On the Enumeration of $(s, s+1, s+2)$-Core Partitions

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Abstract

Anderson established a connection between core partitions and order ideals of certain posets by mapping a partition to its β-set. In this paper, we give a description of the posets $P_{(s,s+1,s+2)}$ whose order ideals correspond to $(s, s+1, s+2)$-core partitions. Using this description, we obtain the number of $(s, s+1, s+2)$-core partitions, the maximum size and the average size of an $(s, s+1, s+2)$-core partition, confirming three conjectures posed by Amdeberhan.

Keywords: core partition, hook length, β-set, poset, order ideal

AMS Subject Classifications: 05A15, 05A17, 06A07

1 Introduction

The objective of this paper is to prove three conjectures of Amdeberhan on $(s, s+1, s+2)$-core partitions.

A partition λ of a positive integer n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that $\lambda_1+\lambda_2+\cdots+\lambda_m = n$. We write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \vdash n$ and we say that n is the size of λ and m is the length of λ. The Young diagram of λ is defined to be an up- and left-justified array of n boxes with λ_i boxes in the i-th row. Each box B in λ determines a hook consisting of the box B itself and boxes directly to the right and directly below B. The hook length of B, denoted $h(B)$, is the number of boxes in the hook of B.

For a partition λ, the β-set of λ, denoted $\beta(\lambda)$, is defined to be the set of hook lengths of the boxes in the first column of λ. For example, Figure 1 illustrates the Young diagram and the hook lengths of a partition $\lambda = (5, 3, 2, 2, 1)$. The β-set of λ

is $\beta(\lambda) = \{9, 6, 4, 3, 1\}$. Notice that a partition λ is uniquely determined by its β-set. Given a decreasing sequence of positive integers $(h_1, h_2, \ldots, h_m)$, it is easily seen that the unique partition λ with $\beta(\lambda) = \{h_1, h_2, \ldots, h_m\}$ is

$$\lambda = (h_1 - (m-1), h_2 - (m-2), \ldots, h_{m-1} - 1, h_m). \tag{1.1}$$

9	7	4	2	1
6	4	1		
4	2			
3	1			
1				

Figure 1: The Young diagram of $\lambda = (5, 3, 2, 2, 1)$.

For a positive integer t, a partition λ is a t-core partition, or simply a t-core, if it contains no box whose hook length is a multiple of t. Let s be a positive integer not equal to t, we say that λ is an (s, t)-core if it is simultaneously an s-core and a t-core. For example, the partition $\lambda = (5, 3, 2, 2, 1)$ in Figure 1 is a $(5, 8)$-core. In general, an $(a_1, a_2, \ldots, a_r)$-core partition can be defined for distinct positive integers $a_1, a_2, \ldots, a_r$. Since a t-core is an s-core if s is a multiple of t, we assume that there is no element in $\{a_1, a_2, \ldots, a_r\}$ that is a multiple of another element.

Let s and t be two coprime positive integers. Anderson [3] showed that the number of (s, t)-core partitions equals $\binom{s+t}{s}/(s+t)$. Ford, Mai and Sze [6] proved that the number of self-conjugate (s, t)-core partitions equals $\binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor}$. Furthermore, Olsson and Stanton [8] proved that there exists a unique (s, t)-core partition with the maximum size $(s^2 - 1)(t^2 - 1)/24$. A simpler proof was provided by Tripathi [12]. Armstrong, Hanusa and Jones [4] conjectured that the average size of an (s, t)-core partition and the average size of a self-conjugate (s, t)-core are both equal to $(s + t + 1)(s - 1)(t - 1)/24$. Stanley and Zanello [11] showed that the average size of an $(s, s+1)$-core equals $\binom{s+1}{3}/2$. Chen, Huang and Wang [5] proved the conjecture for the average size of a self-conjugate (s, t)-core.

Concerning the enumeration of $(s, s+1, s+2)$-core partitions, Amdeberhan [1] posed three conjectures.

Conjecture 1.1 *Let C_k be the k-th Catalan number, that is, $C_k = \frac{1}{k+1}\binom{2k}{k}$. Let s be a positive integer. The number of $(s, s+1, s+2)$-core partitions equals*

$$r(s) = \sum_{k \geq 0} \binom{s}{2k} C_k.$$

Conjecture 1.2 *Let s be a positive integer. The size of the largest $(s,s+1,s+2)$-core partition equals*

$$l(s)=\begin{cases} m\binom{m+1}{3}, & \text{if } s=2m-1,\\ (m+1)\binom{m+1}{3}+\binom{m+2}{3}, & \text{if } s=2m. \end{cases}$$

Conjecture 1.3 *Let s be a positive integer. The sum of the sizes of all $(s,s+1,s+2)$-core partitions equals*

$$h(s)=\sum_{j=0}^{s-2}\binom{j+3}{3}\sum_{i=0}^{\lfloor j/2\rfloor}\binom{j}{2\,i}C_i.$$

Equivalently, the average size of an $(s,s+1,s+2)$-core partition is $\frac{h(s)}{r(s)}$.

Anderson [3] characterized the β-sets of (s,t)-core partitions as order ideals of a poset $P_{(s,t)}$, where

$$P_{(s,t)}=\mathbb{N}^+\setminus\{n\in\mathbb{N}^+ \mid n=k_1s+k_2t \text{ for some } k_1,k_2\in\mathbb{N}\}$$

and $y\geq x$ in $P_{(s,t)}$ if there exist $y=y_0,y_1,y_2,\ldots,y_l=x\in P_{(s,t)}$ such that $y_i-y_{i+1}\in\{s,t\}$. We show that the above characterization can be generalized to $(a_1,a_2,\ldots,a_r)$-core partitions. More precisely, for positive integers $a_1,a_2,\ldots,a_r$, we define

$$P_{(a_1,a_2,\ldots,a_r)}=\mathbb{N}^+\setminus\{n\in\mathbb{N}^+ \mid n=k_1a_1+k_2a_2+\cdots+k_ra_r \text{ for some } k_1,k_2,\ldots,k_r\in\mathbb{N}\},$$

where $y\geq x$ in $P_{(a_1,a_2,\ldots,a_r)}$ if there exist $y=y_0,y_1,y_2,\ldots,y_l=x\in P_{(a_1,a_2,\ldots,a_r)}$ such that $y_i-y_{i+1}\in\{a_1,a_2,\ldots,a_r\}$. It can be shown that β-sets of $(a_1,a_2,\ldots,a_r)$-core partitions are exactly order ideals of the poset $P_{(a_1,a_2,\ldots,a_r)}$. Based on this characterization, we shall prove the above three conjectures.

We note that Conjecture 1.1 was independently proved by Amdeberhan and Leven [2]. In fact, they obtained the generating function for the number $C_s^{(r)}$ of $(s,s+1,\ldots,s+r)$-cores, that is,

$$\sum_{s\geq 0}C_s^{(r)}x^s=\frac{2-2x-A_r(x)-\sqrt{A_r(x)^2-4x^2}}{2x^{r-1}}$$

where

$$A_r(x)=1-x+\frac{x^2-x^{r-1}}{1-x}.$$

2 Proof of Conjecture 1.1

In this section, we show that a partition is an $(a_1,a_2,\ldots,a_r)$-core if and only if its β-set is an order ideal of the poset $P_{(a_1,a_2,\ldots,a_r)}$. We shall use this correspondence to derive a formula for the number of $(s,s+1,s+2)$-core partitions.

