

# Searching for (near) optimal codes\*

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**Abstract.** Formally self-dual (FSD) codes are interesting codes and have received an enormous research effort due to their importance in mathematics and computer science. Danielsen and Parker proved that every self-dual additive code over  $GF(4)$  is equivalent to a graph codes in 2006, and hence graph is an important tool for searching (near) optimal codes. In this paper, we introduce a new method of searching (near) optimal binary (formally self-dual) linear codes and additive codes from circulant graphs.

**Keywords:** graph code, FSD code, additive code, optimal code, circulant graph

## 1 Introduction

Let  $F_2$  be the binary field, and let  $F_2^n$  denote the  $n$ -dimensional binary vector space. A  $k$ -dimensional linear subspace  $\mathcal{C}$  of  $F_2^n$  is called an  $[n, k]$  linear code and vectors in  $\mathcal{C}$  are called *codewords*. Define  $GF(4) = \{0, 1, \omega, \omega^2\}$ , where  $\omega^2 = 1 + \omega$ . An additive code  $\mathcal{C}$  over  $GF(4)$  of length  $n$  is an additive subgroup of  $GF(4)^n$ . It is clear that  $\mathcal{C}$  contains codewords for some  $0 \leq k \leq 2n$ , and can be defined by a  $k \times n$  generator matrix, with entries from  $GF(4)$ , whose rows span  $\mathcal{C}$  additively. We call  $\mathcal{C}$  an  $(n, 2^k)$  additive code. The *Hamming weight* of a vector  $x = (x_1, \dots, x_n)$ , denoted by  $wt(x)$ , is the number of its nonzero coordinates, the *Hamming distance* between two vectors  $x, y$  is equal to the Hamming weight  $wt(x - y)$ . The *minimum distance*  $d$  of a code is defined as the smallest possible distance between pairs of distinct codewords. An  $[n, k]$  linear code with minimum distance  $d$  is denoted as an  $[n, k, d]$  code and an  $(n, 2^k)$  additive code with minimum distance  $d$  is denoted as an  $(n, 2^k, d)$  code. The *weight distribution* of a code  $\mathcal{C}$  is the sequence  $(A_0, A_1, \dots, A_n)$ , where  $A_i$  is

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the number of codewords of weight  $i$  in  $\mathcal{C}$ . The *weight enumerator* of the code is the polynomial  $W(z) = \sum_{i=0}^n A_i z^i$ . The *inner product* of two vectors  $x, y \in F_2^n$  is defined as  $(x, y) = \sum_{i=1}^n x_i y_i$ . The *dual code* of an  $[n, k, d]$  code  $\mathcal{C}$  is defined as  $\mathcal{C}^\perp = \{x \in F_2^n \mid (x, y) = 0, \text{ for all } y \in \mathcal{C}\}$ . A binary code with the same weight distribution as its dual code is called *formally self-dual* (FSD). The *conjugation* of  $x \in GF(4)$  is defined by  $\bar{x} = x^2$ , and the *trace map* is defined by  $Tr(x) = x + \bar{x}$ . The *Hermitian trace inner product* of  $u = (u_1, \dots, u_n)$  and  $v = (v_1, \dots, v_n)$ , where  $u, v \in GF(4)$ , is defined as  $u * v = Tr(u \cdot \bar{v}) = \sum_{i=1}^n Tr(u_i \bar{v}_i) = \sum_{i=1}^n (u_i v_i^2 + u_i^2 v_i)$ . We define the *dual of the additive code*  $\mathcal{C}$  with respect to the Hermitian trace inner product as  $\mathcal{C}^\perp = \{u \in GF(4) \mid u * c = 0 \text{ for all } c \in \mathcal{C}\}$ . Then  $\mathcal{C}$  is *self-orthogonal* if  $\mathcal{C} \subseteq \mathcal{C}^\perp$ , and  $\mathcal{C}$  is *self-dual* if  $\mathcal{C} = \mathcal{C}^\perp$ .

In 2002, Tonchev [21] set up a relationship between an undirected graph and a binary linear code. Given a graph  $\Gamma_n$  on  $n$  vertices with adjacency matrix  $A_n$ , one can define a binary linear code with generator matrix  $G = (I; A_n)$ , where  $I$  is the identity matrix. Such a code is called the *linear graph code* of  $\Gamma_n$ , and a linear graph code is a  $[2n, n]$  FSD code. In 2006, Danielsen and Parker [10] proved that every self-dual additive code over  $GF(4)$  is equivalent to a graph code. Among additive codes over  $GF(4)$ , a graph code is an additive code over  $GF(4)$  that has a generator matrix of form  $\mathcal{C} = \Gamma + \omega I$ , where  $\Gamma$  is the adjacency matrix of a simple undirected graph. An  $[n, k, d]$  (or  $(n, 2^k, d)$ ) code is *optimal* if there is no  $[n, k, d+1]$  (or  $(n, 2^k, d+1)$ ) code, and *near optimal* if there is no  $[n, k, d+2]$  (or  $(n, 2^k, d+2)$ ) code. An  $[n, k, d]$  (or  $(n, 2^k, d)$ ) code is *known best* if  $d$  attains the highest known minimum distance for  $[n, k]$  (or  $(n, 2^k)$ ) codes. For parameters of optimal codes, or lower and upper bounds on minimum distances of optimal codes, see [12].

Formally self-dual codes are important class of codes, they have connections to other mathematical structures such as block designs, lattices, modular forms, and sphere packing. Many works have been done on the FSD codes, and optimal FSD codes of length  $n \leq 28$  have been classified, see [1, 2, 13, 14]. In 2002, Tonchev [21] set up a relationship between an FSD code and the adjacency matrix of an undirected graph, and showed that some interesting codes can be obtained from graphs with high degree of symmetry, such as strongly regular graphs. In 2006, Danielsen and Parker [10] proved that every self-dual additive code over  $GF(4)$  is equivalent to a graph code. In 2012, Danielsen [6] focused his attention on additive codes over  $GF(9)$  and transformed the problem of code equivalence into a problem of graph isomorphism. By an extension technique, they classified all optimal codes of lengths 11 and 12. In fact, computer searching reveals that circulant graph codes usually contain many strong codes, and some of these codes have highly regular graph representations, see [22]. In [6], Danielsen obtained some optimal additive codes from circulant graphs in 2005. Later, Varbanov investigated additive circulant graph codes over  $GF(4)$ , see [22]. Recently, finding optimal codes from graphs has received a wide attention of many researchers, see [6–11, 15, 21, 22]. Inspired by these works, we discuss the construction of (near) optimal FSD codes and additive codes from undirected circulant graphs in this paper.

