ON n-SUMS IN AN ABELIAN GROUP

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Abstract. Let G be an additive abelian group, let $n \geq 1$ be an integer, let S be a sequence over G of length $|S| \geq n+1$, and let $\mathsf{h}(S)$ denote the maximum multiplicity of a term in S. Let $\Sigma_n(S)$ denote the set consisting of all elements in G which can be expressed as the sum of terms from a subsequence of S having length n. In this paper, we prove that either $ng \in \Sigma_n(S)$ for every term g in S whose multiplicity is at least $\mathsf{h}(S)-1$ or $|\Sigma_n(S)| \geq \min\{n+1, |S|-n+|\operatorname{supp}(S)|-1\}$, where $|\operatorname{supp}(S)|$ denotes the number of distinct terms that occur in S. When G is finite cyclic and $n = |G|$, this confirms a conjecture of Y. O. Hamidoune from 2003.

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1. Introduction

Let G be an additive abelian group, let S be a sequence of elements from G, and let $|S|$ denote the length of S. For an integer $n \geq 1$, let $\Sigma_n(S)$ denote the set that consists of all elements in G which can be expressed as the sum of terms from a subsequence of S having length n. The famous Erdős-Ginzburg-Ziv Theorem asserts that, if G is finite and $|S| \geq 2|G|-1$, then $0 \in \Sigma_{|G|}(S)$. This theorem has attracted a lot of attention, and $\Sigma_{|G|}(S)$ has been studied by many authors.

In 1967, Mann [19] extended this theorem by showing that, if $|G|$ is prime and every term of S has multiplicity at most $|S|-|G|+1$, then $\Sigma_{|G|}(S) = G$. In 1977, Olson [21] generalized Mann's result to any finite abelian group and showed that, if $|S| \geq 2|G|-1$ and each coset $x+H$ contains at most $|S|+1-\frac{|G|}{|H|}$ terms of S, for every subgroup H, then $\sum_{|G|}(S) = G$. In 1995, the first author [9] proved that Olson's result is true with the restriction $|S| \geq 2|G|-1$ replaced by $|S| \geq |G|+\mathsf{D}(G)-1$, where $\mathsf{D}(G)$ is the Davenport constant of G, which is the smallest integer d such that every sequence over G of length at least d has a nonempty zero-sum subsequence. Later, in [17], the restriction $|S| \geq |G|+\mathsf{D}(G)-1$ was further weakened to $|S| \geq |G|+\mathsf{d}^*(G)$, where $\mathsf{d}^*(G) = \sum_{i=1}^{r}(n_i-1)$ when $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $n_1 \mid \ldots \mid n_r$ (see also [15, Exercise 15.4]). (Note, it is well-known and rather trivial that $\mathsf{D}(G) \geq \mathsf{d}^*(G)+1$.)

In 1999, Bollobás and Leader [1] proved that, if $|S| \geq |G|+1$, then either $0 \in \Sigma_{|G|}(S)$ or $|\Sigma_{|G|}(S)| \geq |S|-|G|+1$. They further conjectured that the minimum of $|\Sigma_{|G|}(S)|$, assuming $0 \notin \Sigma_{|G|}(S)$, equals the minimum of $|\Sigma(T)|$, assuming T is zero-sum free and $|T| = |S|-|G|+1$, which was confirmed by the first author and Leader [12] in 2005. In 2003, Y. O. Hamidoune [18] noted that the bounds for $|\Sigma_{|G|}(S)|$, assuming $0 \notin \Sigma_{|G|}(S)$, seemed to only be tight for sequences having few distinct terms. To make this specific, he made the following two conjectures (for cyclic groups).

Conjecture 1.1. *Let G be a finite abelian group and let S be a sequence over G of length $|S| \geq |G| + 1$. Suppose the maximum multiplicity of a term of S is at most $|G| - |\operatorname{supp}(S)| + 2$. Then either*

$$|\Sigma_{|G|}(S)| \geq |S| - |G| + |\operatorname{supp}(S)| - 1$$

or there exists a nontrivial subgroup $H \leq G$ with $H \subset \Sigma_{|G|}(S)$, where $|\operatorname{supp}(S)|$ denotes the number of distinct terms in S.

Conjecture 1.2. *Let G be a finite abelian group and let S be a sequence over G of length $|S| \geq |G| + 1$. If $0 \notin \Sigma_{|G|}(S)$, then*

$$|\Sigma_{|G|}(S)| \geq |S| - |G| + |\operatorname{supp}(S)| - 1,$$

where $|\operatorname{supp}(S)|$ denotes the number of distinct terms in S.

In 2005, Conjecture 1.1 was resolved by the second author [15]. Later, it was pointed out by DeVos, Goddyn and Mohar [6] that a similar method actually yields the following stronger generalization of Conjecture 1.1.

Theorem 1.3. *Let G be an abelian group, let $n \geq 1$ be an integer, and let S be a sequence over G of length $|S| \geq n + 1$. Suppose the maximum multiplicity of a term of S is at most $n - |\operatorname{supp}(S)| + 2$. Then either*

$$|\Sigma_n(S)| \geq \min\{n + 1, |S| - n + |\operatorname{supp}(S)| - 1\}$$

or there exists a nontrivial subgroup $H \leq G$ with $ng + H \subset \Sigma_n(S)$ for some $g \in \operatorname{supp}(S)$, where $|\operatorname{supp}(S)|$ denotes the number of distinct terms in S.

In this paper, we show the following similar result to Theorem 1.3 and confirm Conjecture 1.2 as its corollary.

Theorem 1.4. *Let G be an abelian group, let $n \geq 1$ be an integer, let S be a sequence over G of length $|S| \geq n + 1$, and let $\mathsf{h}(S)$ denote the maximum multiplicity of a term from S. Then either*

$$|\Sigma_n(S)| \geq \min\{\ n + 1,\ |S| - n + |\operatorname{supp}(S)| - 1\}$$

or $ng \in \Sigma_n(S)$ for every $g \in G$ whose multiplicity in S is at least $\mathsf{v}_g(S) \geq \mathsf{h}(S) - 1$, where $|\operatorname{supp}(S)|$ denotes the number of distinct terms in S.

Taking G finite and $n = |G|$ in the above theorem, Conjecture 1.2 clearly follows. For some related papers, we refer to [2, 3, 5, 8, 10, 11, 20, 21, 24].

2. Notation and Preliminaries

Let $\mathbb{N}$ denote the set of positive integers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any two integers $a, b \in \mathbb{N}_0$, we set $[a, b] = \{x \in \mathbb{N}_0 : a \leq x \leq b\}$. Throughout this paper, all abelian groups will be written additively.