Let P be a poset. For two elements x and y in P, we say y covers x if $x < y$ and there exists no element $z \in P$ satisfying $x < z < y$. The Hasse diagram of a finite poset P is a graph whose vertices are the elements of P, whose edges are the cover relations, and such that if y covers x then there is an edge connecting x and y and y is placed above x. An order ideal of P is a subset I such that if any $y \in I$ and $x \leq y$ in P, then $x \in I$. Let $J(P)$ denote the set of order ideals of P. For more details on poset, see Stanley [10].

In the following theorem, Anderson [3] established a correspondence between core partitions and order ideals of a certain poset by mapping a partition to its β-set.

Theorem 2.1 *Let s, t be two coprime positive integers, and let λ be a partition of n. Then λ is an s-core (or (s,t)-core) partition if and only if $\beta(\lambda)$ is an order ideal of P_s (or $P_{(s,t)}$).*

For example, let $s = 3$ and $t = 4$. We can construct all $(3,4)$-core partitions by finding order ideals of $P_{(3,4)}$. It is easily checked that $P_{(3,4)} = \{1, 2, 5\}$ with the partial order $5 > 2$ and $5 > 1$. Hence the order ideals of $P_{(3,4)}$ are $\emptyset$, $\{1\}$, $\{2\}$, $\{2, 1\}$ and $\{5, 2, 1\}$. The corresponding $(3,4)$-core partitions are $\emptyset$, (1), (2), $(1,1)$ and $(3,1,1)$, respectively.

Theorem 2.1 can be extended to $(a_1, a_2, \ldots, a_r)$-core partitions.

Theorem 2.2 *Let $a_1, a_2, \ldots, a_r$ be a sequence of positive integers, and let λ be a partition of n. Then λ is an $(a_1, a_2, \ldots, a_r)$-core if and only if $\beta(\lambda)$ is an order ideal of $P_{(a_1,a_2,\ldots,a_r)}$.*

Proof. Assume that λ is an $(a_1, a_2, \ldots, a_r)$-core, we proceed to prove that $\beta(\lambda)$ is an order ideal of $P_{(a_1,a_2,\ldots,a_r)}$. First, we claim that $\beta(\lambda)$ is a subset of $P_{(a_1,a_2,\ldots,a_r)}$. Otherwise, suppose that h is an element in $\beta(\lambda)$ but it is not contained in $P_{(a_1,a_2,\ldots,a_r)}$. By the definition of $P_{(a_1,a_2,\ldots,a_r)}$, there exist nonnegative integers $k_1, k_2, \ldots, k_r$ such that

$$h = k_1a_1 + k_2a_2 + \cdots + k_ra_r.$$

Without loss of generality, we may assume that $k_1 > 0$. Since λ is an $(a_1, a_2, \ldots, a_r)$-core partition, it is an a_r-core partition. By Theorem 2.1, we see that $\beta(\lambda)$ is an order ideal of P_{a_r}. Since $k_1a_1 + k_2a_2 + \cdots + k_{r-1}a_{r-1} \in P_{a_r}$, it is easily seen that $k_1a_1 + k_2a_2 + \cdots + k_{r-1}a_{r-1} \in \beta(\lambda)$. Now, since λ is an a_{r-1}-core partition, we find that $k_1a_1 + k_2a_2 + \cdots + k_{r-2}a_{r-2} \in \beta(\lambda)$. Continuing the above process, we eventually obtain that $k_1a_1 \in \beta(\lambda)$, contradicting the fact that λ is an a_1-core partition. Thus the claim is proved.

To prove that $\beta(\lambda)$ is an order ideal of $P_{(a_1,a_2,\ldots,a_r)}$, we assume that $y \in \beta(\lambda)$ and x is covered by y in $P_{(a_1,a_2,\ldots,a_r)}$. We need to show that $x \in \beta(\lambda)$. Since y covers x in $P_{(a_1,a_2,\ldots,a_r)}$, there exists $1 \leq i \leq r$ such that $y - x = a_i$. From the fact that $\beta(\lambda)$ is an order ideal of P_{a_i}, we see that $x \in \beta(\lambda)$.

Conversely, assume that λ is a partition such that $\beta(\lambda)$ is an order ideal of $P_{(a_1,a_2,\ldots,a_r)}$. We aim to show that λ is an $(a_1, a_2, \ldots, a_r)$-core partition. We now claim that λ is an

a_1-core partition. By Theorem 2.1, it suffices to prove that $\beta(\lambda)$ is an order ideal of P_{a_1}. Notice that $\beta(\lambda)$ is a subset of P_{a_1} since $P_{(a_1,a_2,\ldots,a_r)} \subseteq P_{a_1}$. To prove that $\beta(\lambda)$ is an order ideal of P_{a_1}, we assume that $y \in \beta(\lambda)$, $x \in P_{a_1}$ and $y - x = a_1$. It remains to show that $x \in \beta(\lambda)$. First, we show that $x \in P_{(a_1,a_2,\ldots,a_r)}$. Otherwise, we assume that there exist nonnegative integers $c_1, c_2, \ldots, c_r$ such that

$$x = y - a_1 = c_1a_1 + c_2a_2 + \cdots + c_ra_r,$$

or equivalently,

$$y = (c_1 + 1)a_1 + c_2a_2 + \cdots + c_ra_r.$$

It follows that $y \notin P_{(a_1,a_2,\ldots,a_r)}$, which contradicts the assumption $y \in P_{(a_1,a_2,\ldots,a_r)}$. So we have $x \in P_{(a_1,a_2,\ldots,a_r)}$. Since $\beta(\lambda)$ is an order ideal of $P_{(a_1,a_2,\ldots,a_r)}$ and $y - x = a_1$, we obtain $x \in \beta(\lambda)$. Thus, $\beta(\lambda)$ is an order ideal of P_{a_1}, which implies that λ is an a_1-core. This proves the claim.

Similarly, it can be shown that λ is an a_i-core for $2 \le i \le r$. Hence λ is an $(a_1, a_2, \ldots, a_r)$-core. This completes the proof. ∎

Theorem 2.2 establishes a correspondence between $(s, s+1, s+2)$-core partitions and order ideals of $P_{(s,s+1,s+2)}$. The following description of $P_{(s,s+1,s+2)}$ can be used to compute the number of order ideals of $P_{(s,s+1,s+2)}$. For convenience, we denote $P_{(s,s+1,s+2)}$ by T_s. Given positive integers $a \le b$, we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$.

Theorem 2.3 *Let $s \ge 3$ be a positive integer. Then T_s is graded of length $\lfloor \frac{s}{2} \rfloor - 1$. More precisely, we have*

$$T_s = B_0 \cup B_1 \cup \cdots \cup B_{\lfloor \frac{s}{2} \rfloor - 1},$$

where $B_k = [1 + k(s+2), (k+1)s - 1]$ denotes the set of the elements with rank k. For $1 \le k \le \lfloor \frac{s}{2} \rfloor - 1$, each element b in B_k covers exactly the three elements $b - s, b - (s+1), b - (s+2)$ in B_{k-1}.