The paper is organized as follows. Section 2 recalls some concepts in graph theory. In Section 3, we propose a new method to find (near) optimal binary linear codes from circulant graphs, and construct some (near) optimal or known best binary linear codes by using this method. In Section 4, we propose a new method to find additive optimal codes from circulant graphs.

## 2 Preliminaries

We introduce some concepts of graph theory for latter use in this paper, for more details please see [5]. An *undirected graph*  $\Gamma = (V, E)$  is a set  $V(\Gamma) = \{v_1, v_2, \dots, v_n\}$  of vertices together with a collection  $E(\Gamma)$  of edges, where each edge is an unordered pair of vertices. The vertices  $v_i$  and  $v_j$  are *adjacent* if  $\{v_i, v_j\}$  is an edge. Then  $v_j$  is a *neighbour* of  $v_i$ . All the neighbours of a vertex  $v_i$  in  $\Gamma$  form the *neighbourhood* of  $v_i$ , and it is denoted by  $N_\Gamma(v_i)$ . The *degree of a vertex*  $v$  is the number of vertices adjacent to  $v$ . A graph is *regular* of degree  $k$  if all vertices have the same degree  $k$ . For a graph  $\Gamma = (V, E)$ , suppose that  $V'$  is a nonempty subset of  $V$ . The subgraph of  $\Gamma$  whose vertex set is  $V'$  and whose edge set is the set of those edges of  $\Gamma$  that have both ends in  $V'$  is called the *subgraph of  $\Gamma$  induced by  $V'$* , denoted by  $\Gamma[V']$ . We say that  $\Gamma[V']$  is an induced subgraph of  $\Gamma$ . The *adjacency matrix*  $A = (a_{ij})$  of  $\Gamma = (V, E)$  is a symmetric  $(0, 1)$ -matrix defined as follows:  $a_{i,j} = 1$  if the  $i$ -th and  $j$ -th vertices are adjacent, and  $a_{i,j} = 0$  otherwise.

Circulant graphs and their various applications are the objects of intensive study in computer science and discrete mathematics, see [3, 4, 16, 19]. Recently, Monakhova published a survey paper on this subject, see [18]. Let  $S = \{a_1, a_2, \dots, a_k\}$  be a set of integers such that  $0 < a_1 < \dots < a_k < \frac{n+1}{2}$ , and let the vertices of an  $n$ -vertex graph be labelled as  $0, 1, 2, \dots, n-1$ . Then the *circulant graph*  $C(n, S)$  has  $i \pm a_1, i \pm a_2, \dots, i \pm a_k \pmod{n}$  adjacent to each vertex  $i$ . A *circulant matrix* is obtained by taking an arbitrary first row, and shifting it cyclically one position to the right in order to obtain successive rows. We say that a circulant matrix is generated by its first row. Formally, if the first row of an  $n$ -by- $n$  circulant matrix is  $a_0, a_1, \dots, a_{n-1}$ , then the  $(i, j)^{th}$  element is  $a_{j-i}$ , where subscripts are taken modulo  $n$ . The term circulant graph arises from the fact that the adjacency matrix for such a graph is a circulant matrix.

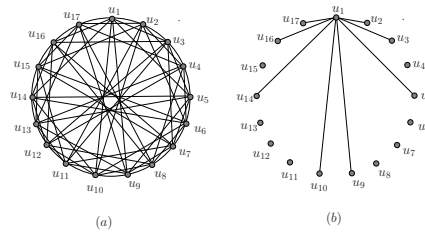


Figure 1: (a) The  $(4, 4)$ -Ramsey graph  $\Gamma$ ; (b) the edge-induced graph  $\Gamma(E_1)$ .

For example, the  $(4, 4)$ -Ramsey graph  $\Gamma$  (see Figure 1) is a famous circulant graph, which can be obtained by regarding the vertices as elements of the field of integers modulo 17, and joining two vertices if their difference is a quadratic residue of 17 (either 1, 2, 4, 8, 9, 13, 15 or 16). For the vertex  $u_1$ , we have  $E_1 = \{u_1u_2, u_1u_3, u_1u_5, u_1u_9, u_1u_{10}, u_1u_{14}, u_1u_{16}, u_1u_{17}\} \subseteq E(\Gamma)$  and a vector  $\alpha_{17} = (0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 1)$ . It is clear that the adjacency matrix  $A_{17}$  of the  $(4, 4)$ -Ramsey graph is generated by  $\alpha_{17}$ , where

$$A_{17} = \begin{pmatrix} 01101000110001011 \\ 10110100011000101 \\ 11011010001100010 \\ \dots\dots\dots \end{pmatrix}.$$

### 3 New binary linear codes searching from circulant graphs

In this section, we discuss the construction of binary linear codes from circulant graphs. Since a circulant graph  $\Gamma_n$  can be uniquely determined by its adjacency matrix  $A_n$ , or by the vector  $\alpha_n$  corresponding to  $E_1$  ( $E_1 \subseteq E(\Gamma_n)$ ), whose elements are incident with the vertex  $u_1 \in V(\Gamma_n)$ . We will make no difference of  $\alpha_n$ ,  $A_n$  or a circulant graph  $\Gamma_n$ , and simply say a circulant graph  $\Gamma_n$  with vector  $\alpha_n$  or a vector  $\alpha_n$  of a circulant graph  $\Gamma_n$ . And in this section, we denote the binary linear code with generator matrix  $G = (I, A_n)$  by  $\mathcal{C}_n$ . According to the relation between a graph code and the adjacency matrix of an undirected graph introduced by [21], we can get a [34, 17, 8] optimal FSD code from the matrix  $(I; A_{17})$ , where  $A_{17}$  is the adjacency matrix of the  $(4, 4)$ -Ramsey graph.