Let G be an abelian group and let $\mathcal{F}(G)$ be the free abelian monoid, multiplicatively written, with basis G. The elements of $\mathcal{F}(G)$ are simply finite (unordered) sequences with terms from G, multiplicatively written. We write sequences $S \in \mathcal{F}(G)$ in the form

$$S = \prod_{g \in G} g^{\mathsf{v}_g(S)}, \text{ with } \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ for all } g \in G.$$

We call $\mathsf{v}_g(G)$ the multiplicity of the term g in S and say that S contains g if $\mathsf{v}_g(S) > 0$. Furthermore, S is called square-free if $\mathsf{v}_g(S) \le 1$ for all $g \in G$. The unit element $1 \in \mathcal{F}(G)$ is called the empty sequence. We use $S_1 \mid S$ to denote that the sequence S_1 is a subsequence of S. In such case, SS_1^{-1} denotes the subsequence of S obtained by removing the terms from S_1. Let $S_1, \cdots, S_r$ be subsequences of S. We say $S_1, \cdots, S_r$ are disjoint subsequences if $S_1 \cdot \ldots \cdot S_r \mid S$. If a sequence $S \in \mathcal{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_\ell$, we tacitly assume that $\ell \in \mathbb{N}_0$ and $g_1, \ldots, g_\ell \in G$.

For a sequence

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g \in G} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G),$$

we call

- $|S| = \ell = \sum_{g \in G} \mathsf{v}_g(G) \in \mathbb{N}_0$ the *length* of S,
- $\mathsf{h}(S) = \max\{\mathsf{v}_g(S) : g \in G\} \in [0, |S|]$ the *maximum* of the *multiplicities* of S,
- $\mathrm{supp}(S) = \{g \in G : \mathsf{v}_g(S) > 0\} \subset G$ the *support* of S,
- $\sigma(S) = \sum_{i=1}^{\ell} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$ the *sum* of S.

If $\phi : G \to G'$ is a map, then $\phi(S) = \phi(g_1) \cdot \ldots \cdot \phi(g_\ell) \in \mathcal{F}(G')$ denotes the sequence over G' obtained by applying ϕ to each term of S. Note $|\phi(S)| = |S|$.

For $r \in \mathbb{Z}$, we define

$$\Sigma_r(S) = \{\sigma(S') : \ S' \mid S \ \text{ and } \ |S'| = r\}.$$

Note $\sigma(S') = 0$ when S' is the empty sequence. For $k \in \mathbb{Z}$, define

$$\Sigma_{\ge k}(S) = \bigcup_{r=k}^{\ell} \Sigma_r(S), \quad \Sigma_{\le k}(S) = \bigcup_{r=1}^{k} \Sigma_r(S) \quad \text{ and } \quad \Sigma(S) = \bigcup_{r=1}^{\ell} \Sigma_r(S)$$

and

$$\Sigma^*_{\le k}(S) = \{0\} \cup \Sigma_{\le k}(S) \quad \text{ and } \quad \Sigma^*(S) = \{0\} \cup \Sigma(S).$$

A sequence S is called

- a *zero-sum sequence* if $\sigma(S) = 0$,
- *zero-sum free* if $0 \notin \Sigma(S)$.

Let A and B be two nonempty subsets of G. Define

$$A + B = \{a + b : a \in A, b \in B\}.$$

If $A = \{x\}$ for some $x \in G$, then we simply denote $A + B$ by $x + B$. For any nonempty subset C of G, let $-C = \{-c : c \in C\}$. We say that $g \in G$ is a unique expression element of $A + B$ if there is precisely one pair $(a, b) \in A \times B$ with $a + b = g$. For a nonempty subset $A \subset G$ and a subgroup H of G, we say that A is H-periodic if A is a union of H-cosets. Let $\mathrm{stab}(A)$ denote the stabilizer of A in G, i.e., $\mathrm{stab}(A) = \{g \in G : g + A = A\}$. Then $\mathrm{stab}(A)$ is the maximal subgroup H for which A is H-periodic. The set A is called *periodic* if $\mathrm{stab}(A)$ is nontrivial. We use $\phi_H : G \to G/H$ for the natural homomorphism.

To prove Theorem 1.4, we need some preliminaries, beginning with a result of Scherk [25].

Lemma 2.1. *Let G be an abelian group and let A and B be two finite subsets of G such that $A+B$ contains a unique expression element. Then $|A+B| \geq |A|+|B|-1$.*

By using Lemma 2.1 repeatedly, one can prove the following result of Bovey, Erdős and Niven [4].

Lemma 2.2. *Let S be a zero-sum free sequence over an abelian group and let $S_1, \cdots, S_k$ be disjoint subsequences of S. Then*

$$|\Sigma(S)| \geq \Sigma_{i=1}^{k} |\Sigma(S_i)| \quad \text{with} \quad |\Sigma(S_i)| \geq |S_i| \quad \text{for all } i.$$

We also need the following result, which is the common corollary of two more general additive results: the DeVos-Goddyn-Mohar Theorem and the Partition Theorem (see [16, Chapters 13-14]).

Theorem 2.3. [6, 16] *Let G be an abelian group. If S is a sequence over G, $n \leq |S|$, and $H = \mathrm{stab}(\Sigma_n(S))$, then*

$$|\Sigma_n(S)| \geq \left(\sum_{g\in G/H} \min\{n, \mathsf{v}_g(\phi_H(S))\} - n + 1\right)|H|,$$

where $\mathsf{v}_g(\phi_H(S))$ denotes the multiplicity of the term $g \in G/H$ in the sequence S when its terms have been reduced modulo H.

Lemma 2.4. *Let G be an abelian group, let $n \geq 1$ be an integer, let $S \in \mathcal{F}(G)$ be a sequence over G with*

$$|\Sigma_n(S)| \leq |S| - n,$$

let $H = \mathrm{stab}(\Sigma_n(S))$, and let $\phi_H : G \to G/H$ be the natural homomorphism.

1. *If $\mathsf{h}(S) \leq n$ and $g \in \mathrm{supp}(S)$ is a term with $\mathsf{v}_{\phi_H(g)}(\phi_H(S)) \geq n$, then*

$$\mathsf{v}_{\phi_H(g)}(\phi_H(S)) \geq n + |H|.$$

2. *If $g \in G$ is a term with near maximum multiplicity $\mathsf{v}_g(S) \geq \mathsf{h}(S) - 1$, then*

$$\mathsf{v}_{\phi_H(g)}(\phi_H(S)) \geq n.$$

Moreover, the above inequality is strict if either $\mathsf{h}(S) \leq n$ or $\mathsf{v}_g(S) = \mathsf{h}(S)$.