Proof. By the definition of $P_{(s,s+1,s+2)}$, it is easily seen that

$$T_s = P_{(s,s+1,s+2)} = B_0 \cup B_1 \cup \cdots \cup B_{\lfloor \frac{s}{2} \rfloor - 1}.$$

We proceed to show that T_s is graded. Examining the definition of T_s, we see that for each element b in B_k, the possible elements covered by b are $b - s, b - (s+1), b - (s+2)$. Since $b \in B_k = [1 + k(s+2), (k+1)s - 1]$, it is easily checked that each of the elements $b - s, b - (s+1)$ and $b - (s+2)$ is in $B_{k-1} = [1 + (k-1)(s+2), ks - 1]$ for $k \ge 1$. Conversely, either $b + s$ or $b + (s+2)$ is in B_{k+1} for $k < \lfloor \frac{s}{2} \rfloor - 1$, so b must be covered by at least one element in B_{k+1}. Hence T_s is graded of length $\lfloor \frac{s}{2} \rfloor - 1$. This completes the proof. ∎

According to Theorem 2.3, the Hasse diagram of T_s can be easily constructed. For example, Figure 2 illustrates the Hasse diagrams of the posets T_8 and T_9.

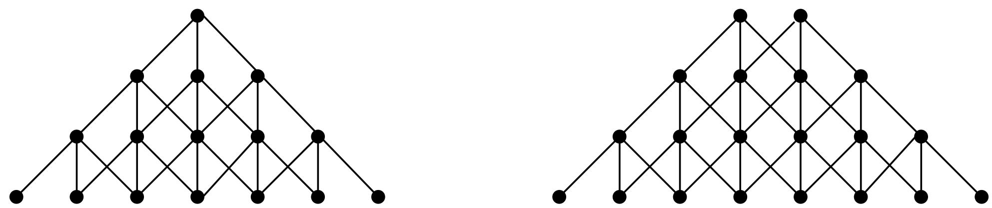

Figure 2: The Hasse diagrams of the posets T_8 and T_9.

Theorem 2.3 enables us to compute the number of order ideals of T_s. To this end, we shall partition $J(T_s)$ according to the smallest missing element of rank 0 in an order ideal. Note that the elements of rank 0 in T_s are just the minimal elements. For $1 \leq i \leq s-1$, let $J_i(T_s)$ denote the set of order ideals of T_s such that i is the smallest missing element of rank 0. Let $J_s(T_s)$ denote the set of order ideals which contain all minimal elements in T_s. Then we can write $J(T_s)$ as

$$J(T_s) = \bigcup_{i=1}^{s} J_i(T_s).$$

Figure 3 gives an illustration of the elements contained in an order ideal in $J_6(T_{12})$. We see that an order ideal $I \in J_6(T_{12})$ must contain the elements labeled by squares, but does not contain any elements represented by open circles. The elements represented by solid circles may or may not appear in I. That is, I can be decomposed into three parts, one is $\{1, 2, 3, 4, 5\}$, one is isomorphic to an order ideal of T_4 and one is isomorphic to an order ideal of T_6.

In general, for $2 \leq i \leq s$ and an order ideal $I \in J_i(T_s)$, we can decompose it into three parts: one is $\{1, 2, \ldots, i-1\}$, one is isomorphic to an order ideal of T_{i-2} and one is isomorphic to an order ideal of T_{s-i}. We shall use this decomposition to prove Conjecture 1.1. Recall that the Motzkin number [7] M_s equals

$$\sum_{k \geq 0} \binom{s}{2k} C_k.$$

By Theorem 2.2, to prove Conjecture 1.1, it suffices to show that the number $r(s)$ of order ideals of T_s equals M_s.

Proof of Conjecture 1.1. It is easily checked that the conclusion is correct when $s = 0, 1, 2$. Suppose now $s \geq 3$. For an order ideal $I \in J_1(T_s)$, I is isomorphic to an order ideal of T_{s-1}. For $2 \leq i \leq s$ and an order ideal $I \in J_i(T_s)$, I can be decomposed into three parts: one is $\{1, 2, \ldots, i-1\}$, one is isomorphic to an order ideal of T_{i-2} and one is isomorphic to an order ideal of T_{s-i}. Hence we have

$$r(s) = r(s-1) + \sum_{i=2}^{s} r(i-2) r(s-i). \tag{2.1}$$

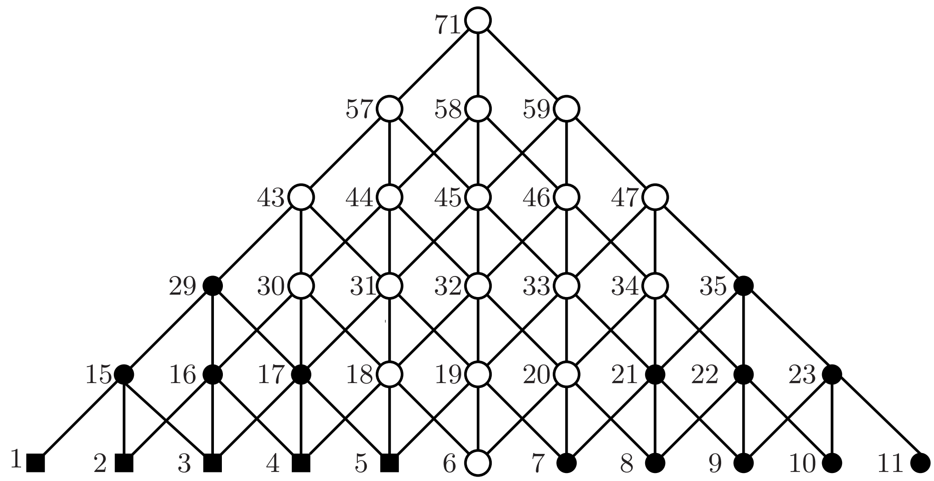

Figure 3: The elements of an order ideal $I \in J_6(T_{12})$.

It is known that the Motzkin number M_s satisfies recurrence relation (2.1) with the same initial conditions as $r(s)$. This yields that $r(s) = M_s$, and hence the proof is complete. ∎

3 Proof of Conjecture 1.2

In this section, we construct a partition κ_s for any integer $s \geq 3$ based on the order ideal consisting of all elements in the poset T_s. It turns out that κ_s is an $(s, s+1, s+2)$-core partition of maximum size. Moreover, we show that if s is even, then κ_s is the unique $(s, s+1, s+2)$-core partition of maximum size, and if s is odd, then there is exactly another $(s, s+1, s+2)$-core partition of maximum size which is the conjugate of κ_s. This leads to a proof of Conjecture 1.2.

We need the following three lemmas to characterize order ideals of T_s corresponding to $(s, s+1, s+2)$-core partitions of maximum size.

Recall that for an order ideal $\beta = \{h_1, h_2, \ldots, h_m\}$ of T_s where the elements are listed in decreasing order, the corresponding $(s, s+1, s+2)$-core partition λ is given by

$$\lambda = (h_1 - (m-1), h_2 - (m-2), \ldots, h_m),$$

whose size is given by

$$|\lambda| = \sum_{i=1}^{m} h_i - \binom{m}{2}. \tag{3.1}$$

For example, $\beta = \{16, 15, 4, 3, 2, 1\}$ is an order ideal of T_{12}, which corresponds to a $(12, 13, 14)$-core partition $\lambda = (11, 11, 1, 1, 1, 1)$ of size 26.