In [6], Danielsen got some optimal additive codes. One of them is the optimal additive code  $(30, 2^{30}, 12)$  obtained from the vector  $\beta_{30} = (\omega, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0)$ , which corresponds to a circulant graph of order 30 with vector  $\alpha_{30} = (0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0)$ . The graph code  $\mathcal{C}_{30}$  with generator matrix  $G = (I; A_{30})$  is a [60, 30, 12] code, where  $A_{30}$  is the circulant matrix generated by  $\alpha_{30}$ . The code  $\mathcal{C}_{30}$  is a known best FSD code. The weight enumerator of  $\mathcal{C}_{30}$  is

$$\begin{aligned} W_{\mathcal{C}_{30}}(z) = & 1 + 4060z^{12} + 24360z^{14} + 294930z^{16} + 1728400z^{18} + 7758400z^{20} \\ & + 26336640z^{22} + 67403540z^{24} + 129936240z^{26} + 192974265z^{28} \\ & + 220819632z^{30} + 192974265z^{32} + 129936240z^{34} + 67403540z^{36} \\ & + 26336640z^{38} + 7758660z^{40} + 1728400z^{42} + 294930z^{44} + 24360z^{46} \\ & + 4060z^{48} + z^{60}. \end{aligned}$$

However, using known circulant graphs in the literature, we only get a few binary linear codes that are optimal, near optimal or known best. So, we need to study a new method for designing good binary linear codes from circulant graphs. To achieve this goal, we give some notations first.

Denote by  $L_n$  the highest known minimum distance of all  $[2n, n]$  codes. For a graph  $\Gamma_n$  with vector  $\alpha_n$ , we let  $\mathcal{C}_n$  be its graph code and  $d_n$  be the distance of  $\mathcal{C}_n$ . If  $d_n \geq L_n$ , then  $\mathcal{C}_n$  is a good FSD code. If  $d_n < L_n$ , then  $\mathcal{C}_n$  is a “poor” code and  $\alpha_n$  is a “poor” vector. For a graph  $\Gamma_n$  with “poor” vector  $\alpha_n$ , we manage to find a new vector  $\alpha'_n$  that gives a binary linear code  $\mathcal{C}'_n = [2n, n, d'_n]$  such that  $\mathcal{C}'_n$  is better than the code  $\mathcal{C}_n$ . The idea of finding a new vector  $\alpha'_n$  from a “poor” vector  $\alpha_n = (b_1, b_2, \dots, b_n)$  is introduced as follows:

1) Let  $A_n$  be the circulant matrix generated by  $\alpha_n$  and denote  $G_n = (I; A_n)$ . Then the  $j$ -th row of  $G_n$  is

$$\begin{aligned} g_j &= (e_j \mid a_j) \\ &= (0, \dots, 0, 1, 0, \dots, 0 \mid a_{j,1}, a_{j,2}, \dots, a_{j,j-1}, a_{j,j}, a_{j,j+1}, \dots, a_{j,n}) \\ &= (0, \dots, 0, 1, 0, \dots, 0 \mid b_{n-j+2}, b_{n-j+3}, \dots, b_n, b_1, b_2, \dots, b_{n-j+1}). \end{aligned}$$

2) We call codeword  $\beta \in \mathcal{C}_n$  “bad” if  $wt(\beta) < L_n$ . Find “bad” codewords  $\beta_1, \beta_2, \dots, \beta_m$  (if exist) such that their weights are  $d_n, d_n + 1, \dots, d_n + (m - 1)$ , where  $m = L_n - d_n$ . If there is no codeword with weight  $d_n + i$  ( $0 \leq i \leq m - 1$ ), then  $\beta_{i-1}$  is not under consideration.

3) Let  $\beta_i = g_{j_1} + g_{j_2} + \dots + g_{j_r} = (u_i \mid v_i)$ , where  $u_i, v_i \in F_2^n$  and  $j_1, j_2, \dots, j_r \in \{1, 2, \dots, n\}$ .

4) From  $g_{j_i} = (e_{j_i} \mid a_{j_i})$ , one can define a column vector set  $\{A_i : i = 1, 2, \dots, n\}$ , where  $A_i = (\lambda_{i,j_1}, \lambda_{i,j_2}, \dots, \lambda_{i,j_r})$  and  $\lambda_{i,j_1} = \sum_{l=1}^r a_{j_\ell, j_1+(i-1)}$ ,  $\lambda_{i,j_2} = \sum_{l=1}^r a_{j_\ell, j_2+(i-1)}$ ,  $\dots$ ,  $\lambda_{i,j_r} = \sum_{l=1}^r a_{j_\ell, j_r+(i-1)}$ . Note that all the additions are proceeding modular  $n$ .

5) Using these  $A_i$  to set up a standard for adjusting a “poor” vector  $\alpha_n = (b_1, b_2, \dots, b_n)$  to  $\alpha'_n$ .

The above method can be realized by the following *Algorithm 1*:

**Step 1.** Give a circulant graph  $\Gamma_n$  with vector  $\alpha_n = (b_1, b_2, \dots, b_n)$ , and generate  $G_n = (I; A_n)$ .

**Step 2.** Calculate the distance  $d_n$  of binary linear code  $\mathcal{C}_n$  (*Algorithm 1-1*). If  $d_n \geq L_n$ , then  $\mathcal{C}_n$  is a good binary linear code, and stop. Or else, go to Step 3.

**Step 3.** Do adjustments of the elements of the “poor” vector  $\alpha_n$  as follows.

**Step 3.1.** Find “bad” codewords  $\beta_1, \beta_2, \dots, \beta_m$  such that their weights are  $d_n, d_n + 1, \dots, d_n + (m - 1)$  by *Algorithm 1-2*, where  $m = L_n - d_n$ . If there is no codeword with weight  $d_n + i$  ( $0 \leq i \leq m - 1$ ), then  $\beta_{i-1}$  is not under consideration.