Proof. Observe that $0 \leq |\Sigma_n(S)| \leq |S| - n$ implies $|S| \geq n$. Applying Theorem 2.3 to $\Sigma_n(S)$, we find that

$$|\Sigma_n(S)| \geq \left(\sum_{g\in G/H} \min\{n, \mathsf{v}_g(\phi_H(S))\} - n + 1\right)|H|. \tag{1}$$

Let $N \geq 0$ denote the number of $g \in G/H$ with $\mathsf{v}_g(\phi_H(S)) \geq n$ and let e denote the number of terms of S not equal modulo H to some $g \in G/H$ with $\mathsf{v}_g(\phi_H(S)) \geq n$. Then (1) can be rewritten as

$$|\Sigma_n(S)| \geq ((N-1)n + e + 1)|H|, \tag{2}$$

and we clearly have

$$|S| \leq \mathsf{h}(S)N|H| + e. \tag{3}$$

If $N = 0$, then $e = |S|$, whence (2) yields $|\Sigma_n(S)| \geq (|S| - n + 1)|H| \geq |S| - n + 1$, contrary to hypothesis. Therefore we may assume

$$N \geq 1.$$

Combining (2), (3) and the hypothesis $|\Sigma_n(S)| \leq |S| - n$ yields

$$((N-1)n + e + 1)|H| \leq |\Sigma_n(S)| \leq |S| - n \leq \mathsf{h}(S)N|H| + e - n. \tag{4}$$

1. Let $x = \mathsf{v}_{\phi_H(g)}(\phi_H(S))$. Then, since $\mathsf{v}_{\phi_H(g)}(\phi_H(S)) \geq n$, we can improve (3) to

$$|S| \leq \mathsf{h}(S)(N-1)|H| + e + x.$$

Thus we can improve (4) to

$$((N-1)n + e + 1)|H| \leq |\Sigma_n(S)| \leq |S| - n \leq \mathsf{h}(S)(N-1)|H| + e + x - n,$$

which rearranges to give

$$\mathsf{v}_{\phi_H(g)}(\phi_H(S)) = x \geq (N-1)|H|(n - \mathsf{h}(S)) + e(|H| - 1) + n + |H|.$$

Since $\mathsf{h}(S) \leq n$, applying the estimates $N \geq 1$ and $e \geq 0$ yields the desired lower bound.

2. If the second conclusion of this lemma is false, then every term of S equal to g is counted by e, i.e.,

$$e \geq \mathsf{v}_g(S) \geq \mathsf{h}(S) - 1.$$

Rearranging (4) and applying the above estimate, we obtain

$$\begin{aligned} 0 &\geq (n - \mathsf{h}(S))N|H| + e(|H| - 1) - n(|H| - 1) + |H| \\ &\geq (n - \mathsf{h}(S))N|H| + (\mathsf{h}(S) - 1)(|H| - 1) - n(|H| - 1) + |H| \\ &= (n - \mathsf{h}(S))(N|H| - |H| + 1) + 1. \end{aligned}$$

Hence, since $N \geq 1$, it follows that $\mathsf{h}(S) \geq n+1$, in which case $\mathsf{v}_{\phi_H(g)}(\phi_H(S)) \geq \mathsf{v}_g(S) \geq \mathsf{h}(S) - 1 \geq n$, a contradiction.

If $\mathsf{h}(S) \leq n$, then part 1 now implies $\mathsf{v}_{\phi_H(g)}(\phi_H(S)) \geq n + |H| \geq n + 1$. On the other hand, if $\mathsf{h}(S) \geq n + 1$ and $\mathsf{v}_g(S) = \mathsf{h}(S)$, then we trivially have $\mathsf{v}_{\phi_H(g)}(\phi_H(S)) \geq \mathsf{v}_g(S) = \mathsf{h}(S) \geq n + 1$, completing the proof. □

The following lemma is crucial in this paper.

Lemma 2.5. *Let G be an abelian group, let $n \geq \lambda \geq 0$ be integers, and let $S = T0^{n-\lambda} \in \mathcal{F}(G)$ be a sequence over G with $|S| \geq n$ and $\mathsf{v}_0(S) \geq \mathsf{h}(S) - 1$. Then either $|\Sigma_n(S)| \geq n + 1$ or*

$$\Sigma_{\geq \lambda}(T) = \Sigma_n(S).$$

Proof. Observe that

$$\Sigma_n(S) = \Sigma_n(T0^{n-\lambda}) = \Sigma_{[\lambda,n]}(T) = \{\sigma(T') : T' \mid T \text{ and } |T'| \in [\lambda, n]\}.$$

Thus $\Sigma_{\geq\lambda}(T) = \Sigma_n(S)$ is trivial unless

$$|T| \geq n + 1,$$

which we now assume. This also shows that $\Sigma_n(S) \subset \Sigma_{\geq\lambda}(T)$, so that it suffices to show $\Sigma_{\geq\lambda}(T) \subset \Sigma_n(S)$. Moreover, we have $|S| \geq |T| \geq n + 1 \geq \lambda + 1$, so that $|T| - \lambda \geq 1$.

Now

$$\Sigma_n(S)=\sigma(S)-\Sigma_{|S|-n}(S)=\sigma(T)-\Sigma_{|T|-\lambda}(S)\qquad\text{and}\qquad\Sigma_{\geq\lambda}(T)=\sigma(T)-\Sigma^*_{\leq|T|-\lambda}(T).$$

Thus to show $\Sigma_{\geq\lambda}(T)\subset\Sigma_n(S)$, it suffices to show

$$\Sigma^*_{\leq|T|-\lambda}(T)\subset\Sigma_{|T|-\lambda}(S),\tag{5}$$

and to show $|\Sigma_n(S)|\geq n+1$, it suffices to show $|\Sigma_{|T|-\lambda}(S)|\geq n+1$. We now assume

$$|\Sigma_{|T|-\lambda}(S)|\leq n=|S|-(|T|-\lambda)\tag{6}$$

and proceed to establish (5).

