

Average size of a self-conjugate (s, t) -core partition

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Abstract

Armstrong, Hanusa and Jones conjectured that if s, t are coprime integers, then the average size of an (s, t) -core partition and the average size of a self-conjugate (s, t) -core partition are both equal to $\frac{(s+t+1)(s-1)(t-1)}{24}$. Stanley and Zanello showed that the average size of an $(s, s+1)$ -core partition equals $\binom{s+1}{3}/2$. Based on a bijection of Ford, Mai and Sze between self-conjugate (s, t) -core partitions and lattice paths in $\lfloor \frac{s}{2} \rfloor \times \lfloor \frac{t}{2} \rfloor$ rectangle, we obtain the average size of a self-conjugate (s, t) -core partition as conjectured by Armstrong, Hanusa and Jones.

Keywords: (s, t) -core partition, self-conjugate partition, lattice path

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1 Introduction

In this paper, employing a bijection of Ford, Mai and Sze between self-conjugate (s, t) -core partitions and lattice paths, we prove a conjecture of Armstrong, Hanusa and Jones on the average size of a self-conjugate (s, t) -core partition.

A partition is called a t -core partition, or simply a t -core, if its Ferrers diagram contains no cells with hook length t . A partition is called an (s, t) -core partition, or simply an (s, t) -core, if it is simultaneously an s -core and a t -core. Let $r = \gcd(s, t)$. If $r > 1$, then each r -core is an (s, t) -core, and so there are infinitely many (s, t) -cores. When s and t are coprime, Anderson [1] showed that the number of (s, t) -core partitions equals

$$\frac{1}{s+t} \binom{s+t}{s}.$$

For coprime integers s and t , Ford, Mai and Sze [4] characterized the set of hook lengths of diagonal cells in self-conjugate (s, t) -core partitions, and they showed that the number of self-conjugate (s, t) -core partitions is

$$\binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor}. \tag{1.1}$$

A partition is of size n if it is a partition of n . Aukerman, Kane and Sze [3] conjectured that the largest size of an (s, t) -core partition for coprime numbers s and t is $\frac{(s^2-1)(t^2-1)}{24}$. Olsson and Stanton [5] proved this conjecture and obtained the following uniqueness property.

Theorem 1.1 *If s and t are coprime, then there is a unique largest (s, t) -core partition of size*

$$\frac{(s^2 - 1)(t^2 - 1)}{24}, \tag{1.2}$$

which turns out to be self-conjugate.

A short proof for the conjecture of Aukerman, Kane and Sze was given by Tripathi [7]. Vandehey [8] obtained the following characterization of the largest (s, t) -core partition.

Theorem 1.2 *There exists a largest (s, t) -core partition λ with respect to the partial order of containment, that is, for each (s, t) -core μ , $\lambda_i \geq \mu_i$ for $1 \leq i \leq l(\mu)$.*

It is clear that the largest (s, t) -core in the above theorem is unique. It is the (s, t) -core of the largest size, and it is also an (s, t) -core of the longest length.

Armstrong, Hanusa and Jones [2] proposed the following conjecture concerning the average size of an (s, t) -core and the average size of a self-conjugate (s, t) -core.

Conjecture 1.3 *Assume that s and t are coprime. Then the average size of an (s, t) -core and the average size of a self-conjugate (s, t) -core are both equal to*

$$\frac{(s + t + 1)(s - 1)(t - 1)}{24}.$$

Stanley and Zanello [6] showed that the conjecture for the average size of an (s, t) -core holds for $(s, s+1)$ -cores. More precisely, they showed that the average size of an $(s, s+1)$ -core equals $\binom{s+1}{3}/2$. In this paper, we prove the case of Conjecture 1.3 concerning the average size of a self-conjugate (s, t) -core.

2 The average size of a self-conjugate (s, t) -core

In this section, we give a proof of the case of Conjecture (1.3) for self-conjugate (s, t) -cores, which is stated as follows.