**Step 3.2.** For each  $\beta_i$  ( $0 \leq i \leq m - 1$ ), we can find a combination of  $\beta_i$  by *Algorithm 1-2*. Suppose  $\beta_i = g_{j_1} + g_{j_2} + \dots + g_{j_r} = (u_i \mid v_i)$ , where  $u_i, v_i \in F_2^n$  and  $j_1, j_2, \dots, j_r \in \{1, 2, \dots, n\}$  and

$$\begin{aligned} g_{j_k} &= (e_{j_k} \mid a_{j_k}) \\ &= (0, \dots, 0, 1, 0, \dots, 0 \mid a_{j_k,1}, a_{j_k,2}, \dots, a_{j_k,j_k-1}, a_{j_k,j_k}, a_{j_k,j_k+1}, \dots, a_{j_k,n}) \\ &= (0, \dots, 0, 1, 0, \dots, 0 \mid b_{n-j_k+2}, b_{n-j_k+3}, \dots, b_n, b_1, b_2, \dots, b_{n-j_k+1}). \end{aligned}$$

**Step 3.3.** Determine whether each element 1 of the generator vertex  $\alpha_n$  is a “bad” element in the following way (since  $b_1 = 0$ , we begin with element  $b_2$ ). If  $b_2 = 1$ , then  $a_{j_1, j_1+1} = a_{j_2, j_2+1} = \dots = a_{j_r, j_r+1} = b_2 = 1$ . We calculate the exact value  $\lambda_{2, j_1} = \sum_{\ell=1}^r a_{j_\ell, j_1+1}$ ,  $\lambda_{2, j_2} = \sum_{\ell=1}^r a_{j_\ell, j_2+1}$ ,  $\dots$ ,  $\lambda_{2, j_r} = \sum_{\ell=1}^r a_{j_\ell, j_r+1}$ .

Note that  $\lambda_{2,j_k} = 0$  or  $\lambda_{2,j_k} = 1$  ( $1 \leq k \leq r$ ). Consider the set  $\Lambda_2 = \{\lambda_{2,j_1}, \lambda_{2,j_2}, \dots, \lambda_{2,j_r}\}$ . If the number of elements with value “0” in  $\Lambda_2$  is larger than the number of elements with value “1”, then the element  $b_2$  is called a “bad” element of the generator vector  $\alpha_n$ . If  $b_2$  is a “bad” element, then we change  $b_2 = 1$  into  $b'_2 = 0$  and obtain a new vector  $\alpha'_n = (b_1, b'_2, \dots, b_n)$ . Then we return to Step 1. If  $b_2$  is not a “bad” element or  $b_2 = 0$ , then we consider  $b_3$  and continue to determining whether  $b_3$  is a “bad” element. The procedure terminates till  $b_n$  has been considered.

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*Algorithm 1-1:* Minimum distance of a binary linear code

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Input: The value of  $n$ , the generator vector  $\alpha_n$  of a binary linear code  $\mathcal{C}_n$

Objective: The minimum distance of binary linear code  $\mathcal{C}_n$

1. Input the value of  $n$ , the generator vector  $\alpha_n = (b_1, b_2, \dots, b_n)$ ;
2. Obtain the generator matrix  $G = (I; A_n)$  of the binary linear code  $\mathcal{C}_n$ ;
3. Get the minimum distance of the binary linear code  $\mathcal{C}_n$ .

For example, let  $n = 19$  and  $\alpha_n = (0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 1)$ . The algorithm details are stated as follows:

*Program:*

```

n = 19;
a = [0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 1];
m = matrix(GF(2), [[a[(i - k)%n] for i in [0..(n - 1)] for k in [0..n - 1]]];
f = lambda s : sum(map(lambda x : m[x], s));
s = [];
for k in [1..8]:
    t = min([list(i).count(1) for i in Subsets(range(n), k).map(f)]);
    s+ = [t];
    print k, t;

```

*Output:*  $s_i$  : 1 2 3 4 5 6 7 8  
 $s'_i$  : 8 6 4 2 2 4 2 2

*Result:* The elements of the first row ( $s_1, s_2, \dots, s_8$ ) are the contribution of the matrix  $I$  for the weight of a codeword. The elements of the second row ( $s'_1, s'_2, \dots, s'_8$ ) are the contribution of the matrix  $A_{19}$  for the weight of a codeword. The value of  $\min\{s_i + s'_i \mid 1 \leq i \leq 8\} = 6$  is the minimum weight of the code  $\mathcal{C}_{19}$  and then the minimum distance of the code  $\mathcal{C}_{19}$  is also 6.

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*Algorithm 1-2:* “Bad” codewords and their combinations

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Input: The value of  $n$ , the generator vector  $\alpha_n$  of a binary linear code  $\mathcal{C}_n$

Objective: “Bad” codewords and their combinations

1. Input the value of  $n$ , the generator vector  $\alpha_n = (b_1, b_2, \dots, b_n)$ ;
2. Obtain the generator matrix  $G = (I; A_n)$  of the binary linear code  $\mathcal{C}_n$ ;
3. Get “bad” codewords  $\beta_1, \beta_2, \dots, \beta_m$  and a combination of each  $\beta_i$  ( $1 \leq i \leq m$ ).

For example, let  $n = 19$  and  $\alpha_n = (0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 1)$ .

The algorithm details are stated as follows:

*Program:*

```

n = 19;
a = [0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1]
m = matrix(GF(2), [[a[(i - k)%n] for i in [0..(n - 1)]] for k in [0..n - 1]]);
f = lambda s : sum(map(lambda x : m[x], s));
g = lambda s : str(sorted(map(lambda x : x + 1, s))).replace("'", ' ');
s = [];
for k in [1..8]:
    t = min([(i, list(f(i)).count(1)) for i in Subsets(range(n), k)]);
    key = lambda x : x[-1];
    s+ = [t];
    print k, t[1], g(t[0]);

```

*Output:*  $s_i \quad s'_i \quad \{j_1, j_2, \dots, j_r\}$  (as defined in Step 3.2)

1	8	{1}
2	6	{1, 9}
3	4	{1, 4, 12}
4	2	{1, 2, 6, 16}
5	2	{1, 2, 8, 11, 14}
6	4	{1, 2, 3, 4, 6, 13}
7	2	{1, 2, 4, 6, 7, 10, 17}
8	2	{1, 2, 3, 6, 7, 8, 12, 16}

Now we show how to use our *Algorithm 1* for searching [38, 19, 8] codes.

