathbf{O}_p(G) \mid p \,\big|\, |G|\rangle$.

Lemma 5.3. *Assume that $\mathbf{O}_2(G) = 1$ and $\mathbf{O}_3(G) = 1$ or $\mathbb{Z}_3$. Then Fitting subgroup of G is cyclic and has exactly two orbits on V, and either $\Gamma \cong \mathsf{K}_{3,3}$ or one of the following holds.*

1. $\mathbb{Z}_n{:}\mathbb{Z}_4$ *and* $\Gamma \cong \mathbf{P}(n, r)$, *where* $r^2 \equiv -1 \,(\mathsf{mod}\ n)$;
2. $G \cong \mathbb{Z}_n{:}\mathbb{Z}_2^2$ *and* $\Gamma \cong \mathbf{M}_n$ *or* $\mathbf{P}(n, r)$, *where* $r^2 \equiv 1 \,(\mathsf{mod}\ n)$;

(3) $G \cong \mathbb{Z}_n{:}\mathbb{Z}_6 \cong \mathrm{D}_{2n}{:}\mathbb{Z}_3$ *and* Γ *is isomorphic to one of the graphs involved in Lemma* 2.3 (3).

Proof. Let F be the Fitting subgroup of G. Noting that $\mathbf{O}_2(G) = 1$ and $\mathbf{O}_p(G) = 1$ or $\mathbb{Z}_p$ for each odd prime p divisor of $|G|$, we conclude that F is cyclic and of odd order. It follows that F is semiregular on V. Since $\mathbf{C}_G(F) \leq F$, we have $\mathbf{C}_G(F) = F$. Then $G/F = \mathbf{N}_G(F)/\mathbf{C}_G(F)$ is isomorphic to a subgroup of $\mathsf{Aut}(F)$, which is abelian.

Suppose that F has at least 3-orbits on V. Then, by Lemma 4.1, Γ is a cover of Γ_F. Thus G/F is isomorphic to a subgroup of $\mathsf{Aut}\Gamma_F$, and so G/F is regular on V_F as it is abelian. Then G is regular on V, which is not the case.

Thus, F has at most two orbits on V. Since F has odd order, F has exactly 2 orbits on V. Since G/F is abelian, G has an abelian Sylow 2-subgroup. If G is not transitive on the arcs of Γ, then $G_\alpha \cong \mathbb{Z}_2$ by Lemma 3.3, and so $G = F{:}\mathbb{Z}_2^2$ or $F{:}\mathbb{Z}_4$. On the other hand, $G_\alpha \cong G_\alpha/F_\alpha \cong FG_\alpha/F \leq G/F$ is abelian. If Γ is G-arc-transitive, then $G_\alpha \cong \mathbb{Z}_3$ by Theorem 3.2, so $G = F{:}\mathbb{Z}_6$. If $G \cong \mathbb{Z}_n{:}\mathbb{Z}_4$ then (1) holds by Lemma 3.4. If $G \ncong \mathbb{Z}_n{:}\mathbb{Z}_4$ then G has a normal regular subgroup $R \cong \mathbb{Z}_n{:}\mathbb{Z}_2$, and so Γ is known by Lemmas 2.1-2.5 and Corollary 2.4. This completes the proof. □

6. Insoluble automorphism groups

Let $\Gamma = (V, E)$ be a connected cubic G-vertex-transitive graph of square-free order $2n$, where $G \leq \mathsf{Aut}\Gamma$. In this section, we assume that G is insoluble.

Recall that the *soluble radical* of a group G is the largest soluble normal subgroup of G. Since G is insoluble, the next lemma is a consequence of Lemma 4.1.

Lemma 6.1. *Let* M *be the soluble radical of* G. *Then* Γ *is a cover of* Γ_M*; in particular,* M *is semiregular on* V *and of odd order.*

Proof. Let V_M be the set of M-orbits on V, and let K be the kernel of G acting on V_M. Then $M \lhd K \lhd G$, and $K = MK_\alpha$. Since $K_\alpha \lhd G_\alpha$ is soluble, so is K, and hence $K = M$. Thus, $G/M \leq \mathsf{Aut}\Gamma_M$ is insoluble, and so Γ_M is cubic. Hence M is semiregular, and $|V_M|$ is even. Since $|V| = |M||V_M|$ is square-free, $|M|$ is odd. □

We first deal with the case where G has trivial soluble radical.

Lemma 6.2. *Suppose that the soluble radical of* G *is trivial. Then* G *is almost simple.*

Proof. Let N be a minimal normal subgroup of G. Then N is insoluble. Let V_N be the set of N-orbits on V, and let K be the kernel of G on V_N. Then $K = NK_\alpha$, and so K/N is soluble. Since $|V|$ is square-free, N is not semiregular on V, and hence the quotient graph Γ_N has valency 0, 1 or 2. Thus, $G/K \leq \mathsf{Aut}\Gamma_N$ is soluble, and so is G/N. Hence N is the only minimal normal subgroup of G. Since $|G|$ is not divisible by p^2 with $p \geq 5$ prime, N is simple, and G is almost simple. □

Lemma 6.3. *Let* G *be almost simple with socle* $\mathrm{soc}(G) = T$. *Assume that* Γ *is* G*-arc-transitive. Then either*

(1) $T = \mathrm{A}_6$, $\mathsf{Aut}\Gamma = \mathrm{P\Gamma L}(2, 9)$ *and* Γ *is isomorphic to the Tutte's* 8*-cage, or*
(2) $T = \mathrm{PSL}(2, p)$ *such that a Sylow* 2*-subgroup of* T *is* $\mathbb{Z}_2^2$, D_8 *or* D_{16}, *and* Γ *is a* 2*-arc-transitive graph; moreover* Γ *is described as in Example* 3.5 *or* 3.6.