Theorem 2.1 *Assume that s and t are coprime. Then the average size of a self-conjugate (s, t) -core equals*

$$\frac{(s + t + 1)(s - 1)(t - 1)}{24}.$$

69	53	37	21	5
47	31	15	-1	-17
25	9	-7	-23	-39
3	-13	-29	-45	-61

Figure 2.1: A lattice path P in the array $A(8, 11)$

Before we present the proof, let us recall a characterization of self-conjugate (s, t) -cores obtained by Ford, Mai and Sze [4]. They introduced an array $A(s, t) = (A_{i,j})_{1 \leq i \leq \lfloor s/2 \rfloor, 1 \leq j \leq \lfloor t/2 \rfloor}$, where

$$A_{i,j} = st - (2j - 1)s - (2i - 1)t. \quad (2.1)$$

Let $\mathcal{P}(A(s, t))$ be the set of lattice paths in $A(s, t)$ from the lower-left corner to the upper-right corner. For example, Figure 2.1 gives an array $A(s, t)$ for $s = 8$ and $t = 11$, where the solid lines represent a lattice path in $\mathcal{P}(A(s, t))$. For a lattice path P in $\mathcal{P}(A(s, t))$, let $M_{A(s,t)}(P)$ denote the set of positive entries $A_{i,j}$ below P along with the absolute values of negative entries above P . The following theorem of Ford, Mai and Sze [4] establishes a connection between self-conjugate (s, t) -cores and lattice paths in $A(s, t)$.

Theorem 2.2 *Assume that s and t are coprime. Let $A(s, t)$ be the array as given in (2.1). Then there is a bijection Φ between the set $\mathcal{P}(A(s, t))$ and the set of self-conjugate (s, t) -core partitions such that for any $P \in \mathcal{P}(A(s, t))$, the set of main diagonal hook lengths of $\Phi(P)$ is given by $M_{A(s,t)}(P)$.*

For example, in Figure 2.1, 5 is the only positive entry below P , while -7 and -13 are the negative entries above P . Thus $M_{A(8,11)}(P) = \{5, 7, 13\}$. So we have $\Phi(P) = (7, 5, 5, 3, 3, 1, 1)$, which is an $(8, 11)$ -core partition.

The following lemma gives a formula for the size of a self-conjugate (s, t) -core partition λ corresponding to a lattice P in $\mathcal{P}(A(s, t))$.

Lemma 2.3 *For any lattice path P in $\mathcal{P}(A(s, t))$, we have*

$$|\Phi(P)| = \frac{(s^2 - 1)(t^2 - 1)}{24} - \sum_{(i,j) \text{ is above } P} A_{i,j}.$$

Proof. For a self-conjugate partition λ , define

$$MD(\lambda) = \{h | h \text{ is the hook length of a cell on the main diagonal of } \lambda\}.$$

Clearly, the main diagonal cells have distinct hook lengths and the size of a self-conjugate partition equals the sum of elements in $MD(\lambda)$. Let P be a lattice path in $\mathcal{P}(A(s, t))$. By

Theorem 2.2, we find that

$$\begin{aligned}
|\Phi(P)| &= \sum_{h \in MD(\Phi(P))} h \\
&= \sum_{(i,j) \text{ is below } P, A_{i,j} > 0} A_{i,j} - \sum_{(i,j) \text{ is above } P, A_{i,j} < 0} A_{i,j} \\
&= \sum_{A_{i,j} > 0} A_{i,j} - \sum_{(i,j) \text{ is above } P} A_{i,j}.
\end{aligned}$$

To show that

$$\sum_{A_{i,j} > 0} A_{i,j} = \frac{(s^2 - 1)(t^2 - 1)}{24}, \quad (2.2)$$

let Q be the lattice path along the left and upper borders of $A(s, t)$. Note that $M_{A(s,t)}(Q)$ consists of positive entries of $A(s, t)$. Let $\lambda = \Phi(Q)$. By Theorem 2.2, the set of main diagonal hook lengths of λ equals $M_{A(s,t)}(Q)$. It follows that