**Step 1.** Among all graphs with 19 vertices, we consider the graph  $\Gamma_{19}$ , which can be generated by the edge set  $E_1 = \{u_1u_2, u_1u_3, u_1u_5, u_1u_{10}, u_1u_{11}, u_1u_{16}, u_1u_{18}, u_1u_{19}\}$ , and then  $\alpha_{19} = (0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1)$ . Obviously,  $b_2 = b_3 = b_5 = b_{10} = b_{11} = b_{16} = b_{18} = b_{19} = 1$ .

**Step 2.** From the Code Tables, we know that the lower bound of the minimum distance of linear code [38, 19] over  $GF(2)$  is 8, that is,  $L_{19} = 8$ . By *Algorithm 1-1*, we obtain that the minimum distance  $d_{19}$  of the code  $(I; A_{19})$  is just 6, that is,  $d_{19} = 6$ . Clearly,  $6 = d_{19} < L_{19} = 8$ .

**Step 3.** Obviously,  $m = L_{19} - d_{19} = 2$ .

**Step 3.1.** From *Algorithm 1-2*, we find two “bad” codewords

$$\begin{aligned}
 \beta_1 &= (1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 \\
 &\quad | 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0), \\
 \beta_2 &= (1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0 \\
 &\quad | 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0).
 \end{aligned}$$

such that their weights are 6 and 7, that is,  $wt(\beta_1) = 6$  and  $wt(\beta_2) = 7$ .

**Step 3.2.** For  $\beta_1$ , we can find a combination of  $\beta_1 = \alpha_{19,1} + \alpha_{19,2} + \alpha_{19,6} + \alpha_{19,16}$  by *Algorithm 2-2*, where





is 8. The code  $\mathcal{C}'_{19}$  is a near optimal and also known best binary linear code over  $GF(2)$ . The weight enumerator of the code  $\mathcal{C}'_{19}$  is

$$\begin{aligned} W_{\mathcal{C}'_{19}}(z) = & 1 + 133z^8 + 2052z^{10} + 10108z^{12} + 36575z^{14} + 85595z^{16} \\ & + 127680z^{18} + 127680z^{20} + 85595z^{22} + 36575z^{24} + 10108z^{26} \\ & + 2052z^{28} + 133z^{30} + z^{38}. \end{aligned}$$

With the above approach and algorithms, we can also find three other near optimal binary linear [38, 19, 8] codes by the generator matrices  $G = (I; A''_{19})$ ,  $G = (I; A'''_{19})$  and  $G = (I; A''''_{19})$ . The circulant matrices  $A''_{19}$ ,  $A'''_{19}$  and  $A''''_{19}$  are separately generated by

$$\begin{aligned} \alpha''_{19} &= (0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1), \\ \alpha'''_{19} &= (0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 1), \\ \alpha''''_{19} &= (0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0). \end{aligned}$$

The weight enumerator of the code  $\mathcal{C}''_{19}$  is

$$\begin{aligned} W_{\mathcal{C}''_{19}}(z) = & 1 + 190z^8 + 1767z^{10} + 10507z^{12} + 36860z^{14} + 84341z^{16} \\ & + 128478z^{18} + 128478z^{20} + 84341z^{22} + 36860z^{24} + 10507z^{26} \\ & + 1767z^{28} + 190z^{30} + z^{38}. \end{aligned}$$

One can also check that the weight enumerators of the codes  $\mathcal{C}''''_{19}$  and  $\mathcal{C}'''_{19}$  are equal to the ones of  $\mathcal{C}''_{19}$  and  $\mathcal{C}'_{19}$ , respectively.

At the end of this section, we list some more binary linear codes constructed from known circulant graphs by applying *Algorithm 1*.

**1.** In [6], Danielsen got a  $(15, 2^{15})$  additive code from the vector  $(\omega, 0, 1, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0)$ , which corresponds to a circulant graph of order 15 with vector  $\alpha_{15} = (0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0)$ . Its graph code  $\mathcal{C}_{15} = [30, 15, 6]$  is a poor code. Applying *Algorithm 1*, we obtain a new vector  $\alpha'_{15} = (0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0)$ . The code  $\mathcal{C}'_{15} = [30, 15, 8]$  generated by  $\alpha'_{15}$  is an optimal binary linear code. The weight enumerator of this code  $\mathcal{C}'_{15}$  is

$$\begin{aligned} W_{\mathcal{C}'_{15}}(z) = & 1 + 450z^8 + 1848z^{10} + 5040z^{12} + 9045z^{14} + 9045z^{16} \\ & + 5040z^{18} + 1848z^{20} + 450z^{22} + z^{30}. \end{aligned}$$

**2.** Extending vector  $\alpha_{17} = (0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 1)$  of the  $(4, 4)$ -Ramsey graph, one can obtain a vector of length 19. For example, let  $\alpha_{19} = (0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1)$ , which corresponds to a circulant graph of order 19. It corresponds to the linear graph code  $\mathcal{C}_{19} = [38, 19, 6]$ , which is a poor code. By *Algorithm 1*, we obtain four new vectors as follows:

$$\begin{aligned} \alpha'_{19} &= (0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1), \\ \alpha''_{19} &= (0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0), \\ \alpha'''_{19} &= (0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 1), \\ \alpha''''_{19} &= (0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1). \end{aligned}$$

The codes  $\mathcal{C}'_{19}$  and  $\mathcal{C}''_{19}$  are [38, 19, 8] codes with the same weight enumerators. The weight enumerator of the code  $\mathcal{C}'_{19}$  is

$$\begin{aligned}
 W_{C'_{19}}(z) = & 1 + 133z^8 + 2052z^{10} + 10108z^{12} + 36575z^{14} + 85595z^{16} \\
 & + 127680z^{18} + 127680z^{20} + 85595z^{22} + 36575z^{24} + 10108z^{26} \\
 & + 2052z^{28} + 133z^{30} + z^{38}.
 \end{aligned}$$

The codes  $C'''_{19}$  and  $C''''_{19}$  are  $[38, 19, 8]$  codes with the same weight enumerators. The weight enumerator of the code  $C'''_{19}$  is

$$\begin{aligned}
 W_{C'''_{19}}(z) = & 1 + 190z^8 + 1767z^{10} + 10507z^{12} + 36860z^{14} + 84341z^{16} \\
 & + 128478z^{18} + 128478z^{20} + 84341z^{22} + 36860z^{24} + 10507z^{26} \\
 & + 1767z^{28} + 190z^{30} + z^{38}.
 \end{aligned}$$

These four binary linear codes are all near optimal and known best.