Proof. By Theorem 3.2, $|G_\alpha|$ is not divisible by $2^5 \cdot 3^2$. Since $|V| = |G : G_\alpha|$ is square-free, $|G|$ is not divisible by 2^6, 3^3 and r^2, where r is a prime with $r > 3$. Inspecting the orders of finite simple groups, we obtain that T is one of A_6, A_7, M_{11}, J_1, $\mathrm{PSL}(2, 2^f)$, $\mathrm{PSL}(2, p)$ for prime $p \geq 5$.

Suppose that $T = \mathrm{PSL}(2, 2^f)$ with $f \geq 3$. Then $f = 3, 4$ or 5. By the information given in the Atlas [8], we conclude that G has no a subgroup of square-free index as listed in Theorem 3.2, which a contradiction.

Suppose that $T = \mathrm{A}_7$. Note that $|G : G_\alpha|$ is even and square-free. Then either $|T_\alpha| = 12$ and T is transitive on V, or $|G_\alpha| = |T_\alpha| = 24$ and T has two orbits on V. Thus, $\varGamma$ is a G-arc-transitive graph of order 210; however, by [6], there exists no such a graph, which is a contradiction.

Suppose that $T = \mathrm{M}_{11}$. Then $G = T$ and $|T_\alpha| = 24$, so $T_\alpha \cong \mathrm{S}_4$. Thus, $T_{\alpha\beta} \cong \mathrm{D}_8$ and $\mathbf{N}_T(T_{\alpha\beta})$ is a Sylow 2-subgroup of T, where $\beta \in \varGamma(\alpha)$. Further, computation using GAP shows that all subgroups of T isomorphic to S_4 are conjugate. Thus we may assume that T_α is contained in a maximal subgroup $M \cong \mathrm{M}_{10}$. So $\mathbf{N}_T(T_{\alpha\beta}) = \mathbf{N}_M(T_{\alpha\beta})$. Then there is no an $x \in \mathbf{N}_T(T_{\alpha\beta})$ with $\langle x, T_\alpha \rangle = T$, which is a contradiction.

Suppose that $T = \mathrm{J}_1$. Then $G = T$ and $T_\alpha \cong \mathrm{D}_{12}$, so $T_{\alpha\beta} \cong \mathbb{Z}_2^2$ for $\beta \in \varGamma(\alpha)$. It follows from the information given in the Atlas [8] that $\mathbf{N}_T(T_{\alpha\beta}) = \mathbb{Z}_2 \times (T_{\alpha\beta}{:}\mathbb{Z}_3) \cong \mathbb{Z}_2 \times \mathrm{A}_4$. Since all elements of order 6 of T are conjugate, all subgroups of T isomorphic to D_{12} are conjugate. Thus, we assume that T_α is contained in a maximal subgroup $M \cong \mathbb{Z}_2 \times \mathrm{A}_5$. Then $\mathbf{N}_M(T_{\alpha\beta}) \cong \mathbb{Z}_2^3$ is the Sylow 2-subgroup of $\mathbf{N}_T(T_{\alpha\beta})$. Thus, there is no a 2-element $x \in \mathbf{N}_T(T_{\alpha\beta})$ with $\langle x, T_\alpha \rangle = T$, which is a contradiction.

Assume that $T = \mathrm{A}_6$. Then 12 divides $|T_\alpha|$, so $T_\alpha \cong \mathrm{A}_4$ or S_4 by checking the subgroups of A_6. If $T_\alpha \cong \mathrm{A}_4$, then T is transitive on V. Hence $\varGamma$ is T-arc-transitive, and so $\mathrm{A}_4 \cong T_\alpha \geq \mathrm{S}_3$ by Theorem 3.2, a contradiction. Thus $T_\alpha \cong \mathrm{S}_4$ and T has exactly two orbits on V, say U and W. Considering the possible permutation representations of A_6 of degree 15, we may assume that each of U and W consists of either the 2-subsets of $\Lambda := \{1, 2, 3, 4, 5, 6\}$, or the partitions with part size 2 of Λ. Noting that, for $\alpha \in U$, the neighborhood $\varGamma(\alpha)$ is a T_α-orbit on W. Since $|\varGamma(\alpha)| = 3$, computation shows that, relabeling if necessary, U consists 2-subsets, and W consists of partitions, such that $\alpha \in U$ is adjacent to $\beta \in W$ if and only if α is a part of β. Thus $\varGamma$ is isomorphic to the Tutte's 8-cage, and then part (1) of this lemma follows.

Now assume that $T = \mathrm{PSL}(2, p)$, for a prime $p \geq 5$. Then $G = \mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$. Inspecting subgroups of G listed in [13, Chapter II, 8.27] and [3], G does not have subgroups isomorphic to $\mathrm{S}_4 \times \mathrm{S}_2$. Thus, G_α is isomorphic to one of S_3, D_{12} and S_4. It follows that either $T_\alpha = G_\alpha$, or $T_\alpha \cong \mathrm{S}_3$ and $G_\alpha \cong \mathrm{D}_{12}$.

First, let $T_\alpha \cong \mathrm{S}_3$. Since $|G : G_\alpha|$ is square-free, so is $|T : T_\alpha|$. Thus, 8 does not divide $|T| = p(p^2-1)/2$, and so $p \equiv \pm 3 \pmod{8}$. Since $|T : T_\alpha|$ is even, T is transitive on V. Hence $\varGamma$ can be written as a coset graph as in Example 3.5 (1).