$$|\lambda| = \sum_{A_{i,j} > 0} A_{i,j}. \quad (2.3)$$

We now proceed to show that

$$|\lambda| = \frac{(s^2 - 1)(t^2 - 1)}{24}. \quad (2.4)$$

By Theorem 1.1, there is a unique (s, t) -core μ with the largest size $\frac{(s^2-1)(t^2-1)}{24}$. To prove (2.4), it suffices to show that $\mu = \lambda$. Let $l(\lambda)$ and $l(\mu)$ denote the lengths of λ and μ respectively. By Theorem 2.2, there is a lattice path $R \in \mathcal{P}(A(s, t))$ such that $\mu = \Phi(R)$. Using Theorem 1.2, we find that

$$l(\mu) \geq l(\lambda) \quad (2.5)$$

and

$$\mu_i \geq \lambda_i \quad (2.6)$$

for all i . Combining (2.5) and (2.6), we obtain that

$$\mu_1 + l(\mu) - 1 \geq \lambda_1 + l(\lambda) - 1. \quad (2.7)$$

Next we show that

$$\lambda_1 + l(\lambda) - 1 \geq \mu_1 + l(\mu) - 1. \quad (2.8)$$

Notice that the largest main diagonal hook length of λ is $\lambda_1 + l(\lambda) - 1$, that is,

$$\max MD(\lambda) = \lambda_1 + l(\lambda) - 1. \quad (2.9)$$

Since $\lambda = \Phi(Q)$, by Theorem 2.2, we deduce that

$$MD(\lambda) = M_{A(s,t)}(Q) = \{A_{i,j} | A_{i,j} > 0, 1 \leq i \leq \lfloor s/2 \rfloor, 1 \leq j \leq \lfloor t/2 \rfloor\}. \quad (2.10)$$

Clearly, $A_{1,1}$ is largest among all positive entries in $A(s, t)$. It follows from (2.9) and (2.10) that

$$A_{1,1} = \lambda_1 + l(\lambda) - 1. \quad (2.11)$$

On the other hand, since $\mu_1 + l(\mu) - 1$ is the hook length of the cell in the upper-left corner of μ , Theorem 2.2 ensures the existence of an entry $A_{i,j}$ of $M_{A(s,t)}(R)$ such that

$$|A_{i,j}| = \mu_1 + l(\mu) - 1. \quad (2.12)$$

We claim that

$$A_{1,1} \geq |A_{i,j}|, \quad (2.13)$$

for any entry $A_{i,j}$. Observe that

$$A_{1,1} > |A_{\lfloor s/2 \rfloor, \lfloor t/2 \rfloor}|, \quad (2.14)$$

since

$$A_{1,1} + A_{\lfloor s/2 \rfloor, \lfloor t/2 \rfloor} = (st - s - t) + (st + s + t - 2t\lfloor s/2 \rfloor - 2s\lfloor t/2 \rfloor) > 0.$$

Notice that $A_{1,1}$ is the largest entry in $A(s, t)$ and $A_{\lfloor s/2 \rfloor, \lfloor t/2 \rfloor}$ is the smallest entry in $A(s, t)$. Thus (2.14) implies (2.13). This proves (2.8).

Combining (2.7) and (2.8), we deduce that

$$\lambda_1 + l(\lambda) - 1 = \mu_1 + l(\mu) - 1. \quad (2.15)$$

In view of (2.11) and (2.15), we see that

$$A_{1,1} = \mu_1 + l(\mu) - 1.$$

Thus $A_{1,1}$ lies in $MD(\mu)$. By Theorem 2.2, $A_{1,1}$ belongs to $M_{A(s,t)}(R)$. Since $A_{1,1} > 0$, R is the lattice path along the left and upper borders, namely, $Q = R$ and $\lambda = \mu$. So we conclude that λ is the largest (s, t) -core. This completes the proof. \blacksquare

As to the case of Conjecture 1.3 for self-conjugate cores, we need some identities on the number of lattice paths in a rectangular region. Let m and n be positive integers, and B_{mn} be an $m \times n$ diagram. The positions of the cells of the first row are $(1, 1), (1, 2), \dots, (1, n)$, and so on. The set of lattice paths from the lower-left corner to the upper-right corner of B_{mn} is denoted by $\mathcal{P}(B_{mn})$. Let $f(i, j)$ be the number of lattice paths in $\mathcal{P}(B_{mn})$ that lie below the cell (i, j) , possibly containing the right or lower border of the cell (i, j) .

