# On sequences over a finite abelian group with zero-sum subsequences of forbidden lengths

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**Abstract**

Let $G$ be an additive finite abelian group. For every positive integer $\ell$, let $\mathrm{disc}_\ell(G)$ be the smallest positive integer $t$ such that each sequence $S$ over $G$ of length $|S| \geq t$ has a nonempty zero-sum subsequence of length not equal to $\ell$. In this paper, we determine $\mathrm{disc}_\ell(G)$ for certain finite groups, including cyclic groups, the groups $G = C_2 \oplus C_{2m}$ and elementary abelian 2-groups. Following Girard, we define $\mathrm{disc}(G)$ as the smallest positive integer $t$ such that every sequence $S$ over $G$ of length $|S| \geq t$ has two nonempty zero-sum subsequences of distinct lengths. We shall prove that $\mathrm{disc}(G) = \max\{\mathrm{disc}_\ell(G) \mid \ell \geq 1\}$ and determine $\mathrm{disc}(G)$ for finite abelian $p$-groups $G$, where $p \geq r(G)$ and $r(G)$ is the rank of $G$.

*Keywords*: Zero-sum subsequence; Davenport constant; $\mathrm{disc}(G)$; $\mathrm{disc}_\ell(G)$

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## 1 Introduction

Throughout this paper, let $G$ be an additive finite abelian group, $C_n$ denote a cyclic group of $n$ elements, and $C_n^k$ denote the direct sum of $k$ copies of $C_n$. It is well known that $|G| = 1$ or $G =$

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$C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid n_2 \cdots \mid n_r$, where $r = r(G)$ is the rank of $G$ and $n_r = \exp(G)$ is the exponent of $G$. Set

$$\mathsf{D}^*(G) = 1 + \sum_{i=1}^{r}(n_i - 1).$$

Let $p$ be a sufficiently large prime. In 1976, Erdős and Szemerédi [1] proved that if $S$ is a sequence of length $|S| = p$ over $C_p$ such that whose support contains at least three distinct terms, then $S$ has two nonempty zero-sum subsequences of distinct lengths, confirming a conjecture of Graham for sufficiently large primes. In 2010, Gao, Hamidoune and Wang [4] extended the above result to all positive integers $n$. A different proof of this result was given by Grynkiewicz [9] in 2011. Girard [8] posed a natural question of determining the smallest positive integer $t$, denoted by $\mathrm{disc}(G)$, such that every sequence $S$ over $G$ of length $|S| \geq t$ has two nonempty zero-sum subsequences of distinct lengths in 2012. Recently, Gao, Zhao and Zhuang [5] determined $\mathrm{disc}(G)$ for all elementary abelian 2-groups, the groups of rank at most two, and some other groups with large exponents. Around 2000, a similar invariant $E_k(G)$ was introduced by the first author and studied successfully by Schmid [16]. The invariant $E_k(G)$ is the smallest positive integer $t$ such that every sequence $S$ over $G$ of length $|S| \geq t$ has a zero-sum subsequence $T$ with $k \nmid |T|$. In this paper we determine $\mathrm{disc}(G)$ for finite abelian $p$-groups $G$ with $p \geq r(G)$ and conduct a further detailed investigation on this problem by introducing the following constant.

**Definition 1.1** *For every positive integer $\ell$, define $\mathrm{disc}_\ell(G)$ to be the smallest positive integer $t$ such that every sequence $S$ over $G$ of length $|S| \geq t$ has a nonempty zero-sum subsequence $T$ of length $|T| \neq \ell$.*

It is easy to see that $\mathrm{disc}(G) = \max\{\mathrm{disc}_\ell(G) \mid \ell \geq 1\}$ (see Proposition 2.1). Let $\mathsf{D}(G)$ denote the Davenport constant of $G$, which is defined as the smallest positive integer $d$ such that every sequence over $G$ of length at least $d$ has a nonempty zero-sum subsequence. Our main results are as follows.

**Theorem 1.2** *Let $G$ be a finite abelian group. Then*

*1). $\mathrm{disc}_\ell(G) = \mathsf{D}(G) + 1$, if $\ell = 1$.*

*2). $\mathrm{disc}_\ell(G) = \begin{cases} \mathsf{D}(G) + 1, & G \text{ is not cyclic} \\ 2\mathsf{D}(G), & G \text{ is cyclic} \end{cases}$, if $\ell = \mathsf{D}(G)$.*

*3). $\mathrm{disc}_\ell(G) = \mathsf{D}(G)$, if $\ell \geq \mathsf{D}(G) + 1$.*

According to the above theorem, it would be sufficient to consider the case that $\ell \in [2, \mathsf{D}(G) - 1]$ when study $\mathrm{disc}_\ell(G)$. We derive the precise values of $\mathrm{disc}_\ell(G)$ for certain groups.

**Theorem 1.3** *Let $\ell \in [2, \mathsf{D}(G) - 1]$ and $m, n$ be positive integers. Then*

*1). $disc_\ell(G) = n + 1$, if $G$ is a cyclic group of order $n \geq 3$.*

*2).* $$disc_\ell(G) = \begin{cases} 2m+3, & \ell \in [2, 2m-2] \text{ and } \ell \text{ is even} \\ 2m+2, & \ell \in [3, 2m-1] \text{ and } \ell \text{ is odd} \\ 4m+1, & \ell = 2m \end{cases}, \text{ if } G = C_2 \oplus C_{2m}.$$

**Theorem 1.4** *Let $G = C_2^r$ with $r \geq 2$, and let $\ell \in [2, r]$. Then*

$$disc_\ell(G) = r + u_1 + 1,$$

*where* $u_1 = \max\{ u \mid 2^{u-1} \mid \ell,\ \ell \cdot \frac{2^u - 1}{2^{u-1}} - u \leq r \}$.