Suppose now that $T_\alpha = G_\alpha \cong \mathrm{D}_{12}$. Since $|G : G_\alpha|$ is even and square-free, 8 divides $|G|$ but 16 does not. Thus, either $G = T = \mathrm{PSL}(2, p)$, $p \equiv \pm 7 \pmod{16}$ and $\varGamma$ is isomorphic to a coset graph in Example 3.5 (2), or $G = \mathrm{PGL}(2, p)$, $p \equiv \pm 3 \pmod{8}$ and $\varGamma$ is isomorphic to a coset graph given in Example 3.6 (1).

In the case where $T_\alpha = G_\alpha = \mathrm{S}_4$, the order $|G|$ is divisible by 16 but not 32 since $|G : G_\alpha|$ is even and square-free. Hence either $G = T = \mathrm{PSL}(2, p)$ with $p \equiv \pm 15 \pmod{32}$

and Γ is isomorphic to the coset graph in Example 3.5 (3), or $G = \mathrm{PGL}(2,p)$ with $p \equiv \pm 7 \pmod{16}$ and Γ is isomorphic to the coset graph in Example 3.6 (2). $\square$

Now we consider the case where G is not transitive on the arcs of Γ. Then $\Gamma \cong \mathsf{Cos}(G, G_\alpha\{x,y\}G_\alpha)$, where x and y are 2-elements such that $\langle x, y, G_\alpha\rangle = G$, $\alpha^x, \alpha^y \in \Gamma(\alpha)$, $x \in \mathbf{N}_G(G_\alpha)$ with $x^2 \in G_\alpha$, $y \in \mathbf{N}_G(G_{\alpha\alpha^y})$ with $y^2 \in G_{\alpha\alpha^y}$.

Lemma 6.4. *Assume that G is almost simple with socle $\mathrm{soc}(G) = T$ and Γ is not G-arc-transitive. Then $T = \mathrm{PSL}(2,p)$, and either $G_\alpha \cong \mathbb{Z}_2^2$, or $G_\alpha = T_\alpha \cong \mathbb{Z}_2$ or D_8; moreover, Γ is isomorphic to a graph given in Examples 3.7 and 3.8.*

Proof. Since Γ is not G-arc-transitive and G is not regular, G_α is a nontrivial 2-group. Then r^2 is not a divisor of $|G|$, where r is an arbitrary odd prime. Checking the orders of finite simple groups, $T = \mathrm{soc}(G)$ is one of J_1, $\mathrm{PSL}(2,p)$ for prime $p \geq 5$, $\mathrm{PSL}(2,2^f)$ with $f \geq 4$, and $\mathrm{Sz}(2^f)$ for odd $f \geq 3$.

Suppose that $T = \mathrm{PSL}(2,2^f)$ with $f \geq 4$ or $\mathrm{Sz}(2^f)$ for $f \geq 3$. Then any two distinct Sylow 2-subgroups of T intersect trivially, see [13, Chapter II, 8.5] and [21]. Now $|T_\alpha| \geq 2^4$ and for $\beta \in \Gamma(\alpha)$, we have $|T_\alpha : T_{\alpha\beta}| \leq 2$, and hence $T_{\alpha\beta} \neq 1$. Thus, T_α and T_β are contained in the same Sylow 2-subgroup Q of T. Since Γ is connected, it follows that $T_\gamma \leq Q$ for all vertices γ of Γ. Hence, Q contains a non-trivial normal subgroup $\langle T_\beta \mid \beta \in V\Gamma\rangle = \langle T_\alpha^g \mid g \in G\rangle$ of T, which is a contradiction.

Suppose that $T = \mathrm{J}_1$. Then $T = G$, and since $|T : T_\alpha|$ is even and square-free, we have $T_\alpha \cong \mathbb{Z}_2^2$. Let $\beta \in \Gamma(\alpha)$ with $T_{\alpha\beta} = \mathbb{Z}_2$. Since Γ is connected, $\langle T_\alpha, x, y\rangle = T$, where $x \in \mathbf{N}_T(T_\alpha)$ with $x^2 \in T_\alpha$, and $y \in \mathbf{N}_T(T_{\alpha\beta})$ with $y^2 \in T_{\alpha\beta}$. By the Atlas [8], $\mathbf{N}_T(T_{\alpha\beta}) \cong \mathbb{Z}_2{\times}\mathrm{A}_5$ and $\mathbf{N}_T(T_\alpha) \cong \mathbb{Z}_2{\times}\mathrm{A}_4$. Then x is contained in the unique Sylow 2-subgroup $\langle T_\alpha, x\rangle$ of $\mathbf{N}_T(T_\alpha)$. Since $T_{\alpha\beta} < \langle T_\alpha, x\rangle \cong \mathbb{Z}_2^3$, we have $x \in \langle T_\alpha, x\rangle < \mathbf{N}_T(T_{\alpha\beta})$. Thus $\langle x, y, G_\alpha\rangle \leq \mathbf{N}_T(T_{\alpha\beta}) \neq T$, which is a contradiction.

Thus, $T = \mathrm{PSL}(2,p)$ for a prime $p \geq 5$. Then $G = \mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$, and a Sylow 2-subgroup of G is a dihedral group.

If $|G_\alpha| = 2$, then $G_\alpha \cong \mathbb{Z}_2$, $G = T = \mathrm{PSL}(2,p)$ with $p \equiv \pm 3 \pmod 8$, and Γ is isomorphic to a coset graph in Example 3.7 (1).

Assume that $|G_\alpha| = 4$. Then, by Lemma 3.3, G_α is not cyclic, so $G_\alpha \cong \mathbb{Z}_2^2$. Hence either $G = T = \mathrm{PSL}(2,p)$ with $p \equiv \pm 7 \pmod{16}$, or $G = \mathrm{PGL}(2,p)$ with $p \equiv \pm 3 \pmod 8$. For the former case, Γ is isomorphic to a coset graph in Example 3.7 (2). The later case implies that $T_\alpha \cong \mathbb{Z}_2$ or $\mathbb{Z}_2^2$ depending on T is or not transitive on V, and so Γ is isomorphic to a coset graph in Example 3.7 (1) or 3.8 (1), respectively.