**3.** In addition, we consider graphs with large number of vertices by similar approach. Let  $C_{25,1}$  and  $C_{25,2}$  be the two codes generated by vectors  $\alpha_{25}^1 = (0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1)$  and  $\alpha_{25}^2 = (0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 1)$ , respectively. These two codes are both poor codes. Then by *Algorithm 1*, we find  $\alpha'_{25} = (0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1)$ , and  $\alpha''_{25} = (0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1)$ . Both  $\alpha'_{25}$  and  $\alpha''_{25}$  give binary linear codes known best. Let  $C'_{25}$  and  $C''_{25}$  be codes of  $\alpha'_{25}$  and  $\alpha''_{25}$ , respectively. It is easy to check that  $C'_{25}$  and  $C''_{25}$  are  $[50, 25, 10]$  codes, and the weight enumerator of the code  $C'_{25}$  and  $C''_{25}$  is

$$\begin{aligned}
 W_{C'_{25}}(z) = & 1 + 225z^{10} + 1250z^{11} + 3825z^{12} + 11525z^{13} + 28050z^{14} + 64005z^{15} \\
 & + 147075z^{16} + 294975z^{17} + 535075z^{18} + 9111100z^{19} + 1409205z^{20} \\
 & + 1999925z^{21} + 2642200z^{22} + 3219675z^{23} + 3623325z^{24} + 377243z^{25} \\
 & + 3621975z^{26} + 3216050z^{27} + 2643475z^{28} + 2009175z^{29} + 1408010z^{30} \\
 & + 904475z^{31} + 535400z^{32} + 292725z^{33} + 147525z^{34} + 68880z^{35} \\
 & + 27975z^{36} + 9775z^{37} + 3500z^{38} + 1125z^{39} + 375z^{40} + 125z^{41} \\
 = & W_{C''_{25}}(z).
 \end{aligned}$$

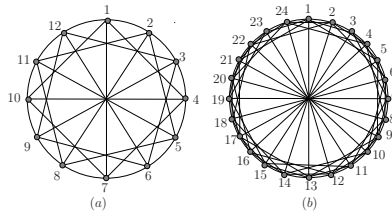


Figure 2: (a) A 5-valent graph  $\Gamma_{12}$ ; (b) Circulant graph  $\Gamma_{24}$ .

### 4 New additive codes searching from circulant graphs

In fact, Glynn et al. [11] obtained an optimal code from a circulant graph, called a 5-valent graph. Recall that  $\Gamma_{12}$  is a circulant graph of order 12; see Figure 2 (a). Let  $V(\Gamma_{12}) = \{u_1, u_2, \dots, u_{12}\}$ . For the vertex  $u_1$ , we let  $E_1 = \{v_1v_2, v_1v_4, v_1v_7, v_1v_{10}, v_1v_{12}\} \subseteq E(\Gamma_{12})$ . For the vertex  $u_2$ , we just rotate the above vertices and edges, that is, we only permit the existence of the edge set  $E_2 = \{v_2v_3, v_2v_4, v_2v_6, v_2v_{10}, v_2u_{12}\} \subseteq E(G)$ . For each vertex  $u_i \in V(\Gamma) \setminus \{u_1, u_2\} = \{u_3, u_4, \dots, u_{12}\}$ , we can also obtain an edge set  $E_i$  ( $3 \leq i \leq 17$ ). Observe that  $E(\Gamma_{12}) = \bigcup_{i=1}^{12} E_i$ . The adjacency matrix of the graph  $\Gamma_{12}$  is the following circulant matrix

$$A_{12} = \begin{pmatrix} 010100100101 \\ 101010010010 \\ 010101001001 \\ \dots\dots\dots \end{pmatrix}.$$

The above matrix can also be obtained by vector  $\alpha_{12} = (0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1)$ . Observe that this vector just corresponds to the set  $E_1$  of edges, which is an expression of the adjacency relation for the vertex  $u_1$ . We conclude that a 5-valent graph can be determined by the edge set  $E_1$ , and the adjacency matrix of this graph is determined by the above vector  $\alpha_{12}$ . Furthermore, the matrix  $A'_{12} = A_{12} + \omega I$  is also a circulant matrix, which can be obtained by vector  $\alpha'_{12} = (\omega, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1)$ . From the matrix  $A_{12} + \omega I$ , we can get a graph code  $\mathcal{C}_{12}$ . From the Code Tables, we know that  $\mathcal{C}_{12}$  is an optimal  $(12, 2^{12}, 6)$  additive code over  $GF(4)$ . The above statement suggests the following method for finding (near) optimal additive codes.

**Step 1.** Given an even integer  $n$ . Denote by  $L_n$  the lower bound of the additive code  $(n, 2^n)$  over  $GF(4)$ . From the Code Tables, we find the exact value of  $L_n$  for given  $n$ . We now construct a circulant graph  $\Gamma_n$  by a set  $E_1$  of edges as follows.