**Theorem 1.5** *Let $p$ be a prime and let $G$ be a finite abelian $p$-group with $r(G) \geq 3$. If $p \geq r(G)$, then $disc(G) = \mathsf{D}(G) + \exp(G)$.*

The paper is organized in the following way. In Section *2* we recall some basic notions, provide several preliminary results and we also give a proof for Theorem 1.2. In Section *3* and Section *4* we determine $\mathrm{disc}_\ell(G)$ on cyclic groups $C_n$, the groups $G = C_2 \oplus C_{2m}$ and elementary abelian 2-groups respectively, and prove our two main results: Theorem 1.3 and Theorem 1.4. Finally, in Section *5* both constants $\mathrm{disc}(G)$ and $\mathrm{disc}_\ell(G)$ for finite abelian $p$-groups are investigated.

## 2 Preliminaries

Throughout the paper, let $\mathbb{N}$ denote the set of positive integers. For real numbers $a \leq b$, we set $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. By a sequence over $G$ we mean a finite sequence of terms from $G$ where the order is disregarded and repetition is allowed. We consider sequences as elements of the free abelian monoid $\mathcal{F}(G)$ over $G$ and our notation and terminology coincides with [3, 7, 10]. A sequence $S = g_1 \cdot \ldots \cdot g_l = \prod_{i=1}^{l} g_i$ over $G$ is called a zero-sum sequence if $\sum_{i=1}^{l} g_i = 0 \in G$. $S$ is called zero-sum free if it contains no nonempty zero-sum subsequence. If $S$ is a zero-sum sequence and each proper subsequence is zero-sum free, then $S$ is called a minimal zero-sum sequence.

Let $S$ be a sequence over $G$. By $\mathrm{supp}(S)$ we denote the subset of $G$ consisting of all elements which occur in $S$. The sum of elements in $S$ is denoted by $\sigma(S)$, and the maximal repetition of a term in S is denoted by $h(S)$. If $T$ is a subsequence of $S$, we denote by $ST^{-1}$ the sequence obtained from $S$ by deleting $T$. Let

$$\sum(S) = \{\sigma(T) \mid 1 \neq T | S\},$$

where $T|S$ means $T$ is a subsequence of $S$, and 1 denotes the empty sequence.

Now we recall some well-known results on Davenport constant, which assert that $\mathsf{D}(G) = \mathsf{D}^*(G)$ if $G$ satisfies any one of the following conditions (see [3], [12], [13]):

1). $G$ has rank at most two;

2). $G$ is an abelian $p$-group;

3). $G = C_2 \oplus C_m \oplus C_n$ with $2 \mid m \mid n$.

We first give several easy observations on $\mathrm{disc}_\ell(G)$ and $\mathrm{disc}(G)$.

**Proposition 2.1** *$disc(G) = \max\{disc_\ell(G) \mid \ell \geq 1\}$.*

*Proof.* By the definition of $\mathrm{disc}(G)$, every sequence $S$ over $G$ of length $\mathrm{disc}(G)$ has two nonempty zero-sum subsequences of distinct lengths, and thus, for every $\ell \geq 1$, $S$ has a nonempty zero-sum subsequence of length not equal to $\ell$, whence $\mathrm{disc}(G) \geq \mathrm{disc}_\ell(G)$, so $\mathrm{disc}(G) \geq \max\{\mathrm{disc}_\ell(G) \mid \ell \geq 1\}$. On the other hand, let $T$ be a sequence over $G$ of length $\mathrm{disc}(G) - 1$ such that $T$ has no two nonempty zero-sum subsequences of distinct lengths. Note the obvious fact that $\mathrm{disc}(G) - 1 \geq \mathsf{D}(G)$, so we infer that all nonempty zero-sum subsequences of $T$ have the same length $\ell_1$ (say). Hence, $\max\{\mathrm{disc}_\ell(G) \mid \ell \geq 1\} \geq \mathrm{disc}_{\ell_1}(G) \geq \mathrm{disc}(G)$. Therefore, $\mathrm{disc}(G) = \max\{\mathrm{disc}_\ell(G) \mid \ell \geq 1\}$. □

**Lemma 2.2** *Let $G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ ($1 < n_1 \mid \cdots \mid n_r$). Then*

$$\max\{\mathsf{D}(G), \mathsf{D}^*(G) + 1\} \leq disc_\ell(G) \leq \min\{\mathsf{D}(G) + \ell, disc(G)\},$$

*where $\ell \in [2, \mathsf{D}(G) - 1]$.*

*Proof.* To prove the right-hand side inequality, we first show $\mathrm{disc}_\ell(G) \leq \mathsf{D}(G) + \ell$. In fact, let $S$ be any sequence over $G$ of length $\mathsf{D}(G) + \ell$, and $S_1$ be a nonempty zero-sum subsequence of $S$. If $|S_1| \neq \ell$, we are done. If $|S_1| = \ell$, we can obtain a nonempty zero-sum subsequence $S_2$ of $SS_1^{-1}$ since $|SS_1^{-1}| = \mathsf{D}(G)$. Thus $S_1S_2$ is a nonempty zero-sum subsequence of length $|S_1S_2| \neq \ell$, implying $\mathrm{disc}_\ell(G) \leq \mathsf{D}(G) + \ell$. In addition, by Proposition 2.1, $\mathrm{disc}_\ell(G) \leq \mathrm{disc}(G)$ for every $\ell \geq 1$. Therefore, $\mathrm{disc}_\ell(G) \leq \min\{\mathsf{D}(G) + \ell,\ \mathrm{disc}(G)\}$.

We now handle the left-hand side inequality. Let $e_1, \cdots, e_r$ be a basis of $G$ with $\mathrm{ord}(e_i) = n_i$ for each $i \in [1, r]$, and $S = (-\sigma(\prod_{i=1}^r e_i^{\ell_i})) \prod_{i=1}^r e_i^{n_i - 1}$ be a sequence over $G$ of length $\mathsf{D}^*(G)$, where $0 \leq \ell_i \leq n_i - 1$, $\sum_{i=1}^r \ell_i = \ell - 1$. Then each nonempty zero-sum subsequence of $S$ is in the form of $(-\sigma(\prod_{i=1}^r e_i^{\ell_i})) \prod_{i=1}^r e_i^{\ell_i}$ with length $\ell$, implying $\mathrm{disc}_\ell(G) \geq \mathsf{D}^*(G) + 1$. Obviously, $\mathrm{disc}_\ell(G) \geq \mathsf{D}(G)$. Therefore, $\mathrm{disc}_\ell(G) \geq \max\{\mathsf{D}(G), \mathsf{D}^*(G) + 1\}$. □