Finally, assume that $G_\alpha = \langle a\rangle{:}\langle b\rangle \cong \mathrm{D}_{2^e}$ for $e \geq 3$. Let $\beta \in \Gamma(\alpha)$ with $G_\alpha \neq G_\beta$. Then $G_{\alpha\beta}$ has index 2 in G_α. If $G_{\alpha\beta}$ contains a cyclic subgroup Z with $|Z| \geq 4$, then Z is characteristic in both G_α and $G_{\alpha\beta}$, which contradicts with Lemma 3.3. Thus $G_{\alpha\beta} \cong \mathbb{Z}_2^2$ and $G_\alpha \cong \mathrm{D}_8$. Suppose that $G_\alpha \neq T_\alpha$. Then $|T_\alpha| = 4$, $G = \mathrm{PGL}(2,p)$, and T is transitive on V. Since T is not regular, $T_{\alpha\beta} \cong \mathbb{Z}_2$, and so $G_{\alpha\beta} \not\leq T$. Thus $\mathbf{N}_G(G_{\alpha\beta}) \cong \mathrm{D}_8$ by [3], so $\mathbf{N}_G(G_{\alpha\beta}) = G_\alpha$. Then there are no $x \in \mathbf{N}_G(G_\alpha)$ and $y \in \mathbf{N}_G(G_{\alpha\beta})$ such that $\langle G_\alpha, x, y\rangle = G$, a contradiction.

Therefore, $G_\alpha = T_\alpha \cong \mathrm{D}_8$. Then either $G = T = \mathrm{PSL}(2,p)$ with $p \equiv \pm 15 \pmod{32}$ and Γ is isomorphic to a coset graph in Example 3.7 (3), or $G = \mathrm{PGL}(2,p)$ with $p \equiv \pm 7 \pmod{16}$ and Γ is isomorphic to a coset graph in Example 3.8 (2). $\square$

By Lemmas 6.3, 6.4 and their proofs, the next result determines some connected cubic Cayley graphs of square-free order which have insoluble automorphism groups.

Corollary 6.5. *Assume that* $T := \mathrm{soc}(G) = \mathrm{PSL}(2,p)$ *for a prime* $p > 5$. *Then* G *contains no regular subgroups unless:*

(1) $G = \mathrm{PGL}(2,7)$, *G has a regular subgroup* $R \cong \mathrm{D}_{14}$, $\mathbf{N}_G(R) = R{:}\mathbb{Z}_3$ *and* Γ *is constructed as in Example* 3.6 (2)*;*
(2) $G = \mathrm{PGL}(2,7)$, *G has a regular subgroup* $R \cong \mathbb{Z}_7{:}\mathbb{Z}_6$, $\mathbf{N}_G(R) = R$ *and* Γ *is constructed as in Example* 3.8 (2)*;*
(3) $G = \mathrm{PGL}(2,11)$, *G has a regular subgroup* $R \cong \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$, $\mathbf{N}_G(R) = R$ *and* Γ *is constructed as in Example* 3.6 (1)*;*
(4) $G = \mathrm{PGL}(2,23)$, *G has a regular subgroup* $R \cong \mathbb{Z}_{23}{:}\mathbb{Z}_{22}$, $\mathbf{N}_G(R) = R$ *and* Γ *is constructed as in Example* 3.6 (2).

Proof. By Lemmas 6.3 and 6.4, T_α (or G_α) and Γ are known and listed as follows:

T_α	G_α	Γ	p
S_3		3.5 (1)	$5, 11$
D_{12}	D_{12}	3.5 (2), 3.6 (1)	$5, 7, 11, 23$
S_4	S_4	3.5 (3), 3.6 (2)	$7, 23, 47$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	3.7 (1)	none
	$\mathbb{Z}_2^2$	3.7 (1)-(2), 3.8 (1)	7
D_8	D_8	3.7 (3), 3.8 (2)	7

Suppose that G has a regular subgroup R. Then Γ is a Cayley graph and, since $|G : T| \leq 2$, we know that T contains a subgroup of order $\frac{|R|}{2}$. Thus T has a subgroup of square-free order $\frac{|T|}{|T_\alpha|}$ or $\frac{|T|}{2|T_\alpha|}$, and such a subgroup has order divided by p as T_α is a $\{2,3\}$-group. Checking the subgroups of T (see [13, 8.27]), we conclude that $p+1$ divides $|T_\alpha|$ or $2|T_\alpha|$. It follows that all possible p are listed at the last column of the above table. If $p = 5$ then Γ is a 2-arc-transitive graph, and so Γ is the Petersen graph, which is not a Cayley graph. If $p = 47$ then $T_\alpha = G_\alpha \cong \mathrm{S}_4$ and Γ is constructed as in Example 3.5 (3); however, $G = T$ has no a subgroup of order $47 \cdot 46$.

Assume that $p = 7$. Then $G_\alpha \cong \mathrm{D}_{12}$, S_4, $\mathbb{Z}_2^2$ or D_8, and Γ is respectively constructed as in Example 3.5 (2), Example 3.6 (2), Example 3.7 (2) or Example 3.8 (2). Note that G has neither subgroups isomorphic to D_{12} and of square-free index, nor subgroups of order $\frac{|G|}{4}$. Then one of items (1) and (2) occurs.

Assume that $p = 11$. Then Γ is a 2-arc-transitive cubic graph of order 110. By [6], such a graph is isomorphic to a bipartite graph. It follows that T is not transitive on the vertices of Γ. Thus item (3) follows.