Let $H\leq G$ denote the stabilizer of $\Sigma_{|T|-\lambda}(S)$. Then, in view of (6) and the hypothesis $\mathsf{v}_0(S)\geq\mathsf{h}(S)-1$, we can apply Lemma 2.4.2 to conclude that

$$\mathsf{v}_0(\phi_H(S))\geq|T|-\lambda.\tag{7}$$

In particular, $\phi_H(T_{G\setminus H})0^{|T|-\lambda}$ is a subsequence of $\phi_H(S)$, where $T_{G\setminus H}\mid T$ denotes the subsequence consisting of all terms from $G\setminus H$. Consequently, since $\Sigma_{|T|-\lambda}(S)$ is H-periodic, we see that, in order to establish (5) (and thus complete the proof), it suffices to show

$$\Sigma^*_{\leq|T|-\lambda}(\phi_H(T_{G\setminus H}))=\Sigma^*_{\leq|T|-\lambda}(\phi_H(T))\subset\Sigma_{|T|-\lambda}(\phi_H(T_{G\setminus H})0^{|T|-\lambda}).$$

Since the above inclusion holds trivially with equality, the proof is complete. □

If $A\subset G$, then we define $\Sigma(A)=\Sigma(S)$ where S is the square-free sequence with $\operatorname{supp}(S)=A$.

Lemma 2.6. *Let S be a subset of an abelian group G with $0\notin\Sigma(S)$. Then*

(1) $|\Sigma(S)|\geq2|S|-1$*;*
(2) *if* $|S|\geq4$*, then* $|\Sigma(S)|\geq2|S|$*;*
(3) *if* $|S|=3$ *and S does not contain exactly one element of order two, then* $|\Sigma(S)|\geq2|S|$.

Proof. 1. and 2. have been proved in [7].

3. If S contains no element of order two, then the result has also been proved in [7]. Now assume that S contains at least two elements of order two. Let $S=\{a,b,c\}$ with $\operatorname{ord}(a)=\operatorname{ord}(b)=2$. If $c=a+b$, then $a+b+c=a+b+a+b=2a+2b=0+0=0$, contradicting that $0\notin\Sigma(S)$. Therefore, $a+b\notin S$. If $a+c=b$, then $a+c+b=2b=0$, likewise a contradiction. Hence, $a+c\notin S$. Similarly, we can prove $b+c\notin S$. Note that $a+b+c\notin\{a,b,c,a+b,b+c,c+a\}$. Therefore, $|\Sigma(S)|=7$ and we are done. □

Lemma 2.7. *Let G be an abelian group and let $S\in\mathcal{F}(G)$ be a zero-sumfree sequence. Then* $|\Sigma(S)|\geq|S|+|\operatorname{supp}(S)|-1$*, and we have strict inequality unless* $|S|\leq2$ *or* $|S|=3$ *with S containing exactly one element of order two.*

Proof. Let S_1 be a square-free subsequence of S with $|S_1|=|\operatorname{supp}(S)|$ and let $S_2=SS_1^{-1}$. Applying Lemma 2.2 to $S=S_1S_2$, we obtain that

$$|\Sigma(S)|\geq|\Sigma(S_1)|+|\Sigma(S_2)|\geq|S_2|+|\Sigma(S_1)|=|S|-|S_1|+|\Sigma(S_1)|.$$

Now the result follows from Lemma 2.6. □

Given subsets $A,\ B \subset G$, we define the restricted sumset to be

$$A\dot{+}B = \{a + b :\ a \in A,\ b \in B,\ a \neq b\}.$$

Lemma 2.8. *Let A be a finite subset of an abelian group with $0 \in A$ and $|A| \geq 3$ and let $H = \langle A\rangle$. If H is an elementary 2-group, also suppose that $A \neq H$. Then $|A\dot{+}A| \geq |A|$.*

Proof. Assume by contradiction that $|A\dot{+}A| \leq |A| - 1$. Clearly, $a + A \setminus \{a\} \subset A\dot{+}A$ for all $a \in A$. Thus

$$a + A \setminus \{a\} = A\dot{+}A = A \setminus \{0\} \tag{8}$$

for all $a \in A$.

If every nonzero element of A has order 2, then H will be an elementary 2-group and $A\dot{+}A = (A + A) \setminus \{0\}$. In this case, (8) implies $A = A + A$, which is easily seen to only be possible if A is itself a subgroup, thus equal to H. As this is contrary to hypothesis, we may now assume there is some $a \in A \setminus \{0\}$ with $\mathrm{ord}(a) \geq 3$.

Now (8) is only possible if

$$A = \{0, a\} \cup B$$

with $B = a + B$ a disjoint $\langle a\rangle$-periodic subset. Since $\langle a\rangle$ is a cyclic group of order at least 3, and since B is $\langle a\rangle$-periodic, it follows that $B\dot{+}B = B + B \subset A\dot{+}A = \{a\} \cup B$ is also $\langle a\rangle$-periodic. Thus $B + B = B$, which is only possible if B is a subgroup of G or the empty set. Since $0 \notin B$, the former is not possible, and since $|A| \geq 3$, the latter is also not possible, a concluding contradiction. □

Lemma 2.9. *Let A be a finite subset of an abelian group with $0 \in A$ and $|A| \geq 4$ and let $H = \langle A\rangle$. Suppose $|A| \leq |H| - 1$ with strict inequality if H is an elementary 2-group. Then $|A\dot{+}A| \geq |A| + 1$ or $A = L \cup (a + L)$ for some cardinality two subgroup $L \leq G$ and $a \in G$.*

Proof. Assume by contradiction that $|A\dot{+}A| \leq |A|$. By Lemma 2.8, we have

$$|A\dot{+}A| = |A|.$$

Clearly, $a + A \setminus \{a\} \subset A\dot{+}A$ for all $a \in A$. Thus

$$a + A \setminus \{a\} \subset A\dot{+}A = (A \setminus \{0\}) \cup \{b\} \tag{9}$$

for all $a \in A$ and some $b \notin A \setminus \{0\}$.

If every nonzero element of A has order 2, then H will be an elementary 2-group and $A\dot{+}A = (A + A) \setminus \{0\}$. In this case, (9) implies $A + A = A \cup \{b\}$, which, in view of $|A| \geq 3$, is only possible if A is itself a subgroup or a subgroup with at most one element removed (being a simple consequence of Kneser's Theorem [16, Chapter 6]). Hence $|A| \geq |H| - 1$, contrary to hypothesis, and we may now assume there is some $a \in A \setminus \{0\}$ with $\mathrm{ord}(a) \geq 3$. Let $K = \langle a\rangle$.

Now (9) is only possible if

$$A = \{0, a\} \cup B \cup B'$$

with $B = B + a$ a disjoint K-periodic subset and B' either empty or a disjoint arithmetic progression with difference a whose last term is $b - a$. Since $\mathrm{ord}(a) \geq 3$, K is a cyclic group of order at least 3.