**Lemma 2.3** *Let $G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$. If $\mathsf{D}(G) = \mathsf{D}^*(G)$, then $disc_{n_i}(G) = \mathsf{D}(G) + n_i$ for each $i \in [1, r]$.*

*Proof.* Let $e_1, e_2, \cdots, e_r$ be a basis of $G$, $\mathrm{ord}(e_j) = n_j$ for each $j \in [1, r]$. Let $i \in [1, r]$ and $S = e_i^{n_i} \prod_{j=1}^r e_j^{n_j - 1}$ be a sequence over $G$ of length $\sum_{j=1}^r (n_j - 1) + n_i = \mathsf{D}(G) + n_i - 1$. Then all nonempty zero-sum subsequences of $S$ have the same length $n_i$, implying $\mathrm{disc}_{n_i}(G) \geq \mathsf{D}(G) + n_i$. On the other hand, by Lemma 2.2, $\mathrm{disc}_{n_i}(G) \leq \mathsf{D}(G) + n_i$. Therefore, $\mathrm{disc}_{n_i}(G) = \mathsf{D}(G) + n_i$. □

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.**

1). Let $T$ be a zero-sum free sequence over $G$ of length $\mathsf{D}(G) - 1$. Then $S = T0$ is a sequence of length $\mathsf{D}(G)$ and $\{0\}$ is the only nonempty zero-sum subsequence of $S$. Hence $\mathrm{disc}_1(G) \ge \mathsf{D}(G) + 1$. On the other hand, every sequence of length $\mathsf{D}(G) + 1$ has at least one nonempty zero-sum subsequence of length $k \ge 2$, and thus $\mathrm{disc}_1(G) \le \mathsf{D}(G) + 1$. Therefore, $\mathrm{disc}_1(G) = \mathsf{D}(G) + 1$.

2). If $G$ is cyclic, then the result follows from Lemma 2.3. Next assume that $G$ is not cyclic. Then $\mathsf{D}(G) \ge \mathsf{D}^*(G) > \exp(G)$. Let $S$ be a sequence over $G$ of length $\mathsf{D}(G)+1$. If all nonempty zero-sum subsequences of $S$ have the same length $\mathsf{D}(G)$, then $|\mathrm{supp}(S)| = 1$. So we have a nonempty zero-sum subsequence of $S$ of length $\exp(G) < \mathsf{D}(G)$, a contradiction. Hence, $\mathrm{disc}_{\mathsf{D}(G)}(G) \le \mathsf{D}(G) + 1$. On the other hand, by the definition of $\mathsf{D}(G)$, there is a minimal zero-sum sequence over $G$ of length $\mathsf{D}(G)$, hence $\mathrm{disc}_{\mathsf{D}(G)}(G) \ge \mathsf{D}(G) + 1$. Therefore, $\mathrm{disc}_{\mathsf{D}(G)}(G) = \mathsf{D}(G) + 1$.

3). Let $\ell \ge \mathsf{D}(G) + 1$. By the definition of $\mathsf{D}(G)$, every sequence $S$ over $G$ of length $\mathsf{D}(G)$ has a nonempty zero-sum subsequence $S_1$ of length $|S_1| \le \mathsf{D}(G) < \ell$. Therefore, $\mathrm{disc}_\ell(G) \le \mathsf{D}(G)$. Clearly, $\mathrm{disc}_\ell(G) \ge \mathsf{D}(G)$. Hence, we have the desired result. □

## 3 $\mathrm{disc}_\ell(G)$ on abelian groups $G$ with rank $r(G) \le 2$

In this section, we determine $\mathrm{disc}_\ell(G)$ for cyclic groups $G$ and for groups $G \cong C_2 \oplus C_{2m}$ for all $\ell, m \in \mathbb{N}$. We first list a few useful lemmas.

**Lemma 3.1** *[4, Theorem 1.1] Let $G$ be a cyclic group of order $|G| = n \ge 2$ and $S$ a sequence over $G$ of length $|S| = n$. If $|\mathrm{supp}(S)| \ge 3$, then $S$ has two nonempty zero-sum subsequences with distinct lengths.*

**Lemma 3.2** *Let $G$ be a cyclic group of order $|G| = n \ge 3$ and $S$ a zero-sum free sequence over $G$ of length $|S| = \ell \ge \frac{n+1}{2}$. Then there is a $g \in G$ with $\mathrm{ord}(g) = n$ such that $S = (n_1 g) \cdot \ldots \cdot (n_\ell g)$ where $1 = n_1 \le \cdots \le n_\ell$, $n_1 + \cdots + n_\ell < n$ and $\sum(S) = \{g, 2g, 3g, \cdots, (n_1 + \cdots + n_\ell)g\}$.*

*Proof.* This result was first proved by Savchev and Chen [15], and Yuan [17] independently, and one can also find a proof in [6, Theorem 5.1.8] . □

**Lemma 3.3** *Let $G$ be a cyclic group of order $|G| = n \ge 2$ and $S$ a sequence over $G$ of length $|S| = n$. Then $S$ has a nonempty zero-sum subsequence $T$ such that $|T| \le h(S)$.*

*Proof.* See [7, Theorem 5.7.3]. □

**Lemma 3.4** *Let $G$ be a cyclic group of order $|G| = n \geq 2$ and $S$ a sequence over $G$ of length $|S| = n+1$ and with $|\operatorname{supp}(S)| = 2$. Then there exist two nonempty zero-sum subsequences of $S$ of distinct lengths.*

*Proof.* Let $g \in G$ with $\operatorname{ord}(g) = n$ and $S = (ag)^s(cg)^t$, where $a,\ c \in [0, n-1]$ distinct, and $s,\ t \in [1, n]$ with $t \leq s$ and $s + t = n + 1$. If $t = 1$, then $S = (ag)^n(cg)$. If $\operatorname{ord}(ag) \neq n$, then $S_1 = (ag)^n$ and $S_2 = (ag)^{\operatorname{ord}(ag)}$ are two nonempty zero-sum subsequences of distinct lengths. If $\operatorname{ord}(ag) = n$, then $\sum((ag)^n) = G$, we conclude that there is a subsequence $S_3$ of $(ag)^n$ such that $\sigma(S_3) = -(cg)$, so $S_3(cg)$ is a nonempty zero-sum subsequence of length not equal to $n$. Next, suppose that $t \geq 2$. Then $t \leq s \leq n-1$. Assume to the contrary that all the nonempty zero-sum subsequences of $S$ have the same length $r$. We distinguish two cases.