Lemma 2.4 *For positive integers m and n , we have*

$$\sum_{1 \leq i \leq m, 1 \leq j \leq n} f(i, j) = \frac{mn}{2} \binom{m+n}{m}. \quad (2.16)$$

Proof. It is clear that the number of lattice paths in $\mathcal{P}(B_{mn})$ below the cell (i, j) equals the number of lattice paths above the cell $(m - i + 1, n - j + 1)$. Hence we have

$$f(i, j) + f(m - i + 1, n - j + 1) = |\mathcal{P}(B_{mn})|.$$

Note that the total number of lattice paths in $\mathcal{P}(B_{mn})$ equals $\binom{m+n}{m}$. So we get

$$f(i, j) + f(m - i + 1, n - j + 1) = \binom{m+n}{m}. \quad (2.17)$$

Summing (2.17) over i and j , we obtain (2.16). \blacksquare

Lemma 2.5 *For positive integers m and n , we have*

$$\sum_{1 \leq i \leq m, 1 \leq j \leq n} if(i, j) = \binom{m+2}{3} \binom{m+n}{m+1} \quad (2.18)$$

and

$$\sum_{1 \leq i \leq m, 1 \leq j \leq n} jf(i, j) = \binom{n+2}{3} \binom{m+n}{n+1}. \quad (2.19)$$

Proof. Let

$$G(m, n) = \sum_{1 \leq i \leq m, 1 \leq j \leq n} if(i, j).$$

To prove (2.18), we claim that for $m, n \geq 2$,

$$G(m, n) = G(m-1, n) + G(m, n-1) + \binom{m+1}{2} \binom{m+n-1}{m}. \quad (2.20)$$

To prove (2.20), let T be the set of triples (P, C_1, C_2) , where P is a path in $\mathcal{P}(B_{mn})$, C_1 and C_2 are two cells above P satisfying that they are in the same column and C_1 is at least as high as C_2 . Notice that C_1 and C_2 are allowed to be the same cell.

We proceed to compute $|T|$ in two ways. It is easily seen that $if(i, j)$ is the number of triples (P, C_1, C_2) in T with $C_1 = (i, j)$. For $m, n \geq 1$, we have $|T| = G(m, n)$.

For a given lattice path P in $\mathcal{P}(B_{mn})$, the cells above P form the Ferrers diagram of a partition, denoted by μ . Let μ' be the conjugate of μ . In the j -th column of the Ferrers diagram of μ , there are $\binom{\mu'_j+1}{2}$ ways to choose C_1 and C_2 such that C_1 is not lower than C_2 . It follows that for given lattice path P in $\mathcal{P}(B_{mn})$, there are $\sum_{1 \leq j \leq \mu_1} \binom{\mu'_j+1}{2}$ choices for C_1 and C_2 . Thus, for $m, n \geq 1$, we have

$$|T| = \sum_{\mu: 1 \leq \mu_1 \leq n, 1 \leq \mu'_1 \leq m} \sum_{1 \leq j \leq \mu_1} \binom{\mu'_j+1}{2}. \quad (2.21)$$

So we deduce that for $m, n \geq 1$,

$$G(m, n) = \sum_{\mu: 1 \leq \mu_1 \leq n, 1 \leq \mu'_1 \leq m} \sum_{1 \leq j \leq \mu_1} \binom{\mu'_j+1}{2}. \quad (2.22)$$

Next, we use the above expression (2.22) for $G(m, n)$ to derive the recurrence relation (2.20). For $m, n \geq 2$, the right hand side of (2.22) can be written as

$$\sum_{\mu: 1 \leq \mu_1 \leq n, \mu'_1 = m} \sum_{1 \leq j \leq \mu_1} \binom{\mu'_j+1}{2} + \sum_{\mu: 1 \leq \mu_1 \leq n, 1 \leq \mu'_1 \leq m-1} \sum_{1 \leq j \leq \mu_1} \binom{\mu'_j+1}{2}. \quad (2.23)$$