Finally, let $p = 23$. Then Γ is constructed as in Example 3.5 (2) or Example 3.6 (2). In this case, by the Atlas [8], G has no subgroups of order $\frac{|G|}{12}$, and then (4) follows. □

Now we can determine the structure of G in the general case.

Let M be the soluble radical of G and let $G^{(\infty)}$ be the smallest normal subgroup of G such that $G/G^{(\infty)}$ is soluble. By Lemma 6.1, M has odd order and Γ is a cover of the quotient Γ_M, so Γ_M is cubic. Moreover, G/M, viewed as a transitive subgroup of $\mathsf{Aut}\Gamma_M$, has trivial soluble radical. Then, by Lemmas 6.2, 6.3 and 6.4, G/M is almost simple with socle A_6 or $\mathrm{PSL}(2,p)$. Set $\mathrm{soc}(G/M) = Y/M$. Then

$G/Y \cong (G/M)/(Y/M)$ is soluble, so $G^{(\infty)} \leq Y$. Thus $Y = MG^{(\infty)}$, and so $G^{(\infty)}/(M \cap G^{(\infty)}) \cong MG^{(\infty)}/M = Y/M \cong \mathrm{A}_6$ or $\mathrm{PSL}(2,p)$.

On the other hand, $\mathsf{Aut}(M)$ is soluble as M has square-free order. Since $G/\mathbf{C}_G(M) = \mathbf{N}_G(M)/\mathbf{C}_G(M)$ is isomorphic to a subgroup of $\mathsf{Aut}(M)$, we have $G^{(\infty)} \leq \mathbf{C}_G(M)$. Then $M \cap G^{(\infty)}$ is the center of $G^{(\infty)}$. Since M has odd order and 3^3 is not a divisor of $|G|$, we conclude that $M \cap G^{(\infty)} = 1$ by checking the Schur multipliers of A_6 and $\mathrm{PSL}(2,p)$. Then $Y = M{\times}T$, and so $G = (M{\times}T).O$, where $T = G^{(\infty)} = \mathrm{A}_6$ or $\mathrm{PSL}(2,p)$, and O lies in the outer automorphism group $\mathsf{Out}(T)$ of T.

Lemma 6.6. *Assume that G is insoluble. Then one of the following holds:*

(1) *G is almost simple with socle isomorphic to A_6 or $\mathrm{PSL}(2,p)$;*
(2) *Γ is not G-arc-transitive, and $G = T{:}\mathrm{D}_{2m}$ such that $T = \mathrm{PSL}(2,p)$, $G_\alpha = T_\alpha \cong \mathbb{Z}_2^2$ is a Sylow 2-subgroup of T, and $(|T|,m) = 1$; G contains no regular subgroups, and Γ can be constructed as in Construction 4.2.*

Proof. Recall that $G = (M{\times}T).O$, where $T = \mathrm{A}_6$ or $\mathrm{PSL}(2,p)$, and $O \leq \mathsf{Out}(T)$.

If $M = 1$, then (1) follows from Lemmas 6.2, 6.3 and 6.4. Thus we assume next that $M \neq 1$. Then $m = |M| \geq 3$ is odd square-free.

Suppose that T has at most two orbits on V. Then M fixes one T-orbit U. By Lemma 6.1, M is semiregular and of odd square-free order. Then $|M| \,\big|\, |U|$, so $|M| \,\big|\, |T|$, and hence $|M|^2 \,\big|\, |G|$. Since $|V| = |G : G_\alpha|$ is square-free for $\alpha \in U$, we have $|M| \,\big|\, |G_\alpha|$. Note that G_α is either a 2-group or isomorphic to one of S_3, D_{12} and S_4. It follows that $|M| = 3$ and $3 \,\big|\, |G_\alpha|$. Thus G_α is 2-transitive on $\Gamma(\alpha)$, and so T_α is transitive on $\Gamma(\alpha)$ as T_α is normal in G_α and T is not semiregular on V; in particular, $3 \,\big|\, |T_\alpha|$. Since $|M| \,\big|\, |V|$ and $|V| = |U|$ or $|2|U|$, we know that 3 divides $|U| = |T : T_\alpha|$. Then $3^2 \,\big|\, |T|$, so $3^3 \,\big|\, |G|$, hence $3^2 \,\big|\, |G_\alpha|$, a contradiction. Thus T has at least 3 orbits on V.

Let K be the kernel of G acting on the T-orbits. Then, by Lemma 4.1, $\Gamma_T \cong \mathbf{C}_l$, $G_\alpha = K_\alpha$ is a 2-group, l is even, and $G/K = \mathrm{D}_l$ acting regularly on T-orbits. Then $M \cong KM/K \cong \mathbb{Z}_{\frac{l}{2}}$ and $l = 2m$. In particular, G is not transitive on the arcs of Γ, and so G/M is not transitive on the arcs of Γ_M. It follows from Lemma 6.4 that $\mathrm{soc}(G/M) \cong \mathrm{PSL}(2,p)$. Since $K \geq T$ and $|G/M| = \frac{|G|}{|M|} = \frac{l|K|}{m} = 2|K|$, we have $G/M \cong \mathrm{PGL}(2,p)$ and $K = T = \mathrm{PSL}(2,p)$. Clearly, $\mathrm{soc}(G/M)$ has two orbits on the vertices of Γ_M. By Lemma 6.4, $(G/M)_\Delta \cong \mathbb{Z}_2^2$ or D_8 for an M-orbit Δ. Let $\alpha \in \Delta$. Then $G_\Delta = MG_\alpha = MT_\alpha$, and so $T_\alpha \cong G_\Delta/M \cong (G/M)_\Delta \cong \mathbb{Z}_2^2$ or D_8. Since $|V| = 2m|T : T_\alpha|$ is square-free, $G_\alpha = T_\alpha$ is a Sylow 2-subgroup of T and m is coprime to $|T|$. Thus, we may assume that $G = M{:}X$ with $T < X \cong \mathrm{PGL}(2,p)$. Then $\mathbf{N}_G(G_\alpha) = M\mathbf{N}_X(T_\alpha)$ and $\mathbf{N}_G(G_{\alpha\beta}) = M\mathbf{N}_X(T_{\alpha\beta})$, where $\beta \in \Gamma(\alpha)$ with $G_\alpha \neq G_\beta$.