Case 1. $r \leq \frac{n+1}{2}$. Choose a nonempty zero-sum subsequence $T$ of $S$. Then $|T| = r$ and $ST^{-1}$ is a zero-sum free sequence with $|ST^{-1}| \geq \frac{n+1}{2}$. By Lemma 3.2, there is a $h \in G$ with $\operatorname{ord}(h) = n$ such that $ST^{-1} = h^u(xh)^v$, where $2 \leq x$ and $u + xv \leq n-1$. We put $T = h^\tau(xh)^w$. Clearly, $u \geq 1$ and $w \geq 1$. We first note that

$$x \geq u + 1. \tag{3.1}$$

Otherwise, $h^{\tau+x}(xh)^{w-1}$ is a nonempty zero-sum subsequence of $S$ of length greater than $r$, a contradiction. We claim that $v \geq 1$. Otherwise, $u = |ST^{-1}| \geq \frac{n+1}{2}$, and by (3.1), we have $x \geq u + 1 \geq n - u + 2 > n - u$, so $n - x < u$. Therefore, $h^{n-x}(xh)$ is a nonempty zero-sum subsequence of length $n - x + 1 \leq n - u = \tau + w - 1 = r - 1$, a contradiction. Now we have $u + vx \geq u + v(u+1) = 2(u+v) + (u-1)(v-1) - 1 \geq n$, a contradiction.

Case 2. $r > \frac{n+1}{2}$, i.e.

$$r \geq \lceil \frac{n}{2} \rceil + 1. \tag{3.2}$$

By Lemma 3.3, we have

$$r \leq s. \tag{3.3}$$

We first assert that $(a, n) = 1$. If $(a, n) \geq 2$, then $(ag)^{\frac{n}{(a,n)}}$ is a nonempty zero-sum subsequence of $S$ of length $\frac{n}{(a,n)} < \frac{n+1}{2} < r$, a contradiction. Hence, $\operatorname{ord}(ag) = \frac{ord(g)}{(ord(g),a)} = n$ and there is a $b \in [2, n-1]$ such that $cg = b(ag)$.

Subcase 2.1. $n - b \leq s$. Clearly, $(ag)^{n-b}(cg)$ is a nonempty zero-sum subsequence of $S$ of length $n - b + 1 = r$. By (3.2) and (3.3), we have $\lceil \frac{n}{2} \rceil + 1 \leq r = n - b + 1 \leq s$, so $n - s + 1 \leq b \leq \lfloor \frac{n}{2} \rfloor$. Thus $0 \leq n - 2b < n - b \leq s$. Now, $(ag)^{n-2b}(cg)^2$ is a nonempty zero-sum subsequence of $S$ of length $n - 2b + 2 \neq n - b + 1$, a contradiction.

Subcase 2.2. $n - b \geq s + 1$. Note that $b \leq n - s - 1 \leq \lfloor \frac{n}{2} \rfloor - 2 < s$ (since $s \geq r \geq \lceil \frac{n}{2} \rceil + 1$).

If $tb < n$, we have $0 < n - tb = t - 1 - tb + s < s$ and $0 < n - tb + b = t - 1 - b(t-1) + s < s$. Thus $(ag)^{n-tb}(cg)^t$ and $(ag)^{n-tb+b}(cg)^{t-1}$ are two nonempty zero-sum subsequences of $S$ of distinct lengths, a contradiction.

If $tb \geq n$, then there is $t_1 \in [1, t-1]$ such that $t_1 b < n$ and $(t_1 + 1)b \geq n$, so $0 < n - t_1 b \leq b < s$. Since $2b < n$, we have $t_1 \geq 2$. Therefore, $(ag)^{n - t_1 b}(cg)^{t_1}$ is a nonempty zero-sum subsequence of $S$

of length $r = n - t_1b + t_1$. By inequality (3.2) we have $2b - 2 \le t_1b - t_1 = n - r \le n - \lceil \frac{n}{2} \rceil - 1$, so $2b \le \lfloor \frac{n}{2} \rfloor + 1 \le s$ and $0 < n - t_1b + b \le b + b \le s$. Therefore, $(ag)^{n-t_1b+b}(cg)^{t_1-1}$ is a nonempty zero-sum subsequence of $S$ of length $n - t_1b + b + t_1 - 1 \ne n - t_1b + t_1$, yielding a contradiction. Therefore, $S$ must have two nonempty zero-sum subsequences of distinct lengths. □

**Lemma 3.5** *Let $G = C_m \oplus C_n$ with $2 \mid m \mid n$. Then*

$$\mathrm{disc}_\ell(G) = \begin{cases} 2m + n - 1, & \ell = m \\ m + n, & \ell \in [2, \mathsf{D}(G) - 1] \text{ and } \ell \text{ is odd} \\ m + 2n - 1, & \ell = n \end{cases}.$$

*Proof.* Note that $\mathsf{D}(G) = m + n - 1 = \mathsf{D}^*(G)$. If $\ell = m$ or $n$, by Lemma 2.3 we derive the desired results.