**Step 1.1.** Arrange the vertices from  $V(\Gamma_n) = \{u_1, u_2, \dots, u_n\}$  in a circular order.

**Step 1.2.** Determine the set  $E_1$  of edges satisfying  $|E_1| = L_n - 1$  or  $|E_1| = L_n + 1$ , where  $E_1 = \{u_iu_j \mid u_i \in N_{\Gamma_n}(u_1)\}$ . If  $|E_1| = L_n + 1$ , then

$$\begin{aligned} E_1 &= \{u_1u_2, u_1u_3\} \cup \{u_1u_{3+2.1}, u_1u_{3+2.2}, \dots, u_1u_{3+2.\frac{L_n-4}{2}}\} \cup \{u_1u_{\frac{n}{2}+1}\} \\ &\quad \cup \{u_1u_{n-1-2.\frac{L_n-4}{2}}, \dots, u_1u_{n-1-2.2}, u_1u_{n-1-2.1}\} \cup \{u_1u_n, u_1u_{n-1}\} \\ &= \{u_1u_2, u_1u_3\} \cup \{u_1u_5, u_1u_7, \dots, u_1u_{L_n-1}\} \cup \{u_1u_{\frac{n}{2}+1}\} \\ &\quad \cup \{u_1u_{n-L_n+3}, \dots, u_1u_{n-5}, u_1u_{n-3}\} \cup \{u_1u_n, u_1u_{n-1}\}. \end{aligned}$$

If  $|E_1| = L_n - 1$ , then

$$\begin{aligned} E_1 &= \{u_1u_2, u_1u_3\} \cup \{u_1u_{3+2.1}, u_1u_{3+2.2}, \dots, u_1u_{3+2.\frac{L_n-6}{2}}\} \cup \{u_1u_{\frac{n}{2}+1}\} \\ &\quad \cup \{u_1u_{n-1-2.\frac{L_n-6}{2}}, \dots, u_1u_{n-1-2.2}, u_1u_{n-1-2.1}\} \cup \{u_1u_n, u_1u_{n-1}\} \\ &= \{u_1u_2, u_1u_3\} \cup \{u_1u_5, u_1u_7, \dots, u_1u_{L_n-3}\} \cup \{u_1u_{\frac{n}{2}+1}\} \\ &\quad \cup \{u_1u_{n-L_n+5}, \dots, u_1u_{n-5}, u_1u_{n-3}\} \cup \{u_1u_n, u_1u_{n-1}\}. \end{aligned}$$

**Step 2.** By the edge set  $E_1$ , we write the vector  $\alpha_n$  corresponding to  $E_1$ . If  $|E_1| = L_n + 1$ , then

$$\alpha_n = (0 \quad u_2 \quad u_3 \quad u_5 \quad u_{L_n-1} \quad u_{\frac{n}{2}+1} \quad u_{n-L_n+3} \quad u_{n-3} \quad u_{n-1} \quad u_n \quad 1 \quad 1).$$

If  $|E_1| = L_n - 1$ , then

$$\alpha_n = (0 \quad u_2 \quad u_3 \quad u_5 \quad u_{L_n-3} \quad u_{\frac{n}{2}+1} \quad u_{n-L_n+5} \quad u_{n-3} \quad u_{n-1} \quad u_n \quad 1 \quad 1).$$

**Step 3.** Change the first component of the vector  $\alpha_n$  into  $\omega$ . Denote by  $\alpha'_n$  the new vector. We generate a circulant matrix  $A'_n$  from  $\alpha'_n$ ,

$$\alpha'_n = (\omega, 1, 1, 0, 1, 0, 1, \dots, 0, 1, 0, 0, \dots, 0, 1, 0, 0, \dots, 0, 1, 0, 1, 0, \dots, 1, 0, 1, 1).$$

**Step 4.** By *Algorithm 2*, we obtain the minimum distance  $d_n$  of the additive code  $\mathcal{C}_n$  and determine whether  $d_n = L_n$ . If so, the code  $\mathcal{C}_n$  is a (near) optimal or at least known best additive code.

Below is an algorithm (running in SAGE). For more details, we refer to [20].

*Algorithm 2:* Minimum distance of a circulant graph code

Input: The value of  $n$ , the generator vector  $\alpha_n$  of a circulant graph code  $\mathcal{C}_n$

Objective: The minimum distance of the circulant graph code  $\mathcal{C}_n$

1. Input the value of  $n$ , the generator vector  $\alpha_n = (b_1, b_2, \dots, b_n)$ ;
2. Obtain the generator matrix  $G$  of the circulant graph code  $\mathcal{C}_n$ ;
3. Get the minimum distance of the circulant graph code  $\mathcal{C}_n$ .

For example, let  $n = 24$  and  $\alpha_n = (\omega, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1)$ . The algorithm details are stated as follows:

*Program:*

```
F. < x >= GF(4, x')
n = 24;
a = [x, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1]
m = matrix(F, n, n, [[a[(i - k)%n] for i in [0..(n - 1)]] for k in [0..n - 1]]);
f = lambda s : sum(map(lambda x : m[x], s));
s = [ ];
for k in [1..8]:
    t = min([n - list(i).count(0) for i in Subsets(range(n), k).map(f)]);
    s+ = [t];
print s;
```

*Output:* [8]

```
[8, 8]
[8, 8, 10]
[8, 8, 10, 8]
[8, 8, 10, 8, 8]
[8, 8, 10, 8, 8, 8]
[8, 8, 10, 8, 8, 8, 8]
[8, 8, 10, 8, 8, 8, 8, 10]
```

*Result:* The minimum element of the last array is the minimum distance  $d_{24}$  of the code  $\mathcal{C}_{24}$ , that is,  $d_{24} = 8$ .

Inspired by the graph code  $\mathcal{C}_{12}$  which corresponds to the 5-valent graph, we hope to find out some other optimal additive codes for  $n = 24$ .

**Step 1.** Recall that  $L_{24}$  is the lower bound of the additive code  $(24, 2^{24})$  over  $GF(4)$ . From the Code Tables, we find that the exact value of  $L_{24}$  is 8, that is,  $L_{24} = 8$ . We now construct a circulant graph  $\Gamma_{24}$  by a set  $E_1$  of edges as follows.

**Step 1.1.** Arrange the vertices from  $V(\Gamma_n) = \{u_1, u_2, \dots, u_{24}\}$  in a circular order.

**Step 1.2.** Determine the set  $E_1$  of edges satisfying  $|E_1| = L_{24} - 1 = 7$  or  $|E_1| = L_{24} + 1 = 9$ , where  $E_1 = \{u_i u_1 \mid u_i \in N_{\Gamma_n}(u_1)\}$ . If  $|E_1| = 9$ , then  $E_1 = \{u_1 u_2, u_1 u_3, u_1 u_5, u_1 u_7, u_1 u_{13}, u_1 u_{19}, u_1 u_{21}, u_1 u_{23}, u_1 u_{24}\}$ . If  $|E_1| = 7$ , then  $E_1 = \{u_1 u_2, u_1 u_3, u_1 u_5, u_1 u_{13}, u_1 u_{21}, u_1 u_{23}, u_1 u_{24}\}$ . In this case, the circulant graph  $\Gamma_{24}$  can be found out; see Figure 2 (b).