Suppose that $G_\alpha = T_\alpha \cong \mathrm{D}_8$. Then $\mathbf{N}_X(G_\alpha) \cong \mathrm{D}_{16}$, $T_{\alpha\beta} = G_{\alpha\beta} \cong \mathbb{Z}_2^2$, $\mathrm{S}_4 \cong \mathbf{N}_X(T_{\alpha\beta}) = \mathbf{N}_T(T_{\alpha\beta})$. Thus $\mathbf{N}_G(G_\alpha) = M{:}\mathrm{D}_{16}$ and $\mathbf{N}_G(G_{\alpha\beta}) = M{\times}\mathrm{S}_4$. Then, for $x \in \mathbf{N}_G(G_\alpha)$ and $y \in \mathbf{N}_G(G_{\alpha\beta})$, either $\langle G_\alpha, x, y\rangle \leq M{\times}T$ or $\langle G_\alpha, x, y\rangle \lesssim \mathrm{PGL}(2,p)$, which contradicts with the connectedness of Γ.

Assume that $G_\alpha = T_\alpha \cong \mathbb{Z}_2^2$. Then $\mathbf{N}_X(G_\alpha) \cong \mathrm{S}_4$ and $\mathbf{N}_X(G_{\alpha\beta}) \cong \mathrm{D}_{2(p-\varepsilon)}$, where $\varepsilon = \pm 1$ such that $4 \,\big\|\, p - \varepsilon$. Note that $G_\alpha \leq \mathbf{N}_X(G_{\alpha\beta})$. Take an involution $b \in \mathbf{N}_X(G_{\alpha\beta})$ with $G_\alpha{:}\langle b\rangle \cong \mathrm{D}_8$. Then $b \in X \setminus T$, $M{:}\langle b\rangle \cong \mathrm{D}_{2m}$, $\mathbf{N}_G(G_\alpha) = (M \times \mathbf{N}_T(T_\alpha))\langle b\rangle$ and $\mathbf{N}_G(G_{\alpha\beta}) = M \times \mathbf{N}_T(T_{\alpha\beta})\langle b\rangle$. Thus Γ can be constructed as in Construction 4.2.

Suppose that G has a regular subgroup. Then, since $|G:MT|=2$, we know that $MT=M\times T$ contains a subgroup of order $\frac{|G:G_\alpha|}{2}=\frac{|MT|}{4}$. Thus T has a subgroup of index 4, which is impossible as T is simple. Then the result follows. $\square$

7. Proof of Theorem 1.1

Let Γ be a connected vertex-transitive cubic graph of square-free order $2n$.

If $\mathsf{Aut}\Gamma$ is insoluble then Γ is known as in parts (2)-(4) of Theorem 1.1 by the argument in Section 6. To complete the proof, we first determine the Cayley graphs which have insoluble automorphism groups. Assume that $\mathsf{Aut}\Gamma$ is insoluble and has a regular subgroup G. By Corollary 6.5 and Lemma 6.6 (2), either

(i) $\mathsf{Aut}\Gamma=\mathrm{PGL}(2,7)$, $G=\langle a\rangle{:}\langle b\rangle\cong \mathrm{D}_{14}$ and $\mathbf{N}_{\mathsf{Aut}\Gamma}(R)=R{:}\mathbb{Z}_3$; or

(ii) $\mathsf{Aut}\Gamma=\mathrm{PGL}(2,p)$, $G=\langle a\rangle{:}\langle b\rangle\cong\mathbb{Z}_p{:}\mathbb{Z}_{p-1}$ and $\mathbf{N}_{\mathsf{Aut}\Gamma}(R)=R$, where $p\in\{7,11,23\}$.

For (i), by Lemma 2.3 (3), $\Gamma\cong \mathrm{Cay}(G,\{ab,a^3b,b\})$ or $\mathrm{Cay}(R,\{ab,a^5b,b\})$. Verified by Magma, $\mathrm{Cay}(R,\{ab,a^3b,b\})\cong\mathrm{Cay}(R,\{ab,a^5b,b\})$, so Line 1 of Table 1 occurs. For (ii), by Lemma 2.5, $\Gamma\cong\mathrm{Cay}(G,\{ab^k,(ab^k)^{-1},b^l\})$ with $a^{b^{\frac{p-1}{2}}}=a^{-1}$, $0<k<\frac{p-1}{2}$ and $(k,\frac{p-1}{2})=1$. Then, verified by Magma, one of Lines 2,4 and 5 of Table 1 occurs.

Now assume that $\mathsf{Aut}\Gamma$ is soluble. Then either Γ is a Cayley graph or a generalized Petersen graph by the argument in Section 5, and hence Γ is known by the argument in Section 2. Assume that $\Gamma\cong\mathbf{P}(n,r)$ is a generalized Petersen graph, where $1\le r<\frac{n}{2}$. If $r^2\equiv 1\,(\mathsf{mod}\ n)$ then, by [11], $\mathsf{Aut}\mathbf{P}(n,r)\cong\mathbb{Z}_n{:}\mathbb{Z}_2^2$ contains a regular subgroup described as in (i), and it is easily shown that $\mathbf{P}(n,r)$ is neither a circulant nor a dihedrant unless $r=1$. For $r^2\equiv -1\,(\mathsf{mod}\ n)$, again by [11], either $\mathsf{Aut}\mathbf{P}(n,r)\cong\mathbb{Z}_n{:}\mathbb{Z}_4$ or $(n,r)=(5,2)$ and Γ is the Petersen graph; moreover, in this case, Γ is not isomorphic to a Cayely graph. Then one of Theorem 1.1 (i) and (vii) occurs.