If $\ell \in [2, \mathsf{D}(G) - 1]$ and $\ell$ is odd, by Lemma 2.2 we obtain that $\mathrm{disc}_\ell(G) \ge \mathsf{D}^*(G) + 1 = m + n$. We now prove $\mathrm{disc}_\ell(G) \le m + n$. Let $S = \prod_{i=1}^{m+n} g_i$ be a sequence over $G$ and assume that all the nonempty zero-sum subsequences of $S$ have the same length $\ell$. We consider another finite abelian group $G' = C_2 \oplus G$ and a new sequence $S' = \prod_{i=1}^{m+n} (x, g_i)$ over $G'$ with $\mathrm{ord}(x) = 2$. Since $\mathsf{D}(G') = m + n = |S'|$, we obtain a nonempty zero-sum subsequence $T' = \prod_{k=1}^{|T'|} (x, g_{i_k})$ of $S'$ with $\mathrm{ord}(x) \mid |T'|$. Clearly, the corresponding sequence $T = \prod_{k=1}^{|T'|} g_{i_k}$ is a nonempty zero-sum subsequence of $S$ with $|T| = |T'|$ and $2 \mid |T|$. Thus $|T| \ne \ell$ as $\ell$ is odd, a contradiction. Hence, $\mathrm{disc}_\ell(G) \le m + n$ and we are done. □

The Erdős-Ginzburg-Ziv constant $\mathsf{s}(G)$ is defined as the smallest positive integer $t \in \mathbb{N}$ such that every sequence $S$ over $G$ of length $|S| \ge t$ has a nonempty zero-sum subsequence $T$ of length $|T| = \exp(G)$. The precise value of $\mathsf{s}(G)$ is known for groups $G$ of rank $r(G) \le 2$ (see [7, Theorem 5.8.3]) and for progress on groups of higher rank we refer to [2]. Here we need this constant of elementary 2-groups.

**Lemma 3.6** *[7, Corollary 5.7.6] For every $r \in \mathbb{N}$, we have $\mathsf{s}(C_2^r) = 2^r + 1$.*

We are now ready to give a proof for Theorem 1.3.

**Proof of Theorem 1.3.**

1). Let $G$ be a cyclic group of order $n$. By Lemma 2.2, we have $\mathrm{disc}_\ell(G) \ge n + 1$.

We next show that $\mathrm{disc}_\ell(G) \le n + 1$. Let $S$ be a sequence over $G$ of length $n + 1$. We show that $S$ has a nonempty zero-sum subsequence of length not equal to $\ell$. We may always assume that $0 \notin S$. If $|\mathrm{supp}(S)| = 1$, set $S = g^{n+1}$, then $g^n$ is a nonempty zero-sum subsequence of length $n \ne \ell$. If $|\mathrm{supp}(S)| \ge 2$, by Lemma 3.1 and Lemma 3.4, we can obtain two nonempty zero-sum subsequences of $S$ of distinct lengths, hence there must be a nonempty zero-sum subsequence of length not equal to $\ell$.

2). The results follow from Lemma 3.5 except for the case when $\ell \in [4, 2m-2]$ *and* $\ell$ is even. Next we consider this case. Let $e_1, e_2$ be a basis of $G$ with $\mathrm{ord}(e_1) = 2$ and $\mathrm{ord}(e_2) = 2m$. Consider the sequence $S_0 = e_2^{2m-1} \cdot (e_1 + (m+1-\frac{\ell}{2})e_2)^3$ over $G$ of length $|S_0| = 2m+2$. Clearly, all nonempty zero-sum subsequences of $S_0$ have the same length $\ell$. Hence, $\mathrm{disc}_\ell(G) \geq 2m+3$. To show the equality, it is sufficient to prove that any sequence $S$ over $G$ of length $2m+3$ has a nonempty zero-sum subsequence of length not equal to $\ell$.

Let $\phi : G = C_2 \oplus C_{2m} \to C_2 \oplus C_2$ be the natural homomorphism with $\ker(\phi) = C_m$ (up to isomorphism). Let $S$ be a sequence over $G$ of length $2m+3$. Applying Lemma 3.6 to $\phi(S)$ repeatedly, we get a decomposition $S = S_1 \cdot \ldots \cdot S_m \cdot S'$ with $|S_i| = 2$, $\sigma(S_i) \in \ker(\phi)$ for $i \in [1, m]$, and $|S'| = 3 (= \mathsf{D}(C_2^2))$, so we can find a subsequence $S_{m+1}$ of $S'$ such that $\sigma(\phi(S_{m+1})) = 0$, i.e. $\sigma(S_{m+1}) \in \ker(\phi)$ and $|S_{m+1}| \in [1, 3]$. Set $T = \prod_{i=1}^{m+1}(\sigma(S_i))$. Then $T$ is a sequence over $\ker(\phi) = C_m$. For $\frac{\ell}{2} \in [2, m-1]$ , by 1) and Theorem 1.2, there is a nonempty zero-sum subsequence $T_1 = \prod_{j=1}^{t}(\sigma(S_{i_j}))$ of $T$ over $C_m$ of length $t \neq \frac{\ell}{2}$. If $|S_{i_j}| = 2$ for all $j \in [1, t]$, then the sequence $\prod_{j=1}^{t} S_{i_j}$ is a nonempty zero-sum subsequence of $S$ of length not equal to $\ell$; otherwise, $\sigma(S_{m+1}) \mid T_1$ and $|S_{m+1}| = 1$ or $|S_{m+1}| = 3$, so $\prod_{j=1}^{t} S_{i_j}$ is a nonempty zero-sum subsequence of $S$ of odd length (not equal to $\ell$).

Therefore, $\mathrm{disc}_\ell(G) = 2m+3$, where $\ell \in [4, 2m-2]$ and $\ell$ is even, and we are done. □

## 4 $\mathrm{disc}_\ell(G)$ on Elementary Abelian 2-Groups

In this section, we determine $\mathrm{disc}_\ell(G)$ for elementary abelian 2-groups. A similar method used in [5] will be adopted to derive the main result.

**Lemma 4.1** *[5, Lemma 4.2] Let $t$ and $r$ be two positive integers with $t \geq 2$, and let $S = e_1 \cdot \ldots \cdot e_r x_1 \cdot \ldots \cdot x_t$ be a sequence of nonzero terms over $C_2^r$ of length $r+t$, where $e_1, \cdots, e_r$ form a basis of $C_2^r$. For each $i \in [1, t]$, let $A_i \subset [1, r]$ be a nonempty subset such that $x_i = \sum_{j \in A_i} e_j$. If every nonempty zero-sum subsequence of $S$ has the same length $\ell$, then $|A_i| = \ell - 1$ and*

$$|\cap_{i \in I} A_i| = \frac{\ell}{2^{|I|-1}}$$

*holds for every subset $I \subset [1, t]$ of cardinality $|I| \in [2, t]$. In particular, we have $\ell \equiv 0 \pmod{2^{t-1}}$.*