Recall that B_k is the set of elements in T_s of rank k, that is,

$$B_k = [1 + k(s + 2), (k + 1)s - 1].$$

Lemma 3.1 *Let λ be an $(s, s+1, s+2)$-core partition of maximum size. If $\beta(\lambda)$ contains an element i that is in B_k, then $\beta(\lambda)$ contains all the elements in $[i, (k + 1)s - 1]$.*

Proof. Assume to the contrary that the lemma is not valid, that is, there exist elements $i, j \in B_k$ such that $i < j$, $i \in \beta(\lambda)$ and $j \notin \beta(\lambda)$. We choose k to be the smallest integer for such B_k and let i be the smallest such number once k is determined. For any $p \in \beta(\lambda)$ such that $p \geq q$ in T_s for some $q \in [i, j - 1]$, we replace it by $p + 1$. We call this process a lift of an order ideal; see Figure 4 for an illustration. This leads us to a new order ideal

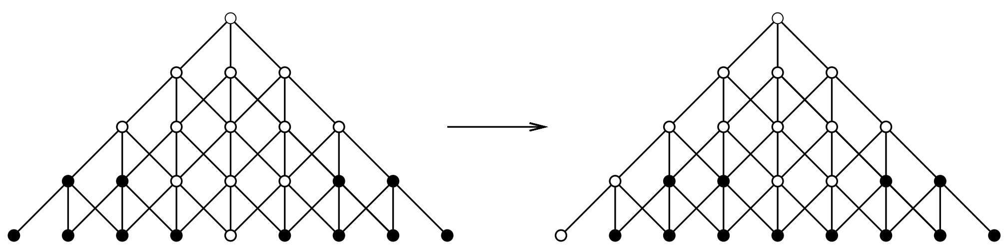

Figure 4: A lift of an order ideal in T_{10}.

β' with the same cardinality as $\beta(\lambda)$ and a larger sum of the elements. By relation (3.1), the size of the $(s, s + 1, s + 2)$-core partition corresponding to β' is larger than that of λ, which contradicts the assumption that λ is of maximum size. This proves the lemma. ∎

For $1 \leq i \leq s - 1$, let $\beta_{i,0}$ be the unique order ideal in T_s that is isomorphic to T_{s-i} and contains all the elements in $[i + 1, s - 1]$. For $1 \leq j \leq \lfloor \frac{s-i+1}{2} \rfloor$, let $\beta_{i,j}$ be the union of $\beta_{i,0}$ and the chain consisting of $i, i + (s + 2), \ldots, i + (j - 1)(s + 2)$. For example, the order ideal $\beta_{4,2}$ of T_{10} is given in Figure 5.

For $1 \leq i \leq s - 1$, $0 \leq j \leq \lfloor \frac{s-i+1}{2} \rfloor$, $\beta_{i,j}$ is an order ideal of T_s. Let $\lambda_{i,j}$ be the unique partition such that $\beta(\lambda_{i,j}) = \beta_{i,j}$. By Theorem 2.2, for each $\beta_{i,j}$, $\lambda_{i,j}$ is an $(s, s+1, s+2)$-core partition. Let λ be an $(s, s + 1, s + 2)$-core partition of maximum size. We shall show that λ equals $\lambda_{i,j}$ for some integers i, j. From Lemma 3.1, we get that $s-1 \in \beta(\lambda)$, so that there exists an integer i such that $[i, s - 1]$ is contained in $\beta(\lambda)$.

Lemma 3.2 *Assume that $s \geq 3$. Let λ be an $(s, s+1, s+2)$-core partition of maximum size. Then there exist some integers $1 \leq i \leq s - 1$ and $0 \leq j \leq \lfloor \frac{s-i+1}{2} \rfloor$ such that $\lambda = \lambda_{i,j}$.*

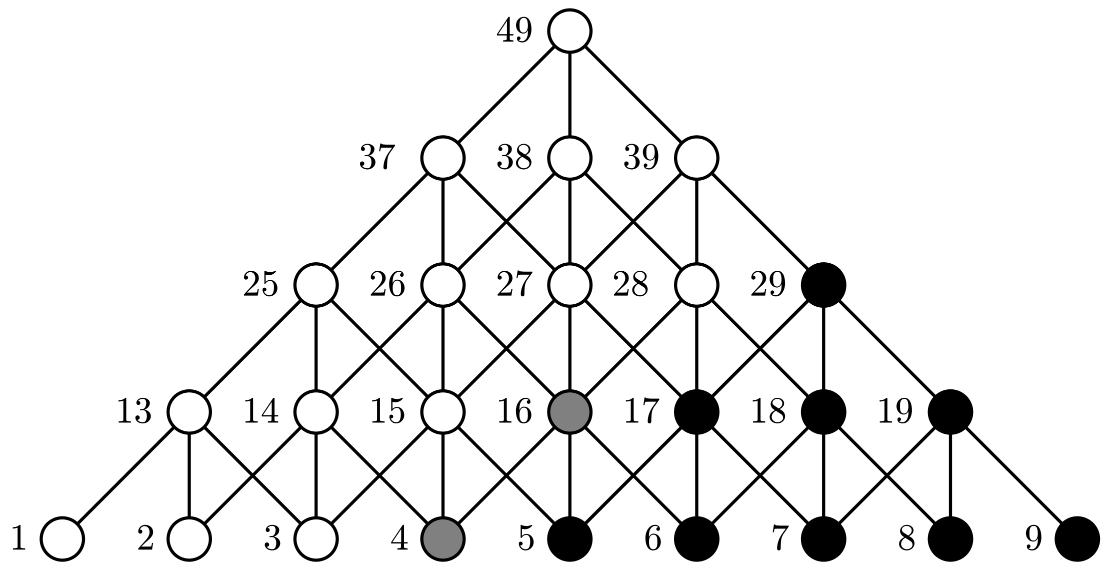

Figure 5: The order ideal $\beta_{4,2}$ of T_{10}.

Proof. Let i be the minimal integer such that $[i, s-1]$ is contained in $\beta(\lambda)$ and j the maximal integer such that $i+(j-1)(s+2) \in \beta(\lambda)$. We proceed to show that $\lambda = \lambda_{i,j}$, or equivalently, $\beta(\lambda) = \beta_{i,j}$.

By the choice of i and j, the proof of Lemma 3.1 shows that $\beta(\lambda) \subseteq \beta_{i,j}$. Hence it remains to show that $\beta_{i,j} \subseteq \beta(\lambda)$. Assume to the contrary that $\beta_{i,j} \not\subseteq \beta(\lambda)$, that is, there exists an element in $\beta_{i,j}$ which is not contained in $\beta(\lambda)$. Let p be the smallest element such that $p \in \beta_{i,j}$ and $p \notin \beta(\lambda)$.

Let β' denote the set $\beta(\lambda) \cup \{p\} \setminus \{i+(j-1)(s+2)\}$. We claim that β' is an order ideal of T_s and it corresponds to an $(s, s+1, s+2)$-core partition of size larger than $|\lambda|$.

First, we show that β' is an order ideal of T_s. Let $\gamma = \beta(\lambda) \cup \{p\}$. To prove that β' is an order ideal of T_s, it is sufficient to show that γ is an order ideal of T_s and $i+(j-1)(s+2)$ is a maximal element in γ. Let q be an arbitrary element of T_s such that $q < p$ in the poset T_s. Notice that $\beta_{i,j}$ is an order ideal of T_s. This implies that $q \in \beta_{i,j}$ since $p \in \beta_{i,j}$ and $q < p$ in T_s. By the choice of p, we see that $q \in \beta(\lambda) \subseteq \gamma$. Hence γ is an order ideal of T_s.