Suppose B is nonempty. Then, since B is K-periodic with K a cyclic group of order $|K| \geq 3$, it follows that $A + B = A\dot{+}B \subset A\dot{+}A = (A \setminus \{0\}) \cup \{b\}$. Since $A + B$ is K-periodic, it must be contained in the maximal K-periodic subset of $(A\setminus\{0\})\cup\{b\}$. We consider two cases depending on whether $b=0$ or $b\neq 0$.

If $b=0$, then $(A\setminus\{0\})\cup\{b\}=A$. In this case, since $|\phi_K(A+B)|\geq|\phi_K(A)|$, we see that the only way $A+B$ can be contained in the maximal K-periodic subset of $A=(A\setminus\{0\})\cup\{b\}$ is if A is itself K-periodic with K cyclic of order $|K|\geq 3$. It follows that $A+A=A\dot{+}A=(A\setminus\{0\})\cup\{b\}=A$, implying that A is itself a subgroup, thus equal to H, which is contrary to hypothesis.

If $b\neq 0$, then $0,\ a\in A\cap K$ ensures that K is a K-coset that intersects $(A\setminus\{0\})\cup\{b\}$ but which is not contained in $(A\setminus\{0\})\cup\{b\}$. Consequently, the maximal K-periodic subset of $(A\setminus\{0\})\cup\{b\}$ is contained in $(A+K)\setminus K$, and thus has size at most $|\phi_K(A)|-1$. But this makes it impossible for $A+B$ to be contained in this maximal K-periodic subset in view of $|\phi_K(A+B)|\geq|\phi_K(A)|$. So we may now assume B is empty.

Since B is empty and $|A|\geq 4$, we have

$$A=\{0,a\}\cup B'=\{0,a\}\cup\{x,x+a,\ldots,x+ta\},$$

for some $x\in G$, where $t=|A|-3\geq 1$ and $b=x+(t+1)a$. Thus

$$\begin{aligned} A\dot{+}A &= \{a\}\cup\{x,x+a,\ldots,x+(t+1)a\}\cup\{2x+a,2x+2a,\ldots,2x+(2t-1)a\} && (10)\\ &= \{a\}\cup\{x,x+a,\ldots,x+ta,x+(t+1)a\}, && (11)\end{aligned}$$

with the latter equality from (9) and the elements listed in (11) distinct.

Since $1\leq t\leq 2t-1$, it follows that the element $2x+ta$, from the third set in (10), must also lie in the set $\{a\}\cup\{x,x+a,\ldots,x+(t+1)a\}$ from (11). If $2x+ta=x+ja$ for some $j\in[0,t]$, then $0=x+(t-j)a\in\{x,x+a,\ldots,x+ta\}$, contradicting that these are all elements of A distinct from 0 and a. If $2x+ta=x+(t+1)a$, then this implies $x=a$, contradicting that $x,\ a\in A$ are distinct elements of A. Therefore the only remaining possibility is that

$$2x+ta=a. \tag{12}$$

Suppose $|A|\geq 5$, which is equivalent to assuming $t\geq 2$. In this case, (10) and (12) ensure that $2a=2x+(t+1)a\in A\dot{+}A$. Comparing this with (11), we see that $2a\in A\dot{+}A$ forces $x=2a$, which combined with (12) yields $(t+3)a=0$. Since $x=2a$ and $(t+3)a=0$, it follows that $A=\{0,a,x,x+a,\ldots,x+ta\}=\{0,a,2a,\ldots,(t+2)a\}=H$, contrary to hypothesis. So it only remains to consider the case $|A|=4$.

For $|A|=4$, we have $A=\{0,a\}\cup\{x,x+a\}$. In this case,

$$A\dot{+}A=\{a\}\cup\{x,x+a,x+2a\}\cup\{2x+a\}.$$

Since $A=\{0,a\}\cup\{x,x+a\}$ are the distinct elements of A with $\mathrm{ord}(a)\geq 3$, it is easily verified that the elements $\{x,x+a,x+2a\}$ are distinct from each other as well as from a and $2x+a$. Thus $|A\dot{+}A|\geq 5=|A|+1$ follows unless $a=2x+a$. However, if $a=2x+a$, then $A=\{0,x\}\cup(a+\{0,x\})$ with $\{0,x\}=L\leq G$ a subgroup of order two, also as desired. □

Note that Lemmas 2.8 and 2.9 both may be paraphrased as concluding that either $|A\dot{+}A|$ is large or A is a large subset of a periodic subset. Unlike the case of ordinary sumsets, this latter conclusion does not force $A+A$ to be itself periodic. As yet, there is no Kneser-type extension of the Erdős-Heilbronn Conjecture to an arbitrary abelian group (see [16, Chapter 22]). Lemmas 2.8 and 2.9 may be viewed as the first easily verified cases in whatever this extension should be.

3. Proof of Theorem 1.4

Proof of Theorem 1.4. Assume by contradiction that we have some $g \in G$ with $\mathsf{v}_g(S) \geq \mathsf{h}(S) - 1$ and $ng \notin \Sigma_n(S)$. Note that this theorem is translation invariant, so we may assume that $g = 0$. Hence

$$0 = n0 \notin \Sigma_n(S) \qquad \text{and} \qquad \mathsf{v}_0(S) \geq \mathsf{h}(S) - 1.$$

If $\mathsf{v}_0(S) \geq n$, then $0 = n0 \in \Sigma_n(S)$ holds trivially, contrary to assumption. So we may assume that

$$\mathsf{v}_0(S) = n - \lambda \quad \text{for some } \lambda \in [1, n].$$

Let

$$S = 0^{n-\lambda}T$$

with $0 \nmid T$. We need to show

$$|\Sigma_n(S)| \geq \min\{\, n + 1,\ |S| - n + |\operatorname{supp}(S)| - 1\}.$$

Assume by contradiction that

$$|\Sigma_n(S)| \leq n.$$

Then, by Lemma 2.5,

$$\Sigma_{\geq\lambda}(T) = \Sigma_n(S). \tag{13}$$

So it suffices to prove that

$$|\Sigma_{\geq\lambda}(T)| \geq |S| - n + |\operatorname{supp}(S)| - 1.$$

Let T_0 be a maximal (in length) subsequence of T with $\sigma(T_0) = 0$ (T_0 is the empty sequence if T is zero-sum free). Since $0 \notin \Sigma_n(S) = \Sigma_{\geq\lambda}(T)$, we have

$$|T_0| \leq \lambda - 1.$$

Let $T_1 = TT_0^{-1}$, so

$$T = T_0T_1 \quad \text{with} \quad |T_1| = |T| - |T_0| \geq |T| - \lambda + 1 = |S| - n + 1. \tag{14}$$

Then, in view of the maximality of T_0, it follows that

$$T_1 \text{ is zero-sum free.}$$

Claim 1. $\Big(\operatorname{supp}(T_0) \setminus \operatorname{supp}(T_1)\Big) \cap \Sigma(T_1) = \emptyset.$

Assume to the contrary that $x = \sigma(V_1) \in \operatorname{supp}(T_0) \setminus \operatorname{supp}(T_1)$ for some nontrivial subsequence $V_1 \mid T_1$. Then $|V_1| \geq 2$ (else $x \in \operatorname{supp}(T_1)$, contrary to assumption). Therefore, $T_0x^{-1}V_1$ is a zero-sum subsequence of T of length $|T_0| - 1 + |V_1| > |T_0|$, contradicting the maximality of T_0. This proves Claim 1.