It is evident from (2.22) that the second double sum in (2.23) equals $G(m-1, n)$. The first double sum in (2.23) can be rewritten as

$$\sum_{\mu: 1 \leq \mu_1 \leq n, \mu'_1 = m} \sum_{2 \leq j \leq \mu_1} \binom{\mu'_j + 1}{2} + \sum_{\mu: 1 \leq \mu_1 \leq n, \mu'_1 = m} \binom{m+1}{2}. \quad (2.24)$$

Clearly, the number of partitions μ with $1 \leq \mu_1 \leq n$ and $\mu'_1 = m$ equals the number of lattice paths from the lower-left corner to the upper-right corner in $B_{m, n-1}$, which is $\binom{m+n-1}{m}$. Hence the second sum in (2.24) simplifies to

$$\binom{m+1}{2} \binom{m+n-1}{m}. \quad (2.25)$$

To compute the double sum in (2.24), let $\tilde{\mu}$ denote the partition obtained from μ by deleting the first column of the Ferrers diagram of μ . In this notation, we have

$$\sum_{\mu: 1 \leq \mu_1 \leq n, \mu'_1 = m} \sum_{2 \leq j \leq \mu_1} \binom{\mu'_j + 1}{2} = \sum_{\tilde{\mu}: 0 \leq \tilde{\mu}_1 \leq n-1, \tilde{\mu}'_1 \leq m} \sum_{1 \leq j \leq \tilde{\mu}_1} \binom{\tilde{\mu}'_j + 1}{2}. \quad (2.26)$$

Notice that the right hand side of (2.26) equals $G(m, n-1)$. Combining (2.25) and (2.26), we see that the first double sum in (2.23) equals

$$G(m, n-1) + \binom{m+1}{2} \binom{m+n-1}{m}.$$

This proves the recurrence relation (2.20).

For $m, n \geq 1$, let

$$F(m, n) = \binom{m+2}{3} \binom{m+n}{m+1}.$$

Clearly, $F(1, n) = G(1, n)$ and $F(m, 1) = G(m, 1)$ for $m, n \geq 1$. Moreover, it is easily verified that $F(m, n)$ also satisfies the recurrence relation (2.20). So we obtain (2.18), which can be rewritten in the form of (2.19). This completes the proof. \blacksquare

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $SC(s, t)$ denote the set of self-conjugate (s, t) -cores. We aim to show that

$$\sum_{\lambda \in SC(s, t)} |\lambda| = \frac{(s+t+1)(s-1)(t-1)}{24} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor}. \quad (2.27)$$

By Theorem 2.2, we find that

$$\sum_{\lambda \in SC(s, t)} |\lambda| = \sum_{P \in \mathcal{P}(A(s, t))} |\Phi(P)|. \quad (2.28)$$

Using Lemma 2.3, we get

$$\sum_{P \in \mathcal{P}(A(s, t))} |\Phi(P)| = \frac{(s^2-1)(t^2-1)}{24} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor} - \sum_{P \in \mathcal{P}(A(s, t))} \sum_{(i, j) \text{ is above } P} A_{i, j}. \quad (2.29)$$

Combining (2.28) and (2.29), we obtain that

$$\sum_{\lambda \in SC(s,t)} |\lambda| = \frac{(s^2-1)(t^2-1)}{24} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor} - \sum_{P \in \mathcal{P}(A(s,t))} \sum_{(i,j) \text{ is above } P} A_{i,j}. \quad (2.30)$$