Therefore, we assume next that $\Gamma=\mathrm{Cay}(G,S)$ is a Cayley graph. If G has a subgroup isomorphic to $\mathbb{Z}_n$ then $G\cong\mathbb{Z}_n{:}\mathbb{Z}_2$, hence $\mathsf{Aut}\Gamma=\bar{G}{:}\mathsf{Aut}(G,S)$ and one of (i)-(v) occurs by Lemmas 2.2-2.5, Corollary 2.4 and the argument in Section 5.

Suppose that G has no subgroups isomorphic to $\mathbb{Z}_n$. By Lemmas 2.1 and 2.5, we may assume that $n>3$, $\Gamma=\mathrm{Cay}(G,S_k)$ and $\mathsf{Aut}(G,S_k)=1$, where $G=\langle c\rangle\times(\langle a\rangle{:}\langle b\rangle)$, $o(b)=2l>2$, $\mathbf{Z}(G)=\langle c\rangle$, $G'=\langle a\rangle$, $a^{b^l}=a^{-1}$, $S_k=\{cab^k,(cab^k)^{-1},b^l\}$, $1<k<l$ and $(k,l)=1$. Then, by the argument in Section 5 , either $\mathsf{Aut}\Gamma=\bar{G}$ or $\mathsf{Aut}\Gamma\cong\mathbb{Z}_n{:}\mathbb{Z}_6\cong\mathrm{D}_{2n}{:}\mathbb{Z}_3$. We next show Theorem 1.1 (vi) occurs, it suffices to show that $\mathsf{Aut}\Gamma\cong\mathbb{Z}_n{:}\mathbb{Z}_6$ if and only if G and k are described as in Line 3 of Table 1.

Suppose that $G=\langle a\rangle{:}\langle b\rangle$ with $o(b)=6$ and $a^b=a^t$ such that $t^2-t+1\equiv 0\,(\mathsf{mod}\ n)$. Let $\Gamma=\mathrm{Cay}(G,S)$, where $S=\{ab,(ab)^{-1},b^3\}$. Define a map

$$\pi: G\to,\ a^ib^j\mapsto\begin{cases} a^{it^2}, & \text{if } j\equiv 0\,(\mathsf{mod}\ 6);\\ a^{it^2-t+1}b^2, & \text{if } j\equiv 2\,(\mathsf{mod}\ 6);\\ a^{it^2-t}b^4, & \text{if } j\equiv 4\,(\mathsf{mod}\ 6);\\ a^{-it}b^5, & \text{if } j\equiv 1\,(\mathsf{mod}\ 6);\\ a^{-it+1}b, & \text{if } j\equiv 3\,(\mathsf{mod}\ 6);\\ a^{-it-t+1}b^3, & \text{if } j\equiv 5\,(\mathsf{mod}\ 6).\end{cases}$$

It is easily shown π is an automorphism of Γ and fixes the vertex 1. Note that all Cayley graphs with insoluble automorphism groups are known, whose order is either

42 or not divisible 3. If $|G| = 42$ then, verified by Magma, $\mathsf{Aut}\Gamma$ is soluble and has order 126. Thus we conclude that $\mathsf{Aut}\Gamma$ is soluble. By the argument in Section 5, we conclude that $\mathsf{Aut}\Gamma \cong \mathbb{Z}_n{:}\mathbb{Z}_6$.

Suppose now that $\mathsf{Aut}\Gamma \cong \mathbb{Z}_n{:}\mathbb{Z}_6$. Then $\mathsf{Aut}\Gamma$ has a unique $\{2,3\}'$-Hall subgroup L. Clearly, L is cyclic and normal in $\mathsf{Aut}\Gamma$. Consider the subgroup $X := L\bar{G}$ of $\mathsf{Aut}\Gamma$. Since X is transitive on the vertices of Γ, we have $X = \bar{G}X_\alpha$ for some vertex α. Then $\frac{|L||G|}{|L\cap\bar{G}|} = |L\bar{G}| = |X| = |G||X_\alpha| = |G|$ or $3|G|$, yielding $L < \bar{G}$. Thus L is a cyclic normal subgroup of $\bar{G}$. Let N be the Fitting subgroup of $\bar{G}$. Then $L \le N$. Since $\bar{G}$ has square-free order, N is cyclic. It is easily shown that $N = \langle\bar{c}\rangle\times\langle\bar{a}\rangle$. Then $2l = |\bar{G}{:}N|$ divides $|\bar{G}{:}L|$, so $|\bar{G}{:}L| \ge 2l \ge 6$. Note that L is a $\{2,3\}'$-Hall subgroup of $\bar{G}$. Thus $|\bar{G}{:}L|$ divides 6, and so $2l$ divides 6. Thus $2l = 6$ as $l > 1$, and hence $L = N$. Since $0 < k < l = 3$, we have $k = 1$ or 2.

Consider the normal quotient graph Γ_N. We know that $\Gamma_N \cong \mathrm{Cay}(\langle b\rangle, \{b^k, b^{-k}, b^3\})$. Then either $\Gamma_N \cong \mathsf{K}_{3,3}$ for $k = 1$, or $\Gamma_N \cong \mathbf{P}(3,1)$ for $k = 2$. Since N is normal in $\mathsf{Aut}\Gamma$ and Γ is arc-transitive, Γ_N is also arc-transitive. It follows that $k = 1$.