**Lemma 4.2** *Let $G = C_2^r$ with $r \geq 2$ and let $\ell \in [2, r]$. For $u \in \mathbb{N}$, if $2^{u-1} \mid \ell$ and $\ell \cdot \frac{2^u-1}{2^{u-1}} - u \leq r$, then there is a sequence $S$ over $G$ of length $r+u$ such that all nonempty zero-sum subsequences of $S$ have the same length $\ell$.*

*Proof.* To construct the sequence $S$, let us first take a basis $e_1, \cdots, e_r$ of $C_2^r$. The assumption that $\ell \cdot \frac{2^u-1}{2^{u-1}} - u \leq r$ allows us to find $2^u - 1$ disjoint subsets of $[1, r]$, each of which is labeled by a nonempty subset $I \subset [1, u]$ and denoted by $E_I$, satisfying the following conditions:

1). $|E_I| = \frac{\ell}{2^{u-1}}$ if $|I| \in [2, u]$.

2). $|E_I| = \frac{\ell}{2^{u-1}} - 1$ if $|I| = 1$.

( We note that if $l = 2^{u-1}$ there are exactly $u$ empty subsets among the above $2^u - 1$ disjoint subsets, which are denoted by $E_{\{1\}}, E_{\{2\}}, \cdots, E_{\{u\}}$.)

We now define $u$ subsets $A_1, \cdots, A_u$ of $[1, r]$ in the following way: each $j \in [1, r]$ belongs to $A_k$ if and only if there is a subset $I \subset [1, u]$ containing $k$ such that $j \in E_I$. It follows that $A_i = \cup_{i \in I \subset [1,u]} E_I$ for each $i \in [1, u]$. Therefore,

$$|A_i| = \sum_{i \in I \subset [1,u]} |E_I| = \frac{\ell}{2^{u-1}} - 1 + \sum_{i \in I \subset [1,u], |I| \geq 2} \frac{\ell}{2^{u-1}} = \ell - 1 \tag{4.1}$$

for every $i \in [1, u]$, and

$$| \cap_{i \in J} A_i| = \sum_{J \subset I \subset [1,u]} |E_I| = \frac{\ell}{2^{u-1}} \cdot 2^{u-|J|} = \frac{\ell}{2^{|J|-1}}, \tag{4.2}$$

for every $J \subset [1, u]$ with $|J| \geq 2$. Let

$$x_i = \sum_{j \in A_i} e_j$$

for each $i \in [1, u]$, and let

$$S = e_1 \cdot \ldots \cdot e_r x_1 \cdot \ldots \cdot x_u.$$

If $T$ is any nonempty zero-sum subsequence of $S$, then $T$ is of the form

$$T = \prod_{i \in I} x_i \prod_{j \in A} e_j$$

for some nonempty subset $I$ of $[1, u]$ and some subset $A$ of $[1, r]$.

We next show the following result.

**Claim.**
$$|A| = \sum_{k=1}^{|I|} (-2)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq u,\ i_1, \cdots, i_k \in I} |A_{i_1} \cap \cdots \cap A_{i_k}|.$$

In fact, let $j \in \cup_{i \in I} A_i$ and $\lambda(j) = \left|\{i \in I \mid j \in A_i\}\right|$. Since $T$ is zero-sum, clearly, $j \in A$ if and only if $\lambda(j)$ is odd. Let $r_j$ be the number of the times that $j$ is counted in the right side of the equality in the above Claim. Then,

$$r_j = \binom{\lambda(j)}{1} - 2\binom{\lambda(j)}{2} + 2^2\binom{\lambda(j)}{3} - \cdots + (-2)^{\lambda(j)-1}\binom{\lambda(j)}{\lambda(j)} = \frac{1 - (1-2)^{\lambda(j)}}{2}.$$

Therefore, $r_j = 1$ if $\lambda(j)$ is odd and $r_j = 0$ if $\lambda(j)$ is even. This proves the Claim.

By the above claim, (4.1) and (4.2), we obtain

$$\begin{aligned}|A| &= \sum_{k=1}^{|I|}(-2)^{k-1}\sum_{1\le i_1<\cdots<i_k\le u,\ i_1,\ \cdots,\ i_k\in I}|A_{i_1}\cap\cdots\cap A_{i_k}|\\ &= \binom{|I|}{1}(\ell-1)-2\binom{|I|}{2}\cdot\frac{\ell}{2}+\cdots+(-2)^{|I|-1}\cdot\binom{|I|}{|I|}\frac{\ell}{2^{|I|-1}}\\ &= \ell-|I|,\end{aligned}$$

namely, $|A|+|I|=\ell$. Thus all the nonempty zero-sum subsequences of $S$ have the same length $\ell$ and we are done. □

We now give a proof for the third main result.

**Proof of Theorem 1.4.**

By Lemma 4.2, we have $\mathrm{disc}_\ell(G)\ge r+u_1+1$. Next we show $\mathrm{disc}_\ell(G)\le r+u_1+1$. Let $S$ be a sequence over $G$ of length $|S|=r+u_1+1$. We show that $S$ contains a nonempty zero-sum subsequence of length not equal to $\ell$. Assume to the contrary that every nonempty zero-sum subsequence of $S$ has the same length $\ell$. We may assume that $S$ does not contain 0. Suppose $\langle\mathrm{supp}(S)\rangle=C_2^{r_1}\subset G$. Then $r_1\le r$ and $|S|=r_1+t_1=r+u_1+1$, where $t_1\ge u_1+1\ge 2$. Let $S=e_1\cdot\ldots\cdot e_{r_1}x_1\cdot\ldots\cdot x_{t_1}$ with $e_1,\cdots,e_{r_1}$ being a basis of $C_2^{r_1}$. For each $i\in[1,t_1]$, let $A_i\subset[1,r_1]$ be a nonempty subset such that $x_i=\sum_{j\in A_i}e_j$. Applying Lemma 4.1 on $S$ we obtain that

$$\begin{aligned}r_1 &\ge |\cup_{i=1}^{t_1}A_i|\\ &= \sum_{1\le i\le t_1}|A_i|-\sum_{1\le i<j\le t_1}|A_i\cap A_j|+\cdots+(-1)^{t_1-1}|\cap_{i=1}^{t_1}A_i|\\ &= t_1(\ell-1)-\binom{t_1}{2}\cdot\frac{\ell}{2}+\cdots+(-1)^{t_1-1}\cdot\frac{\ell}{2^{t_1-1}}\\ &= \ell\cdot\frac{2^{t_1}-1}{2^{t_1-1}}-t_1.\end{aligned}$$

and

$$\ell\equiv 0 \pmod{2^{t_1-1}}$$

Since $2^{t_1-1}\mid\ell$ and $t_1\ge u_1+1$, according to the definition of $u_1$, we get $r_1>r$, a contradiction. Therefore, $S$ contains a nonempty zero-sum subsequence of length not equal to $\ell$. So $\mathrm{disc}_\ell(G)\le r+u_1+1$, and thus $\mathrm{disc}_\ell(G)=r+u_1+1$.