In view of (14) and the hypothesis $|S| \geq n+1$, choose a subsequence V of T_1 with

$$|V| = |S| - n - 1 \tag{15}$$

and let $U = T_1V^{-1}$. Observe that $|U| = |T_1| - |V| = |T| - |T_0| - (|S| - n - 1) = \lambda - |T_0| + 1$, so

$$T_1 = UV \quad \text{with} \quad |U| = \lambda - |T_0| + 1 \geq 2. \tag{16}$$

Furthermore, choose V as above so that $|\text{supp}(V) \cap \text{supp}(U)|$ is maximal.

Let

$$A = \{0\} \cup -\Big(\text{supp}(T_0) \setminus \text{supp}(T_1)\Big).$$

Since $\sigma(T_0) = 0$, we have

$$A \subset \{0\} \cup -\text{supp}(T_0) = \Sigma_{\geq |T_0|-1}(T_0). \tag{17}$$

Let

$$B = \sigma(U) + \Sigma^*(V).$$

Since $UV = T_1$, (16) implies that

$$B \subset \Sigma_{\geq \lambda - |T_0|+1}(T_1). \tag{18}$$

Since $T_0 \mid T$ with $0 \nmid T$, and since $V \mid T_1$ with T_1 zero-sum free, we clearly have

$$|A| = |\text{supp}(T_0) \setminus \text{supp}(T_1)| + 1 \qquad \text{and} \qquad |B| = 1 + |\Sigma(V)|. \tag{19}$$

Since $T = T_0T_1$, (17) and (18) imply that

$$A + B \subset \Sigma_{\geq \lambda}(T). \tag{20}$$

Let

$$C = \Sigma_{|U|-1}(U) = \sigma(U) - \text{supp}(U).$$

Then

$$|C| = |\text{supp}(U)|. \tag{21}$$

For any $x \in C$, there is some subsequence $U_x \mid U$ with

$$\sigma(U_x) = x \qquad \text{and} \qquad |U_x| = |U| - 1 = \lambda - |T_0|.$$

Since $\sigma(T_0) = 0$, it follows that $\sigma(U_xT_0) = \sigma(U_x) + \sigma(T_0) = x$ with $|U_xT_0| = |U_x| + |T_0| = \lambda$. Since $U_x \mid U$, $U \mid T_1$ and $T = T_1T_0$, it follows that $U_xT_0 \mid T$. As this is true for any $x \in C$, we conclude that

$$C \subset \Sigma_{\lambda}(T) \subset \Sigma_{\geq \lambda}(T). \tag{22}$$

Claim 2. $|A + B| \geq |A| + |B| - 1$.

Since $0 \in A$ and $\sigma(U) \in B$, we have $\sigma(U) \in A + B$. If $\sigma(U)$ is not a unique expression element of $A + B$, then we deduce that $\sigma(U) = -x + \sigma(U) + \sigma(V_1)$ for some $x \in \text{supp}(T_0) \setminus \text{supp}(T_1)$ and some nontrivial subsequence V_1 of $V \mid T_1$. It follows that $\sigma(V_1) = x$, contrary to Claim 1. Therefore, $\sigma(U)$ is a unique expression element of $A + B$, and Claim 2 follows from Lemma 2.1.

Claim 3. $(A + B) \cap C = \emptyset$.

Assume to the contrary that Claim 3 is false. We have the following possibilities:

(a) $\sigma(U) - x = \sigma(U) + \sigma(V_1)$ with $x \in \operatorname{supp}(U)$ and $V_1 \mid V$; or

(b) $\sigma(U) - x = \sigma(U) - z + \sigma(V_1)$ with $x \in \operatorname{supp}(U)$, $z \in \operatorname{supp}(T_0) \setminus \operatorname{supp}(T_1)$ and $V_1 \mid V$.

Possibility (a) implies that $\sigma(xV_1) = 0$. Since $V_1 \mid V$, $T_1 = UV$ and $x \in \operatorname{supp}(U)$, we must have $xV_1 \mid T_1$. But this contradicts that T_1 is zero-sum free. Possibility (b) implies that $\sigma(xV_1) = z \in \operatorname{supp}(T_0) \setminus \operatorname{supp}(T_1)$. As before, $xV_1 \mid T_1$, and now we have a contradiction to Claim 1. This proves Claim 3.

Now, from (20), (22) and Claim 3, (21), Claim 2, (19), Lemma 2.7 applied to $\Sigma(V)$ (note $V \mid T_1$ with T_1 zero-sum free, so V is also zero-sum free), (15) and the inclusion-exclusion principle, $T_1 = UV$, $T = T_1T_0$, $\operatorname{supp}(S) \setminus \{0\} \subset \operatorname{supp}(T)$ (which follows from the definition of T), and the trivial estimate $|\operatorname{supp}(U) \cap \operatorname{supp}(V)| \geq 0$, we obtain

$$\begin{aligned}
|\Sigma_{\geq\lambda}(T)| &\geq |A + B| + |C| \\
&= |A + B| + |\operatorname{supp}(U)| \\
&\geq |A| + |B| - 1 + |\operatorname{supp}(U)| \\
&= |\operatorname{supp}(T_0) \setminus \operatorname{supp}(T_1)| + 1 + |\Sigma(V)| + |\operatorname{supp}(U)| \\
&\geq |\operatorname{supp}(T_0) \setminus \operatorname{supp}(T_1)| + |V| + |\operatorname{supp}(V)| + |\operatorname{supp}(U)| \\
&= |\operatorname{supp}(T_0) \setminus \operatorname{supp}(T_1)| + |S| - n - 1 + |\operatorname{supp}(UV)| + |\operatorname{supp}(U) \cap \operatorname{supp}(V)| \\
&= |S| - n - 1 + |\operatorname{supp}(T_0) \setminus \operatorname{supp}(T_1)| + |\operatorname{supp}(T_1)| + |\operatorname{supp}(U) \cap \operatorname{supp}(V)| \\
&= |S| - n - 1 + |\operatorname{supp}(T)| + |\operatorname{supp}(U) \cap \operatorname{supp}(V)| \\
&\geq |S| - n - 2 + |\operatorname{supp}(S)| + |\operatorname{supp}(U) \cap \operatorname{supp}(V)| \\
&\geq |S| - n - 2 + |\operatorname{supp}(S)|.
\end{aligned}$$