To prove that β' is an order ideal of T_s, it remains to show that $i+(j-1)(s+2)$ is a maximal element of the order ideal γ. By the definition of $\beta_{i,j}$, $i+(j-1)(s+2)$ is a maximal element of $\beta_{i,j}$. Since $\gamma \subseteq \beta_{i,j}$ and $i+(j-1)(s+2) \in \gamma$, we obtain that $i+(j-1)(s+2)$ is a maximal element of γ. Hence γ is an order ideal of T_s and $i+(j-1)(s+2)$ is a maximal element in γ. So we deduce that β' is an order ideal of T_s.

Let μ be the partition determined by $\beta(\mu) = \beta'$. By Theorem 2.2, μ is an $(s, s+1, s+2)$-core. We aim to show that $|\mu| > |\lambda|$. Because of relation (3.1), it suffices to show that $p > i + (j-1)(s+2)$. Assume to the contrary that $p \le i + (j-1)(s+2)$. Since $p \in \beta_{i,j}$, we obtain that $p \in B_k \cap \beta_{i,j} = [i + k(s+2), (k+1)s - 1]$ for some integer $0 \le k \le j-1$. Notice that $i+(j-1)(s+2) \in \beta(\lambda)$ and $\beta(\lambda)$ is an order ideal of T_s. Since $i + k(s+2) \le i + (j-1)(s+2)$ in T_s, we have $i + k(s+2) \in \beta(\lambda)$. From Lemma 3.1 we see that $[i+k(s+2), (k+1)s-1] \subseteq \beta(\lambda)$. It follows that $p \in \beta(\lambda)$, which contradicts the assumption that $p \notin \beta(\lambda)$. Thus $p > i + (j-1)(s+2)$, that is, $|\mu| > |\lambda|$, contradicting the condition that λ is of maximum size. This proves that $\beta_{i,j} \subseteq \beta(\lambda)$. So we conclude that $\lambda = \lambda_{i,j}$, and this completes the proof. ∎

Lemma 3.3 *Given $1 \le i \le s-1$, we have $|\lambda_{i,j}| \le |\lambda_{i,\lfloor \frac{s-i+1}{2} \rfloor}|$ for $0 \le j \le \lfloor \frac{s-i+1}{2} \rfloor$, with equality holding if and only if $j = \lfloor \frac{s-i+1}{2} \rfloor$, or s is odd, $i = 1$ and $j = 0$.*

Proof. By relation (3.1), the size of $\lambda_{i,j}$ equals

$$\begin{aligned}
|\lambda_{i,j}| &= \sum_{h \in \beta_{i,j}} h - \binom{|\beta_{i,j}|}{2} \\
&= \sum_{h \in \beta_{i,0}} h + \sum_{p=0}^{j-1} \big(i + p(s+2)\big) - \binom{|\beta_{i,0}| + j}{2} \\
&= \sum_{h \in \beta_{i,0}} h - \binom{|\beta_{i,0}|}{2} + \sum_{p=0}^{j-1} \big(i + p(s+2)\big) - \binom{|\beta_{i,0}| + j}{2} + \binom{|\beta_{i,0}|}{2} \\
&= |\lambda_{i,0}| + ij + (s+1)\binom{j}{2} - |\beta_{i,0}| j. \qquad (3.2)
\end{aligned}$$

By the definition of $\beta_{i,0}$ and Theorem 2.3, we obtain that

$$|\beta_{i,0}| = |T_{s-i}| = \begin{cases} k^2 + k, & \text{if } i = s - 2k - 1, \\ k^2, & \text{if } i = s - 2k. \end{cases}$$

Hence

$$|\lambda_{i,j}| = \begin{cases} |\lambda_{i,0}| + ij + (s+1)\binom{j}{2} - (k^2+k)j, & \text{if } i = s - 2k - 1, \\ |\lambda_{i,0}| + ij + (s+1)\binom{j}{2} - k^2 j, & \text{if } i = s - 2k. \end{cases} \qquad (3.3)$$

In particular, for $j = \lfloor \frac{s-i+1}{2} \rfloor$, we have

$$|\lambda_{i,\lfloor \frac{s-i+1}{2} \rfloor}| - |\lambda_{i,0}| = \begin{cases} (\frac{ik}{2} + i)(k+1), & \text{if } i = s - 2k - 1, \\ \frac{(i-1)(k^2+k)}{2}, & \text{if } i = s - 2k, \end{cases} \qquad (3.4)$$

which implies that

$$|\lambda_{i,\lfloor \frac{s-i+1}{2} \rfloor}| \ge |\lambda_{i,0}|. \qquad (3.5)$$

For fixed integers i and s, we see that $|\lambda_{i,j}|$ is a quadratic function of j with a positive leading coefficient. Hence the maximum value of $|\lambda_{i,j}|$ is obtained at $j=0$ or $j=\lfloor\frac{s-i+1}{2}\rfloor$ when j ranges over $[0,\lfloor\frac{s-i+1}{2}\rfloor]$. In view of (3.5), we conclude that

$$|\lambda_{i,j}| \leq |\lambda_{i,\lfloor\frac{s-i+1}{2}\rfloor}| \tag{3.6}$$

for $0\leq j\leq\lfloor\frac{s-i+1}{2}\rfloor$. Moreover, we have

$$|\lambda_{i,j}| < |\lambda_{i,\lfloor\frac{s-i+1}{2}\rfloor}|$$

for $0<j<\lfloor\frac{s-i+1}{2}\rfloor$. Hence (3.6) holds with equality only when $j=0$ or $j=\lfloor\frac{s-i+1}{2}\rfloor$. Assume that (3.6) holds with equality for $j=0$, that is $|\lambda_{i,0}|=|\lambda_{i,\lfloor\frac{s-i+1}{2}\rfloor}|$. It follows from (3.4) that $i=1$ and $s=2k+1$ for some k. Conversely, for $i=1$, $j=0$ and $s=2k+1$, by (3.4) we see that (3.6) holds with equality, namely, $|\lambda_{1,0}|=|\lambda_{1,\lfloor\frac{s}{2}\rfloor}|$. This completes the proof. ∎

The following theorem provides a characterization of $(s,s+1,s+2)$-core partitions of maximum size in terms of corresponding order ideals of T_s under the map β. We shall use the common notation λ' for the conjugate of a partition λ.

Theorem 3.4 *Assume that $s\geq 3$. Let κ_s be the $(s,s+1,s+2)$-core partition such that $\beta(\kappa_s)=T_s$. Then κ_s is an $(s,s+1,s+2)$-core partition of maximum size. Moreover, if s is even, then κ_s is the unique $(s,s+1,s+2)$-core partition of maximum size, which is self-conjugate. If s is odd, then there is exactly another $(s,s+1,s+2)$-core partition of maximum size, which is the conjugate of κ_s.*

Proof. Let λ be an $(s,s+1,s+2)$-core partition of maximum size. We aim to show that $\lambda=\kappa_s$ if s is even, and $\lambda=\kappa_s$ or κ_s' if s is odd. From Lemma 3.2 we see that $\lambda=\lambda_{i,j}$ for some integers i,j. To determine the values of i,j, we consider the following two cases.