□

## 5 $\mathrm{disc}_\ell(G)$ and $\mathrm{disc}(G)$ on Abelian $p$-Groups

Let $p$ be a prime, and $G$ be a finite abelian $p$-group. In this section, we investigate both $\mathrm{disc}_\ell(G)$ and $\mathrm{disc}(G)$.

**Definition 5.1** *Let $S = g_1 \cdot \ldots \cdot g_l \in \mathcal{F}(G)$ be a sequence of length $|S| = l \in \mathbb{N}_0$ and let $g \in G$.*

*(a) For every $k \in \mathbb{N}_0$ let*

$$\mathbf{N}_g^k(S) = \Big|\{I \subset [1, l] \mid \sum_{i \in I} g_i = g \text{ and } |I| = k\}\Big|$$

*denote the number of subsequences $T$ of $S$ having sum $\sigma(T) = g$ and length $|T| = k$ (counted with the multiplicity of their appearance in $S$).*

*(b) We define*

$$\mathbf{N}_g(S) = \sum_{k \geq 0} \mathbf{N}_g^k(S), \quad \mathbf{N}_g^+(S) = \sum_{k \geq 0} \mathbf{N}_g^{2k}(S) \quad \text{and} \quad \mathbf{N}_g^-(S) = \sum_{k \geq 0} \mathbf{N}_g^{2k+1}(S).$$

*Thus $\mathbf{N}_g(S)$ denotes the number of subsequences $T$ of $S$ having sum $\sigma(T) = g$, $\mathbf{N}_g^+(S)$ denotes the number of all such subsequences of even length, and $\mathbf{N}_g^-(S)$ denotes the number of all such subsequences of odd length (each counted with the multiplicity of its appearance in S).*

**Lemma 5.2** *[12, Theorem 1] Let $G$ be a finite abelian $p$-group for some prime $p$ and let $S$ be a sequence over $G$ of length $|S| \geq \mathsf{D}(G)$. Then $\mathbf{N}_g^+(S) \equiv \mathbf{N}_g^-(S) \pmod{p}$ for all $g \in G$.*

*Proof.* See [7, proposition 5.5.8]. □

The following congruence is first used by Lucas [11], we give a proof for the convenience of the reader.

**Lemma 5.3** *Let $p$ be a prime and let $a, b$ be positive integers with the $p$-adic expansions: $a = a_n p^n + \cdots + a_1 p + a_0$ and $b = b_n p^n + \cdots + b_1 p + b_0$, respectively. Define $\binom{k}{0} = 1$ for $k \geq 0$. Then*

$$\binom{a}{b} \equiv \binom{a_n}{b_n}\binom{a_{n-1}}{b_{n-1}} \cdots \binom{a_0}{b_0} \pmod{p}.$$

*Proof.* We have

$$\begin{aligned} (1+x)^a &= (1+x)^{a_n p^n + \cdots + a_1 p + a_0} \\ &\equiv (1+x^{p^n})^{a_n} \cdots (1+x^p)^{a_1} (1+x)^{a_0} \pmod{p} \end{aligned}$$

Since $0 \leq a_i \leq p-1$, by comparing the coefficients of $x^b$ on both sides of the above equation, we obtain the desired result. □

Now we are ready to prove Theorem 1.5.

**Proof of Theorem 1.5.**

Let $G = C_{p^{n_1}} \oplus \cdots \oplus C_{p^{n_r}}$, where $1 \le n_1 \le \cdots \le n_r$. Then $\mathsf{D}(G) = \sum_{i=1}^{r}(p^{n_i} - 1) + 1$. By Proposition 2.1 and Lemma 2.3 we have $\mathrm{disc}(G) \ge \mathsf{D}(G) + \exp(G)$. So, it suffices to show that $\mathrm{disc}(G) \le \mathsf{D}(G) + \exp(G)$.

Let $S$ be a sequence over $G$ of length $|S| = \mathsf{D}(G)+\exp(G)$. We need to show that $S$ contains two nonempty zero-sum subsequences of distinct lengths. Assume to the contrary that every nonempty zero-sum subsequence has the same length $\ell$. By Theorem 1.2 we have $\ell \le \mathsf{D}(G) - 1$. Therefore, $|S| - \mathsf{D}(G) + 1 \le \ell \le \mathsf{D}(G) - 1$, i.e., $p^{n_r} + 1 \le \ell \le \sum_{i=1}^{r}(p^{n_i} - 1)$.

Let $\phi : G \to G \oplus < e > \cong G \oplus C_{p^{n_r}}$, where $\mathrm{ord}(e) = p^{n_r}$, be the map defined by $\phi(g) = g + e$. Since $|\phi(S)| = |S| > \mathsf{D}^*(G \oplus C_{p^{n_r}}) = \mathsf{D}(G \oplus C_{p^{n_r}})$, there is a subsequence $S_1$ of $S$ such that $0 = \sigma(\phi(S_1)) = \sigma(S_1) + |S_1|e$. Whence $S_1$ is a nonempty zero-sum subsequence of $S$ whose length is divisible by $\mathrm{ord}(e) = p^{n_r}$, so $\ell = kp^{n_r}$ for some $k \in [2, t]$, where $t = \lfloor \frac{\sum_{i=1}^{r}(p^{n_i}-1)}{p^{n_r}} \rfloor \le r - 1$.