If $|\Sigma_{\geq\lambda}(T)| \geq |S| - n + |\operatorname{supp}(S)| - 1$, then the proof is complete. Otherwise, it forces equality in all estimates used above. In particular,

$$\operatorname{supp}(U) \cap \operatorname{supp}(V) = \emptyset \quad \text{and} \quad |\Sigma(V)| = |V| + |\operatorname{supp}(V)| - 1. \tag{23}$$

Now $\operatorname{supp}(U) \cap \operatorname{supp}(V) = \emptyset$ is only possible, in view of the maximality of $|\operatorname{supp}(U) \cap \operatorname{supp}(V)|$, if

$$V \quad \text{is the empty sequence} \quad \text{or} \quad T_1 = UV \quad \text{is square-free.}$$

If V is empty, then (15) gives $|S| = n + |V| + 1 = n + 1$. Clearly,

$$|\Sigma_n(S)| = |\Sigma_{|S|-1}(S)| = |\sigma(S) - \operatorname{supp}(S)| = |\operatorname{supp}(S)| = |S| - n + |\operatorname{supp}(S)| - 1,$$

and we are done. So we may instead assume

$$|V| \geq 1 \quad \text{and} \quad T_1 = UV \quad \text{is square-free.}$$

The estimate $|\Sigma(V)| = |V| + |\operatorname{supp}(V)| - 1$ from (23) can only hold, according to Lemma 2.7, if

$$|S| - n - 1 = |V| \leq 3, \tag{24}$$

where the first equality follows from (15). This gives us three remaining cases based on the size of $|V| \in [1, 3]$.

If $|V| = |S| - n - 1 = 3$, then (14) ensures that $|T_1| \geq |S| - n + 1 = 5$. Consequently, since $T_1 = UV$ is square-free, we can choose V such that V either contains no element with order two or at least two elements with order two (while still preserving that $|\operatorname{supp}(V) \cap \operatorname{supp}(U)| = 0$ is maximal for the definition of U and V). But now Lemma 2.7 ensures that $|\Sigma(V)| \geq |V| + |\mathrm{supp}(V)|$, contrary to (23). Therefore it remains to consider the cases when

$$2 \leq |V| + 1 = |S| - n \leq 3. \tag{25}$$

Note that $|\Sigma_{\geq\lambda}(T)| = |\sigma(T) - \Sigma^*_{\leq|T|-\lambda}(T)| = |\Sigma^*_{\leq|T|-\lambda}(T)| = |\{0\} \cup \Sigma_{\leq|S|-n}(T)|$ with $|S| - n \in [2, 3]$. It thus suffices to prove that

$$|\{0\} \cup \Sigma_{\leq|S|-n}(T)| \geq |S| - n + |\mathrm{supp}(S)| - 1 \tag{26}$$

in the two remaining cases. Let $D = \{0\} \cup \mathrm{supp}(T_1)$. Since T_1 is square-free and zero-sum free, we have

$$|D| = |T_1| + 1 \qquad \text{and} \qquad D\dot{+}D = \Sigma_{\leq 2}(T_1). \tag{27}$$

Since $0 \notin \mathrm{supp}(T)$ (per definition of T) with $T = T_0T_1$, we have $0 \notin \mathrm{supp}(T_0) \setminus \mathrm{supp}(T_1)$. Since T_1 zero-sum free, we have $0 \notin \Sigma_{\leq 2}(T_1)$. Thus, in view of $T = T_0T_1$ and Claim 1, it follows that $\mathrm{supp}(T_0) \setminus \mathrm{supp}(T_1)$ and $\Sigma_{\leq 2}(T_1)$ are both disjoint subsets of $\Sigma_{\leq 2}(T)$ that do not contain 0. Combining this with (25) and (27), we obtain

$$\begin{aligned} |\{0\} \cup \Sigma_{\leq|S|-n}(T)| \geq |\{0\} \cup \Sigma_{\leq 2}(T)| &\geq 1 + |\mathrm{supp}(T_0) \setminus \mathrm{supp}(T_1)| + |\Sigma_{\leq 2}(T_1)| \\ &= 1 + |\mathrm{supp}(T_0) \setminus \mathrm{supp}(T_1)| + |D\dot{+}D|. \end{aligned} \tag{28}$$

It remains to estimate $|D\dot{+}D|$ using Lemmas 2.8 and 2.9.

Suppose $|S| - n = 2$. Then, in view of (27) and (14), we have $|D| = |T_1| + 1 \geq |S| - n + 2 = 4$. If $\mathrm{supp}(T_1) \cup \{0\} = D = \langle D \rangle$ is an elementary 2 group, then $0 \in \Sigma_3(T_1)$, contradicting that T_1 is zero-sum free. Therefore we may assume otherwise, in which case Lemma 2.8 and (27) together imply $|D\dot{+}D| \geq |D| = |T_1| + 1 \geq |\mathrm{supp}(T_1)| + 1$. Applying this estimate in (28), and recalling that $T = T_0T_1$ with $|\mathrm{supp}(T)| \geq |\mathrm{supp}(S)| - 1$, we obtain

$$\begin{aligned} |\{0\} \cup \Sigma_{\leq|S|-n}(T)| &\geq 1 + |\mathrm{supp}(T_0) \setminus \mathrm{supp}(T_1)| + |\mathrm{supp}(T_1)| + 1 \\ &= 2 + |\mathrm{supp}(T)| \geq 1 + |\mathrm{supp}(S)| = |S| - n + |\mathrm{supp}(S)| - 1. \end{aligned}$$

Thus (26) is established in this case, as desired.