By the definition (2.1) of the array $A(s, t)$, we deduce that

$$\begin{aligned} \sum_{P \in \mathcal{P}(A(s,t))} \sum_{(i,j) \text{ is above } P} A_{i,j} &= \sum_{P \in \mathcal{P}(A(s,t))} \sum_{(i,j) \text{ is above } P} (st + s + t - 2sj - 2ti) \\ &= (st + s + t) \sum_{1 \leq i \leq \lfloor \frac{s}{2} \rfloor, 1 \leq j \leq \lfloor \frac{t}{2} \rfloor} f(i, j) - 2s \sum_{1 \leq i \leq \lfloor \frac{s}{2} \rfloor, 1 \leq j \leq \lfloor \frac{t}{2} \rfloor} jf(i, j) \\ &\quad - 2t \sum_{1 \leq i \leq \lfloor \frac{s}{2} \rfloor, 1 \leq j \leq \lfloor \frac{t}{2} \rfloor} if(i, j). \end{aligned} \quad (2.31)$$

Using Lemma 2.4 and Lemma 2.5 with $m = \lfloor \frac{s}{2} \rfloor$ and $n = \lfloor \frac{t}{2} \rfloor$, (2.31) becomes

$$\begin{aligned} \sum_{P \in \mathcal{P}(A(s,t))} \sum_{(i,j) \text{ is above } P} A_{i,j} &= (st + s + t) \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor} \frac{\lfloor \frac{s}{2} \rfloor \lfloor \frac{t}{2} \rfloor}{2} - 2s \binom{\lfloor \frac{t}{2} \rfloor + 2}{3} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor - 1} \\ &\quad - 2t \binom{\lfloor \frac{s}{2} \rfloor + 2}{3} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor - 1}. \end{aligned} \quad (2.32)$$

We claim that

$$\begin{aligned} \frac{(s^2-1)(t^2-1)}{24} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor} &= \frac{(s+t+1)(s-1)(t-1)}{24} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor} \\ &\quad + (st + s + t) \frac{\lfloor \frac{s}{2} \rfloor \lfloor \frac{t}{2} \rfloor}{2} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor} \\ &\quad - 2t \binom{\lfloor \frac{s}{2} \rfloor + 2}{3} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{t}{2} \rfloor - 1} \\ &\quad - 2s \binom{\lfloor \frac{t}{2} \rfloor + 2}{3} \binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor - 1}, \end{aligned} \quad (2.33)$$

which simplifies to

$$\frac{st(s-1)(t-1)}{24} = (st + s + t) \frac{\lfloor \frac{s}{2} \rfloor \lfloor \frac{t}{2} \rfloor}{2} - \frac{t}{3} \left(\lfloor \frac{s}{2} \rfloor + 2 \right) \lfloor \frac{s}{2} \rfloor \lfloor \frac{t}{2} \rfloor - \frac{s}{3} \left(\lfloor \frac{t}{2} \rfloor + 2 \right) \lfloor \frac{s}{2} \rfloor \lfloor \frac{t}{2} \rfloor.$$

When s and t are coprime, at least one of s and t is odd. Thus, we may assume, without loss of generality, that s is odd. In this case, the above relation can be easily verified. So the claim holds. Combining (2.30), (2.32) and (2.33), we arrive at (2.27), and hence the proof is complete. \blacksquare

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References

- [1] J. Anderson, Partitions which are simultaneously t_1 - and t_2 -core, *Discrete Math.*, 248 (2002), 237–243.
- [2] D. Armstrong, C. Hanusa and B. Jones, Results and conjectures on simultaneous core partitions, *European J. Combin.*, 41 (2014), 205–220.
- [3] D. Aukerman, B. Kane and L. Sze, On simultaneous s -cores/ t -cores, *Discrete Math.*, 309 (2009), 2712–2720.
- [4] B. Ford, H. Mai and L. Sze, Self-conjugate simultaneous p - and q - core partitions and blocks of A_n , *J. Number Theory*, 129 (2009), 858–865.
- [5] J.B. Olsson and D. Stanton, Block inclusions and cores of partitions, *Aequationes Math.*, 74 (2007), 90–110.
- [6] R.P. Stanley and F. Zanello, The Catalan case of Armstrong’s conjecture on core partitions, *SIAM J. Discrete. Math.*, to appear.
- [7] A. Tripathi, On the largest size of a partition that is both s -core and t -core, *J. Number Theory*, 129 (2009), 1805–1811.
- [8] J. Vandehey, Containment in (s, t) -core partitions, arXiv:0809.2134.