By Lemma 5.3, $\mathsf{Aut}\Gamma$ has a normal regular subgroup $R \cong \mathrm{D}_{2n}$. Note that each Sylow 2-subgroup of $\mathsf{Aut}\Gamma \cong \mathbb{Z}_n{:}\mathbb{Z}_6$ has order 2. It follows that all involutions in $\mathsf{Aut}\Gamma$ are conjugate. Thus we may choose R such that $\bar{b}^3 \in R$. Recalling $L = N = \langle\bar{c}, \bar{a}\rangle$ is the $\{2,3\}'$-Hall subgroup of $\mathsf{Aut}\Gamma$, we have $N = \langle\bar{c}, \bar{a}\rangle < R$. Then $\bar{c}^{\bar{b}^3} = \bar{c}^{-1}$, yielding $o(c) = o(\bar{c}) = 1$ as $\bar{c}\bar{b} = \bar{b}\bar{c}$. Thus $o(a) = \frac{n}{3}$ and $\bar{G} \cong G = \langle a, b\rangle$ has trivial center. Moreover, $R = \langle\bar{a}z, \bar{b}^3\rangle$ for some z with $o(z) = 3$ and $z\bar{a} = \bar{a}z$. It is easily shown that $\langle\bar{a}z\rangle \cap \langle\bar{b}\rangle \le \mathbf{Z}(\bar{G})$. Then $\langle\bar{a}z\rangle \cap \langle\bar{b}\rangle = 1$, and so $\mathsf{Aut}\Gamma = \langle\bar{a}z\rangle{:}\langle\bar{b}\rangle = R{:}\langle\bar{b}^2\rangle$.

Assume that $\theta \in \mathsf{Aut}\Gamma$ has order 3. Note that $\mathsf{Aut}\Gamma$ has an abelian Sylow 3-subgroup $\langle z, \bar{b}^2\rangle$. Then $\theta \in \langle z, \bar{b}^2\rangle^{\bar{a}^i}$ for some i. Assume further that θ fixes the vertex 1 of Γ. Then, replacing z by z^{-1} if necessary, we may set $\theta = z\bar{g}$ for $g = a^{-i}b^{\pm 2}a^i$. Thus $1 = 1^\theta = 1^z g$, and so $1^z = g^{-1}$. Since $z\bar{g} = \bar{g}z$, we have $1 = 1^\theta = 1^{\bar{g}z} = g^z$, and so $1^{z^{-1}} = g$. Let $a^b = a^r$ for some r coprime to $\frac{n}{3}$. Then $r^6 \equiv 1 \,(\mathsf{mod}\ \frac{n}{3})$ and $r^3 \equiv -1 \,(\mathsf{mod}\ \frac{n}{3})$. Thus $(b^3)^\theta = 1^{\bar{b}^3 z\bar{g}} = 1^{z^{-1}\bar{b}^3\bar{g}} = gb^3g = a^{-i(r+1)^2}b$ or $a^{-i(r^2-1)^2}b^{-1}$. Since Γ is arc-transitive, $\langle\theta\rangle$ is transitive on $\{ab, (ab)^{-1}, b^3\}$. Then $(b^3)^\theta = ab$ or $(ab)^{-1}$. Therefore, either $a^{-i(r+1)^2}b = ab$ or $a^{-i(r^2-1)^2}b^{-1} = (ab)^{-1} = a^{-r}b^{-1}$. Then $-i(r+1)^2 \equiv 1 \,(\mathsf{mod}\ \frac{n}{3})$ or $-i(r^2-1)^2 \equiv -r \,(\mathsf{mod}\ \frac{n}{3})$, it follows that $(r+1, \frac{n}{3}) = 1$. Since $r^3 \equiv -1 \,(\mathsf{mod}\ \frac{n}{3})$, we have $r^2 - r + 1 \equiv 0 \,(\mathsf{mod}\ \frac{n}{3})$.

Since $\langle\bar{a}z\rangle$ is normal in $\mathsf{Aut}\Gamma$, we set $(\bar{a}z)^{\bar{b}} = (\bar{a}z)^t$ for some t coprime to n. Then $(\bar{a}z)^{t^3} = (\bar{a}z)^{\bar{b}^3} = \bar{a}^{\bar{b}^3}z^{\bar{b}^3} = \bar{a}^{-1}z^{-1} = (\bar{a}z)^{-1}$, so $t^3 \equiv -1 \,(\mathsf{mod}\ n)$, hence $t^3 \equiv -1 \,(\mathsf{mod}\ \frac{n}{3})$. Note that $\bar{a}^t z^t = (\bar{a}z)^t = (\bar{a}z)^{\bar{b}} = \bar{a}^{\bar{b}} z^{\bar{b}} = \bar{a}^r z^{\bar{b}^4\bar{b}^3} = \bar{a}^r z^{-1}$. It follows that $t \equiv r \,(\mathsf{mod}\ \frac{n}{3})$ and $t \equiv -1 \,(\mathsf{mod}\ 3)$. Since $t \equiv -1 \,(\mathsf{mod}\ 3)$, we know that $3 \,\big|\, (t^2 - t + 1)$. Since $r^2 - r + 1 \equiv 0 \,(\mathsf{mod}\ \frac{n}{3})$ and $t \equiv r \,(\mathsf{mod}\ \frac{n}{3})$, we have $t^2 - t + 1 \equiv 0 \,(\mathsf{mod}\ \frac{n}{3})$. Then, since $(3, \frac{n}{3}) = 1$, we have $t^2 - t + 1 \equiv 0 \,(\mathsf{mod}\ n)$. Thus Theorem 1.1 (vi) occurs. This completes the proof.

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