Case 1: s is even. As a consequence of Lemma 3.3, we have $\lambda_{i,k}<\lambda_{i,\lfloor\frac{s-i+1}{2}\rfloor}$ for $0\leq k<\lfloor\frac{s-i+1}{2}\rfloor$. Hence $j=\lfloor\frac{s-i+1}{2}\rfloor$, that is, $\lambda=\lambda_{i,\lfloor\frac{s-i+1}{2}\rfloor}$ for some i. We claim that $i=1$. Suppose to the contrary that $i>1$, that is, $i-1\geq 1$. By the definition of $\lambda_{i,j}$, we find that $\lambda_{i,\lfloor\frac{s-i+1}{2}\rfloor}=\lambda_{i-1,0}$. Since s is even, by Lemma 3.3, we obtain that

$$|\lambda|=|\lambda_{i,\lfloor\frac{s-i+1}{2}\rfloor}|=|\lambda_{i-1,0}|<|\lambda_{i-1,\lfloor\frac{s-i+2}{2}\rfloor}|,$$

contradicting the fact that λ is of maximum size. Hence we have $i=1$, and so $\lambda=\lambda_{1,\lfloor\frac{s}{2}\rfloor}$.

Case 2: s is odd. We claim that $i\leq 2$. Suppose that $i>2$. By Lemma 3.3, we have $j=\lfloor\frac{s-i+1}{2}\rfloor$, that is, $\lambda=\lambda_{i,\lfloor\frac{s-i+1}{2}\rfloor}$ for some i. Since $i-1>1$ and $\lambda_{i,\lfloor\frac{s-i+1}{2}\rfloor}=\lambda_{i-1,0}$, using Lemma 3.3 we get

$$|\lambda|=|\lambda_{i,\lfloor\frac{s-i+1}{2}\rfloor}|=|\lambda_{i-1,0}|<|\lambda_{i-1,\lfloor\frac{s-i+2}{2}\rfloor}|,$$

which contradicts the fact that λ is of maximum size. This proves the claim, namely, $i = 1$ or 2. By Lemma 3.3, we obtain $\lambda = \lambda_{1,\lfloor \frac{s}{2} \rfloor}$, $\lambda_{1,0}$ or $\lambda_{2,\lfloor \frac{s-1}{2} \rfloor}$. By the definition of $\lambda_{i,j}$, we see that $\lambda_{1,0} = \lambda_{2,\lfloor \frac{s-1}{2} \rfloor}$. Thus, $\lambda = \lambda_{1,\lfloor \frac{s}{2} \rfloor}$ or $\lambda_{1,0}$. Again, using Lemma 3.3, we have $|\lambda_{1,\lfloor \frac{s}{2} \rfloor}| = |\lambda_{1,0}|$. So we find that $\lambda_{1,\lfloor \frac{s}{2} \rfloor}$ and $\lambda_{1,0}$ are the only two $(s, s+1, s+2)$-core partitions of maximum size.

Notice that in both cases, $\lambda_{1,\lfloor \frac{s}{2} \rfloor}$ is an $(s, s+1, s+2)$-core partition of maximum size. To prove that κ_s is of maximum size, it suffices to show that $\kappa_s = \lambda_{1,\lfloor \frac{s}{2} \rfloor}$. By the definitions of $\lambda_{i,j}$ and $\beta_{i,j}$, it can be verified that $\beta(\lambda_{1,\lfloor \frac{s}{2} \rfloor}) = \beta_{1,\lfloor \frac{s}{2} \rfloor} = T_s$. Since $\beta(\kappa_s) = T_s$, we have $\beta(\kappa_s) = \beta(\lambda_{1,\lfloor \frac{s}{2} \rfloor})$. This implies that $\kappa_s = \lambda_{1,\lfloor \frac{s}{2} \rfloor}$. So we reach the conclusion that κ_s is an $(s, s+1, s+2)$-core partition of maximum size.

It remains to show that when s is even, κ_s is self-conjugate, and when s is odd, $\lambda_{1,0} = \kappa_s'$.

Clearly, the conjugate of an $(s, s+1, s+2)$-core partition is still an $(s, s+1, s+2)$-core partition of the same size. Since κ_s is an $(s, s+1, s+2)$-core partition of maximum size, κ_s' is also an $(s, s+1, s+2)$-core partition of maximum size.

When s is even, since $\kappa_s = \lambda_{1,\lfloor \frac{s}{2} \rfloor}$ is the unique $(s, s+1, s+2)$-core partition of maximum size, we have $\kappa_s' = \kappa_s$, that is, κ_s is self-conjugate.

When s is odd, we have shown that κ_s and $\lambda_{1,0}$ are the only two $(s, s+1, s+2)$-core partitions of maximum size. To prove that $\lambda_{1,0} = \kappa_s'$, it suffices to show that $\kappa_s' \neq \kappa_s$, that is, κ_s is not self-conjugate. To this end, we aim to prove that the length of κ_s is not equal to the largest part of κ_s.

Assume that $s = 2m - 1$ for some $m \geq 2$. Note that the length of κ_s equals $|\beta(\kappa_s)| = |T_s|$. In view of Theorem 2.3, we obtain that

$$|T_s| = \sum_{k=0}^{m-2} (2m - 2 - 2k) = m^2 - m. \tag{3.7}$$

Thus the length of κ_s equals $m^2 - m$. Since $\beta(\kappa_s) = T_s$, from (1.1) and Theorem 2.3, it can be seen that the largest part of κ_s equals

$$\left\lfloor \frac{s}{2} \right\rfloor s - 1 - (|T_s| - 1) = (m-1)(2m-1) - (m^2 - m) = m^2 - 2m + 1.$$

Since $m \geq 2$, we have $m^2 - m \neq m^2 - 2m + 1$, so that the length of κ_s is not equal to the largest part of κ_s. This completes the proof. ∎

Theorem 3.4 says that the partition κ_s corresponding to the order ideal T_s is of maximum size. This leads to a proof of Conjecture 1.2 which gives an explicit formula for the maximum size of an $(s, s+1, s+2)$-core partition.

Corollary 3.5 *Let s be a positive integer. The maximum size of an $(s, s+1, s+2)$-core* *partition equals*

$$l(s) = \begin{cases} m\binom{m+1}{3}, & \text{if } s = 2m-1, \\ (m+1)\binom{m+1}{3} + \binom{m+2}{3}, & \text{if } s = 2m. \end{cases}$$

Proof. It is easily checked that the corollary holds for $s \leq 2$. We now assume that $s \geq 3$. By Theorem 3.4, we know that the partition κ_s such that $\beta(\kappa_s) = T_s$ is of maximum size. Using (3.1), we get

$$l(s) = |\kappa_s| = \sum_{h \in T_s} h - \binom{|T_s|}{2}. \tag{3.8}$$

If s is odd, that is, $s = 2m - 1$ for some $m \geq 2$, by Theorem 2.3, we find that

$$\begin{aligned} \sum_{h \in T_s} h &= \sum_{k=0}^{m-2} \sum_{i=1}^{2m-2k-2} \big(k(2m+1) + i\big) \\ &= \sum_{k=0}^{m-2} (-4mk^2 + 4m^2k - 6mk + k + 2m^2 - 3m + 1) \\ &= \frac{2}{3}m^4 - m^3 - \frac{1}{6}m^2 + \frac{1}{2}m. \end{aligned} \tag{3.9}$$

Substituting (3.7) and (3.9) into (3.8), we obtain that

$$l(s) = m\binom{m+1}{3}.$$

If s is even, that is, $s = 2m$ for some $m \geq 2$, by Theorem 2.3, we obtain that

$$\begin{aligned} \sum_{h \in T_s} h &= \sum_{k=0}^{m-1} \sum_{i=1}^{2m-2k-1} \big(k(2m+2) + i\big) \\ &= \sum_{k=0}^{m-1} (-4mk^2 - 2k^2 + 4m^2k - 2mk - k + 2m^2 - m) \\ &= \frac{2}{3}m^4 + \frac{1}{3}m^3 - \frac{1}{6}m^2 + \frac{1}{6}m. \end{aligned} \tag{3.10}$$

Again, by Theorem 2.3, we get

$$|T_s| = \sum_{k=0}^{m-1} (2m - 2k - 1) = m^2. \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.8) gives

$$l(s) = (m+1)\binom{m+1}{3} + \binom{m+2}{3}.$$

This completes the proof. ∎

4 Proof of Conjecture 1.3

In this section, we shall give a proof of Conjecture 1.3 on the total sum $h(s)$ of sizes of $(s, s+1, s+2)$-core partitions. By the correspondence between $(s, s+1, s+2)$-core partitions and the order ideals of T_s, we can express $h(s)$ in terms of the sums of elements of order ideals of T_s. Then we obtain an explicit formula for the generating function of $h(s)$, which leads to a proof of Conjecture 1.3.