It remains to consider the case when $|S| - n = 3$. Then, in view of (27) and (14), we have $|D| = |T_1| + 1 \geq |S| - n + 2 = 5$. Let $H = \langle D \rangle$. If H is an elementary 2-group, then $|D| \geq 5$ ensures that it must have size $|H| \geq 8$. Consequently, if $|D| = |\mathrm{supp}(T_1) \cup \{0\}| \geq |H| - 1$, then it is easily seen that T_1 will contain a 3-term zero-sum subsequence, contradicting that T_1 is zero-sum free. On the other hand, if H is not an elementary 2-group and $D = H$, then there will be some $a \in D \setminus \{0\} = \mathrm{supp}(T_1)$ with $\mathrm{ord}(a) \geq 3$. Since $\{0\} \cup \mathrm{supp}(T_1) = D = H$ ensures that we also have $-a \in \mathrm{supp}(T_1)$, and since $a \neq -a$ in view of $\mathrm{ord}(a) \geq 3$, it follows that T_1 contains a 2-term zero-sum, again contradicting that T_1 is zero-sum free. Finally, since $|D| \geq 5$, we cannot have $D = L \cup (a + L)$ with $L \leq G$ an order 2 subgroup. As a result, Lemma 2.9 and (27) together imply $|D\dot{+}D| \geq |D| + 1 = |T_1| + 2 \geq |\mathrm{supp}(T_1)| + 2$. Applying this estimate in (28), and recalling that $T = T_0T_1$ with $|\mathrm{supp}(T)| \geq |\mathrm{supp}(S)| - 1$, we obtain

$$\begin{aligned} |\{0\} \cup \Sigma_{\leq|S|-n}(T)| &\geq 1 + |\mathrm{supp}(T_0) \setminus \mathrm{supp}(T_1)| + |\mathrm{supp}(T_1)| + 2 \\ &= 3 + |\mathrm{supp}(T)| \geq 2 + |\mathrm{supp}(S)| = |S| - n + |\mathrm{supp}(S)| - 1. \end{aligned}$$

Thus (26) is established in the final case, completing the proof. □

4. Concluding Remarks

Let G be a finite abelian group with exponent $\exp(G)$. Let S be a sequence over G with $|S| \geq |G|+1$ and $0 \notin \Sigma_{|G|}(S)$. When G is non-cyclic, $|\operatorname{supp}(S)| \leq |S| - |G| + 1$ and $|S| \geq |G| + \exp(G) - 1$, we can get better lower bounds for $|\Sigma_{|G|}(S)|$ than those from Conjecture 1.2 (see Proposition 4.4). We need the following results.

Proposition 4.1. *(Gao and Leader, 2005) Let G be a finite abelian group and let S be a sequence over G with $|S| \geq |G| + 1$ and $0 \notin \Sigma_{|G|}(S)$. Then there is a zero-sum free sequence T over G such that $|T| = |S| - |G| + 1$ and $|\Sigma_{|G|}(S)| \geq |\Sigma(T)|$.*

For every integer $k \in [1, \mathsf{D}(G) - 1]$, let

$$f_G(k) = \min\{|\Sigma(T)| : \ T \in \mathcal{F}(G),\ |T| = k \text{ and } 0 \notin \Sigma(T)\}.$$

Proposition 4.2. *Let G be a finite abelian group that is noncyclic with exponent $\exp(G)$.*

1. *If $k \geq \exp(G)$, then $f_G(k) \geq 2k - 1$. (Olson and White, 1975; Sun, 2007)*
2. *If $k \geq \exp(G) + 1$, then $f_G(k) \geq 3k - 1$. (Gao, Li, Peng and Sun , 2008)*

Proposition 4.3. *(Pixton, 2009) Let G be a finite abelian group and let T be a zero-sum free sequence over G.*

1. *If the rank of $\langle \operatorname{supp}(T) \rangle$ is at least 3, then $|\Sigma(T)| \geq 4|T| - 5$.*
2. *If the rank of $\langle \operatorname{supp}(T) \rangle$ is at least r, then $|\Sigma(T)| \geq 2^r|T| - (r - 1)2^r - 1$.*

Let G be a finite abelian group of rank $r = r(G)$. For every $t \in [1, r]$, define

$$\mathsf{d}_t(G) = \max\{\mathsf{D}(H) : H \leq G, r(H) = t\},$$

where the maximum is taken as H runs over all subgroups of G of rank t.

Proposition 4.4. *Let G be a finite abelian group that is noncyclic, let $r = \mathsf{r}(G)$ be the rank of G, and let S be sequence over G with $|S| \geq |G| + 1$ and $0 \notin \Sigma_{|G|}(S)$.*

1. *If $|S| \geq |G| + \exp(G) - 1$, then $|\Sigma_{|G|}(S)| \geq 2|S| - 2|G| + 1$.*
2. *If $|S| \geq |G| + \exp(G)$, then $|\Sigma_{|G|}(S)| \geq 3|S| - 3|G| + 2$.*
3. *If $|S| \geq |G| + \mathsf{d}_{t-1}(G) - 1$ with $t \in [2, r]$, then $|\Sigma_{|G|}(S)| \geq 2^t|S| - 2^t|G| + (t - 2)2^t - 1$.*
4. *If $|S| \geq |G| + \mathsf{d}_2(G) - 1$, then $|\Sigma_{|G|}(S)| \geq 4|S| - 4|G| - 1$.*

Proof. We only prove Conclusion 3 here. The other three conclusions can be proved in a similar way. By Proposition 4.1, there is a zero-sum free sequence T over G with $|T| = |S| - |G| + 1$ and $|\Sigma_{|G|}(S)| \geq |\Sigma(T)|$. Since $|T| = |S| - |G| + 1 \geq \mathsf{d}_{t-1}(G)$ and T is zero-sum free, the rank of $\langle T \rangle$ is at least t. It follows from Proposition 4.3 that $|\Sigma_{|G|}(S)| \geq |\Sigma(T)| \geq 2^t|T| - (t - 1)2^t - 1 = 2^t(|S| - |G| + 1) - (t - 1)2^t - 1 = 2^t|S| - 2^t|G| - (t - 2)2^t - 1$. □

Given a fixed (and arbitrary) finite abelian group G, it would be very difficult to give a sharp lower bound for $|\Sigma_{|G|}(S)|$ involving $|\operatorname{supp}(S)|$ in general. Indeed, even finding sharp lower bounds when G is not fixed would be difficult, though it would be expected that the improvement be at least quadratic in $|\operatorname{supp}(S)|$, rather than linear. We end this section with the following open problem.

Conjecture 4.5. *Let G be a finite abelian group and let S be a sequence over G with $|S| \geq |G| + 1$ and $0 \notin \Sigma_{|G|}(S)$. Then there is a zero-sum free sequence T over G of length $|T| = |S| - |G| + 1$ such that $|\Sigma_{|G|}(S)| \geq |\Sigma(T)|$ and $|\operatorname{supp}(T)| \geq \min\{|S| - |G| + 1, |\operatorname{supp}(S)| - 1\}$.*

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