Let $T$ be a subsequence of $S$ of length $|T| \ge \sum_{i=1}^{r}(p^{n_i} - 1) + 1 = \mathsf{D}(G)$. By Lemma 5.2, $1 + (-1)^k \mathbf{N}_0^{kp^{n_r}}(T) = \mathbf{N}_0^{+}(T) - \mathbf{N}_0^{-}(T) \equiv 0 \pmod p$. Therefore,

$$\mathbf{N}_0^{kp^{n_r}}(T) \equiv (-1)^{k+1} \pmod p$$

for every $T \mid S$ with $|T| \ge \sum_{i=1}^{r}(p^{n_i} - 1) + 1$, and especially,

$$\mathbf{N}_0^{kp^{n_r}}(S) \equiv (-1)^{k+1} \pmod p.$$

Hence, by Lemma 5.3

$$\begin{aligned}\sum_{T|S,|T|=\sum_{i=1}^{r}(p^{n_i}-1)+1} \mathbf{N}_0^{kp^{n_r}}(T) &\equiv \sum_{T|S,|T|=\sum_{i=1}^{r}(p^{n_i}-1)+1} (-1)^{k+1} \\ &= \binom{\sum_{i=1}^{r}(p^{n_i} - 1) + p^{n_r} + 1}{\sum_{i=1}^{r}(p^{n_i} - 1) + 1}(-1)^{k+1} \\ &\equiv (t + 1)(-1)^{k+1} \pmod p.\end{aligned}$$

Note that for every nonempty zero-sum subsequence $W$ of $S$ of length $|W| = kp^{n_r}$, there exist $\binom{\sum_{i=1}^{r}(p^{n_i}-1)+1+p^{n_r}-kp^{n_r}}{\sum_{i=1}^{r}(p^{n_i}-1)+1-kp^{n_r}}$ subsequences $T$ of $S$ such that $W \mid T \mid S$ and $|T| = \sum_{i=1}^{r}(p^{n_i} - 1) + 1$. Thus, by Lemma 5.3

$$\begin{aligned}\sum_{T|S,|T|=\sum_{i=1}^{r}(p^{n_i}-1)+1} \mathbf{N}_0^{kp^{n_r}}(T) &= \binom{\sum_{i=1}^{r}(p^{n_i} - 1) + 1 + p^{n_r} - kp^{n_r}}{\sum_{i=1}^{r}(p^{n_i} - 1) + 1 - kp^{n_r}}\mathbf{N}_0^{kp^{n_r}}(S) \\ &\equiv (t + 1 - k)(-1)^{k+1} \pmod p.\end{aligned}$$

Therefore,

$$(t + 1 - k)(-1)^{k+1} \equiv (t + 1) \cdot (-1)^{k+1} \pmod p.$$

Thus $k \equiv 0 \pmod p$, a contradiction to $k \in [2, t]$ and $p \ge r(G) = r > t$. □

We next present some results on $\mathrm{disc}_\ell(G)$ for finite abelian $p$-groups.

**Lemma 5.4** *[8, Corollary 2.4] Let $G$ be a finite abelian $p$-group, and let $S$ be a sequence over $G$ with $|S| = \mathsf{D}(G) + i - 1$, where $i \geq 1$. If $i \geq 2$ and $S$ contains a zero-sum subsequence $S'$ with $p \nmid |S'|$, then $S$ has two nonempty zero-sum subsequences of distinct lengths.*

**Theorem 5.5** *Let $p$ be a prime, $G$ be a finite abelian $p$-group and let $\ell \in [2, \mathsf{D}(G) - 1]$ with $p \nmid \ell$. Then $disc_\ell(G) = \mathsf{D}(G) + 1$.*

*Proof.* The result follows from Lemma 2.2 and Lemma 5.4 with $i = 2$. □

**Theorem 5.6** *Let $\alpha \geq 1, r \geq 3$ be two integers, and let $p$ be a prime such that $p \geq r$. Then $(r+1)(p^\alpha - 1) + 1 \leq disc_{kp^\alpha}(C_{p^\alpha}^r) \leq (r+1)(p^\alpha - 1) + 2$, where $k \in [2, \lceil \frac{r}{2} \rceil]$.*

*Proof.* Let $G = C_{p^\alpha}^r$ and let $e_1, e_2, \cdots, e_r$ be a basis of $G$ with $\mathrm{ord}(e_i) = p^\alpha$ for every $i \in [1, r]$. Since $k \leq \lceil \frac{r}{2} \rceil$, we have $2k - 1 \leq r$. Let $S = \prod_{i=1}^r e_i^{p^\alpha - 1} \cdot (\sum_{j=1}^k e_j + \sum_{t=k+1}^{2k-1} (p^\alpha - 1)e_t)^{p^\alpha - 1}$ be a sequence over $G$ of length $(r+1)(p^\alpha - 1)$. It is easy to show that all the nonempty zero-sum subsequences of $S$ have the same length $kp^\alpha$. Hence, $\mathrm{disc}_{kp^\alpha}(G) \geq (r+1)(p^\alpha - 1) + 1$. It follows from Theorem 1.5 and Lemma 2.2 that $\mathrm{disc}_{kp^\alpha}(G) \leq (r+1)(p^\alpha - 1) + 2$ as desired. □

We close the paper by making the following conjecture together with a remark on $\mathrm{disc}(G)$ for finite abelian $p$-groups.

**Conjecture 5.7** *For any finite abelian group $G$, there is an integer $t = t(G)$ depending only on $G$ such that, if $S$ is a sequence over $G$ of length $disc(G) - 1$ and every nonempty zero-sum subsequence of $S$ has the same length $\ell$, then $\ell = t(G)$.*

**Remark 5.8** *In terms of the results of [5] we know that the above conjecture holds true for finite elementary abelian 2-groups, finite abelian groups of rank at most two, and some finite abelian groups with large exponent. It seems that it might be very difficult to determine the precise value of $disc(G)$ for a general finite abelian $p$-group with $p < r(G)$. Even if $p = 2$ and $G = C_{2^\alpha}^r$ with $r \geq 3$, the invariant $disc(G)$ has only recently been determined for the special case of $\alpha = 1$ (with a somewhat complicated proof [5]).*

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