By Theorem 2.2 and relation (3.1), we have

$$h(s) = \sum_{I \in J(T_s)} \left(\sum_{a \in I} a - \binom{|I|}{2} \right). \tag{4.1}$$

Let ρ_s denote the rank function of the poset T_s. By Theorem 2.3, we see that $\rho_s(a) = k$ for $a \in B_k = [1 + k(s+2), (k+1)s - 1]$. In order to derive the generating function of $h(s)$, we need the following two functions

$$f(s) = \sum_{I \in J(T_s)} |I|,$$

$$g(s) = \sum_{I \in J(T_s)} \sum_{a \in I} \rho_s(a).$$

Let $F(x)$, $G(x)$ and $H(x)$ be the ordinary generating functions of the numbers $f(s)$, $g(s)$ and $h(s)$, that is,

$$F(x) = \sum_{s \geq 0} f(s) x^s,$$

$$G(x) = \sum_{s \geq 0} g(s) x^s,$$

$$H(x) = \sum_{s \geq 0} h(s) x^s.$$

The following lemma gives recurrence relations for $f(s)$, $g(s)$ and $h(s)$, which lead to the generating functions $F(x)$, $G(x)$ and $H(x)$. In fact, we shall use the generating functions $F(x)$ and $G(x)$ to compute $H(x)$.

Lemma 4.1 *For $s \geq 2$, we have*

$$f(s) = f(s-1) + \sum_{i=2}^{s} \Big(2M_{s-i} f(i-2) + (i-1) M_{s-i} M_{i-2} \Big), \tag{4.2}$$

$$g(s) = g(s-1) + \sum_{i=2}^{s} M_{s-i} \Big(2g(i-2) + f(i-2) \Big), \tag{4.3}$$

$$h(s) = h(s-1) + f(s-1) + g(s-1)$$

$$+\sum_{i=2}^{s}\Big(2M_{s-i}h(i-2)+(s+4-i)M_{s-i}f(i-2)$$
$$+2(s-i+2)M_{s-i}g(i-2)+(i-1)M_{s-i}M_{i-2}-f(i-2)f(s-i)\Big). \tag{4.4}$$

Proof. We shall only give a proof of (4.4). Relations (4.2) and (4.3) can be verified in the same manner.

Let

$$h_i(s)=\sum_{I\in J_i(T_s)}\left(\sum_{a\in I}a-\binom{|I|}{2}\right).$$

Since

$$J(T_s)=\bigcup_{i=1}^{s}J_i(T_s),$$

in view of (4.1), we have

$$h(s)=\sum_{i=1}^{s}h_i(s).$$

To compute $h_i(s)$, we recall the decomposition of an order ideal of T_s as given in Section 2. For an order ideal $I\in J_i(T_s)$, we can express I as

$$I=\{1,2,\ldots,i-1\}\cup I'\cup I'', \tag{4.5}$$

where I' is isomorphic to an order ideal I_1 of T_{i-2} and I'' is isomorphic to an order ideal I_2 of T_{s-i}. Here we set T_{-1} to be the empty set. Conversely, an order ideal I_1 of T_{i-2} and an order ideal I_2 of T_{s-i} uniquely determine an order ideal $I\in J_i(T_s)$. To be more specific, we have

$$I'=\{a+s+2+(s+2-i)\rho_{i-2}(a)\mid a\in I_1\}$$

and

$$I''=\{a+i+i\rho_{s-i}(a)\mid a\in I_2\}.$$

The above decomposition implies that for $I\in J_i(T_s)$,

$$\sum_{a\in I}a=\binom{i}{2}+\sum_{a\in I'}a+\sum_{a\in I''}a$$
$$=\binom{i}{2}+\sum_{a\in I_1}\Big(a+s+2+(s+2-i)\rho_{i-2}(a)\Big)+\sum_{a\in I_2}\Big(a+i+i\rho_{s-i}(a)\Big).$$

From the proof of Conjecture 1.1, we see that M_s equals the number of order ideals of T_s. Let

$$p(s)=\sum_{I\in J(T_s)}\sum_{a\in I}a.$$

Using the decomposition (4.5) of an order ideal $I \in J_i(T_s)$, $h_i(s)$ can be computed as follows. For $i = 1$, we have

$$\begin{aligned} h_1(s) &= \sum_{I \in J_1(T_s)} \left(\sum_{a \in I} a - \binom{|I|}{2} \right) \\ &= \sum_{I_2 \in J(T_{s-1})} \left(\sum_{a \in I_2} \big(a + 1 + \rho_{s-1}(a)\big) - \binom{|I_2|}{2} \right) \\ &= h(s-1) + f(s-1) + g(s-1). \end{aligned} \tag{4.6}$$

For $2 \leq i \leq s$, we find that

$$\begin{aligned} h_i(s) &= \sum_{I \in J_i(T_s)} \left(\sum_{a \in I} a - \binom{|I|}{2} \right) \\ &= \sum_{\substack{I_1 \in J(T_{i-2}) \\ I_2 \in J(T_{s-i})}} \left(\binom{i}{2} + \sum_{a \in I_1} \big(a + s + 2 + (s + 2 - i)\rho_{i-2}(a)\big) \right. \\ &\qquad \left. + \sum_{a \in I_2} \big(a + i + i\rho_{s-i}(a)\big) - \binom{|I_1| + |I_2| + i - 1}{2} \right) \\ &= \sum_{I_1 \in J(T_{i-2})} M_{s-i} \left(\sum_{a \in I_1} \big(a + s + 2 + (s + 2 - i)\rho_{i-2}(a)\big) + \binom{i}{2} \right) \\ &\qquad + \sum_{I_2 \in J(T_{s-i})} M_{i-2} \sum_{a \in I_2} \big(a + i + i\rho_{s-i}(a)\big) \\ &\qquad - \sum_{\substack{I_1 \in J(T_{i-2}) \\ I_2 \in J(T_{s-i})}} \binom{|I_1| + |I_2| + i - 1}{2} \\ &= M_{s-i}\big(p(i-2) + (s+2)f(i-2) + (s-i+2)g(i-2)\big) \\ &\qquad + M_{s-i}M_{i-2}\binom{i}{2} + M_{i-2}\big(p(s-i) + if(s-i) + ig(s-i)\big) \\ &\qquad - \sum_{\substack{I_1 \in J(T_{i-2}) \\ I_2 \in J(T_{s-i})}} \binom{|I_1| + |I_2| + i - 1}{2}. \end{aligned} \tag{4.7}$$