**Step 2.** By the edge set  $E_1$ , we write the vector  $\alpha_{24}$  corresponding to  $E_1$ . If  $|E_1| = 9$ , then  $\alpha_{24} = (0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 1)$ . If  $|E_1| = 7$ , then  $\alpha_{24} = (0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1)$ .

**Step 3.** Change the first component of the vector  $\alpha_{24}$  into  $\omega$ . Denote by  $\alpha'_{24}$  the new vector. We generate a circulant matrix  $A'_{24}$  ( $A''_{24}$ ) from  $\alpha'_{24}$  ( $\alpha''_{24}$ ):

$$\begin{aligned} \alpha'_{24} &= (\omega, 1, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 1), \\ \alpha''_{24} &= (\omega, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1). \end{aligned}$$

**Step 4.** By *Algorithm 2*, we obtain the minimum distance  $d'_{24} = d''_{24} = 8$ . Therefore, both  $\mathcal{C}'_{24}$  and  $\mathcal{C}''_{24}$  are known best additive codes over  $GF(4)$ . The weight enumerators of the codes  $\mathcal{C}'_{24}$  and  $\mathcal{C}''_{24}$  are

$$\begin{aligned} W_{\mathcal{C}'_{24}}(z) &= 1 + 528z^8 + 13992z^{10} + 171276z^{12} + 1118040z^{14} + 3773517z^{16} \\ &\quad + 6218520z^{18} + 4413948z^{20} + 1034088z^{22} + 33306z^{24}, \\ W_{\mathcal{C}''_{24}}(z) &= 1 + 648z^8 + 13032z^{10} + 174636z^{12} + 1111320z^{14} + 3781917z^{16} \\ &\quad + 6211800z^{18} + 4417308z^{20} + 1033128z^{22} + 33426z^{24}. \end{aligned}$$

Applying the above method, we can obtain (near) optimal additive codes over  $GF(4)$  from the first two generator vectors.

$n$	$d$	$L_n$	First row of generator matrix	about the code
8	4	4	$(\omega, 1, 1, 0, 1, 0, 1, 1)$	optimal
8	4	4	$(\omega, 1, 0, 0, 1, 0, 0, 1)$	optimal
10	4	4	$(\omega, 1, 1, 0, 0, 1, 0, 0, 1, 1)$	optimal
10	4	4	$(\omega, 1, 0, 0, 0, 1, 0, 0, 0, 1)$	optimal
16	6	6	$(\omega, 1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1)$	optimal
22	8	8	$(\omega, 1, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 1)$	optimal
24	8	8	$(\omega, 1, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 1)$	known best
24	8	8	$(\omega, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1)$	known best

From the above analysis, we see that the circulant graphs under our consideration are all relatively sparse. So one may think that only sparse circulant graphs

produce optimal graph codes. However, the following fact gives it a negative answer. Danielsen [6] obtained an optimal additive code  $(30, 2^{30}, 12)$  from the vector  $\alpha_{30} = (\omega, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0)$ , which corresponds to a circulant graph of order 30 such that its adjacency matrix  $A_{30}$  is generated by the first row

$$\alpha_{30} = (0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0).$$

It is clear that the circulant graph is 17-regular. Since the order of this graph is 30, it follows that the degree of each vertex is relatively large, i.e., it is a relatively dense graph. Let

$\alpha_n = (\omega, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, \overbrace{1, 1, \dots, 1}^{n-21}, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0)$ . For  $n = 32$  and  $n = 34$ , we have the following vectors:

$$\begin{aligned} \alpha_{32} &= (\omega, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0), \\ \alpha_{34} &= (\omega, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0). \end{aligned}$$

By *Algorithm 2*, we know that  $C_{32}$  and  $C_{34}$  are  $(32, 2^{32}, 10)$  and  $(34, 2^{34}, 10)$  additive codes, respectively, and both are known best additive codes over  $GF(4)$ .

The weight enumerators of the codes  $C_{32}$  and  $C_{34}$  are

$$\begin{aligned} W_{C_{32}}(z) &= 1 + 1325z^{10} + 41973z^{12} + 745155z^{14} + 8030541z^{16} + 53150370z^{18} \\ &\quad + 213875634z^{20} + 510617670z^{22} + 691665390z^{24} \\ &\quad + 491629473z^{26} + 159600905z^{28} + 17838471z^{30} + 286740z^{32}, \\ W_{C_{34}}(z) &= 1 + 492z^{10} + 14373z^{12} + 291849z^{14} + 3494061z^{16} + 26279603z^{18} \\ &\quad + 123536402z^{20} + 357928154z^{22} + 620714798z^{24} + 614055698z^{26} \\ &\quad + 319190777z^{28} + 75747789z^{30} + 6157556z^{32} + 72095z^{34}. \end{aligned}$$

## 5 Concluding remarks

In recent years, graphs have been used to construct codes. But, usually they cannot produce good codes. Proper graphs have to be chosen in order to construct good codes. Because the structure of a circulant graph is very symmetric, the row vectors of its adjacency matrix may span a subspace with the property that the minimum Hamming distance among the vectors of the subspace is comparatively large. Our paper uses this possibility to develop two algorithms for searching for good binary linear codes and additive codes. Starting from a circulant graph, the algorithms modify the generator vector of the circulant graph successively, and get a good code at some step, or fail when all “1”’s of the generator vector have been checked. Therefore, sometimes our algorithms can find good codes, but sometimes they would fail to produce any good code. In general, the running time of our algorithms is exponential in the order  $n$  of the circulant graph, since we have to generate all the vectors of the subspace spanned by the row vectors of an  $n \times n$  or  $n \times 2n$  generator matrix, and this complexity cannot be lower down generally. However, as is seen in the above sections, the row vectors of a circulant are generated by a single vector—its first row vector. This gives us some

reasonable hope to use the property to lower down the complexity of generating the subspace spanned by the row vectors of a circulant, which will be left for us to further study.

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