Since

$$\sum_{I_1 \in J(T_{i-2})} |I_1| = f(i-2),$$

$$\sum_{I_2\in J(T_{s-i})}|I_2| = f(s-i),$$

we have

$$\begin{aligned}
&\sum_{\substack{I_1\in J(T_{i-2})\\ I_2\in J(T_{s-i})}}\binom{|I_1|+|I_2|+i-1}{2}\\
=&\sum_{\substack{I_1\in J(T_{i-2})\\ I_2\in J(T_{s-i})}}\left(\binom{|I_1|}{2}+\binom{|I_2|}{2}+(i-1)|I_1|+(i-1)|I_2|+|I_1||I_2|+\binom{i-1}{2}\right)\\
=&\sum_{I_1\in J(T_{i-2})}M_{s-i}\left(\binom{|I_1|}{2}+(i-1)|I_1|\right)+\sum_{I_2\in J(T_{s-i})}M_{i-2}\left(\binom{|I_2|}{2}+(i-1)|I_2|\right)\\
&+\left(\sum_{I_1\in J(T_{i-2})}|I_1|\right)\left(\sum_{I_2\in J(T_{s-i})}|I_2|\right)+M_{s-i}M_{i-2}\binom{i-1}{2}\\
=&M_{s-i}\sum_{I_1\in J(T_{i-2})}\binom{|I_1|}{2}+(i-1)M_{s-i}f(i-2)+M_{i-2}\sum_{I_2\in J(T_{s-i})}\binom{|I_2|}{2}\\
&+(i-1)M_{i-2}f(s-i)+f(i-2)f(s-i)+M_{s-i}M_{i-2}\binom{i-1}{2}. \qquad (4.8)
\end{aligned}$$

Note that

$$h(s) = p(s) - \sum_{I\in J(T_s)}\binom{|I|}{2},$$

Substituting (4.8) into (4.7), we obtain that

$$\begin{aligned}
h_i(s) =&M_{s-i}\Big(p(i-2)+(s+2)f(i-2)+(s-i+2)g(i-2)\Big)\\
&+M_{s-i}M_{i-2}\binom{i}{2}+M_{i-2}\Big(p(s-i)+if(s-i)+ig(s-i)\Big)\\
&-M_{s-i}\sum_{I_1\in J(T_{i-2})}\binom{|I_1|}{2}-(i-1)M_{s-i}f(i-2)\\
&-M_{i-2}\sum_{I_2\in J(T_{s-i})}\binom{|I_2|}{2}-(i-1)M_{i-2}f(s-i)\\
&-f(i-2)f(s-i)-M_{s-i}M_{i-2}\binom{i-1}{2}\\
=&M_{s-i}h(i-2)+(s+3-i)M_{s-i}f(i-2)\\
&+(s-i+2)M_{s-i}g(i-2)+(i-1)M_{s-i}M_{i-2}
\end{aligned}$$

$$+ M_{i-2}h(s-i) + M_{i-2}f(s-i)$$
$$+ iM_{i-2}g(s-i) - f(i-2)f(s-i).$$

Summing over i, we deduce that

$$\begin{aligned}
h(s) =& \sum_{i=1}^{s} h_i(s) \\
=& h(s-1) + f(s-1) + g(s-1) \\
&+ \sum_{i=2}^{s} \Big(M_{s-i}h(i-2) + (s+3-i)M_{s-i}f(i-2) \\
&+ (s-i+2)M_{s-i}g(i-2) + (i-1)M_{s-i}M_{i-2} \\
&+ M_{i-2}h(s-i) + M_{i-2}f(s-i) \\
&+ iM_{i-2}g(s-i) - f(i-2)f(s-i) \Big) \\
=& h(s-1) + f(s-1) + g(s-1) \\
&+ \sum_{i=2}^{s} \Big(2M_{s-i}h(i-2) + (s+4-i)M_{s-i}f(i-2) \\
&+ 2(s-i+2)M_{s-i}g(i-2) \\
&+ (i-1)M_{s-i}M_{i-2} - f(i-2)f(s-i) \Big).
\end{aligned}$$

This proves (4.4). ∎

From the recurrence relations in Lemma 4.1, we get the following explicit formula for the generating function $H(x)$.

Theorem 4.2 *We have*

$$H(x) = \frac{x^2}{(1-2x-3x^2)^{5/2}}. \tag{4.9}$$

Proof. Let $M(x)$ be the ordinary generating function of the Motzkin numbers, that is,

$$M(x) = \sum_{s\geq 0} M_s x^s.$$

It follows from (4.4) that

$$\begin{aligned}
H(x) =& xH(x) + xF(x) + xG(x) + 2x^2M(x)H(x) + 4x^2M(x)F(x) \\
&+ x^3M'(x)F(x) + 4x^2M(x)G(x) + 2x^3M'(x)G(x) \\
&+ x^3M'(x)M(x) + x^2M^2(x) - x^2F^2(x).
\end{aligned} \tag{4.10}$$

It is known that

$$M(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}. \tag{4.11}$$

To derive a formula for $H(x)$, we proceed to compute $F(x)$ and $G(x)$.

From recurrence relation (4.2) we get

$$F(x) = xF(x) + 2x^2 M(x)F(x) + x^3 \left(M'(x) + \frac{M(x)}{x} \right) M(x). \tag{4.12}$$

Substituting (4.11) into (4.12), we obtain

$$F(x) = \frac{\left(-1 + x + \sqrt{1 - 2x - 3x^2}\right)^2}{4x^2(1 - 2x - 3x^2)}. \tag{4.13}$$

Similarly, from recurrence relation (4.3) we deduce that

$$G(x) = xG(x) + 2x^2 M(x)G(x) + x^2 M(x)F(x),$$

which implies that

$$G(x) = \frac{x^2 M(x)F(x)}{1 - x - 2x^2 M(x)}. \tag{4.14}$$

Substituting (4.11) and (4.13) into (4.14), we get

$$G(x) = -\frac{\left(-1 + x + \sqrt{1 - 2x - 3x^2}\right)^3}{8x^2(1 - 2x - 3x^2)^{3/2}}. \tag{4.15}$$

Based on the formulas for $F(x)$ and $G(x)$, the formula (4.9) for $H(x)$ immediately follows from (4.10). This completes the proof. ∎

Theorem 4.2 leads to an explicit formula for $h(s)$, which confirms Conjecture 1.3.

Corollary 4.3 *Let s be a positive integer. The sum of the sizes of all the $(s, s+1, s+2)$-core partitions is*

$$h(s) = \sum_{j=0}^{s-2} \binom{j+3}{3} \sum_{i=0}^{\lfloor j/2 \rfloor} \binom{j}{2i} C_i. \tag{4.16}$$

Proof. Based on the expression for $H(x)$ as in (4.9), it can be verified that $H(x)$ satisfies the following differential equation

$$(-x + 2x^2 + 3x^3)H'(x) + (2 + x + 9x^2)H(x) = 0,$$

which implies that for $s \geq 3$,

$$(2-s)h(s) + (2s-1)h(s-1) + (3s+3)h(s-2) = 0. \tag{4.17}$$

By exchanging the order of summations, it is easily seen that the sum on the right hand side of (4.16) equals

$$\sum_{i\geq 0} \frac{(2i+3)!}{6\, i!(i+1)!} \binom{s+2}{2i+4}. \tag{4.18}$$

Using the Zeilberger algorithm, see [9], we find that the sum in (4.18) also satisfies the same recurrence relation (4.17) as $h(s)$. Taking the initial values into consideration, we arrive at (4.16). This completes the proof. ∎

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