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Rainbow 2-Connection Numbers of Cayley Graphs ☆

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A path in an edge colored graph is said to be a rainbow path if no two edges on this path share the same color. For an l-connected graph Γ and an integer k with $1 \le k \le l$, the rainbow k-connection number of Γ is the minimum number of colors required to color the edges of Γ such that any two distinct vertices of Γ are connected by k internally disjoint rainbow paths. In this paper, a method is provided for bounding the rainbow 2-connection numbers of graphs with certain structural properties. Using this method, we consider the rainbow 2-connection numbers of Cayley graphs, especially, those defined on abelian groups and dihedral groups.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the notation and terminology of [2] for those not described here.

For a graph Γ, we denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and edge set of Γ, respectively. An *edge-coloring* of a graph Γ is a mapping from $E(\Gamma)$ to some finite set of colors. A path in an edge colored graph is said to be a *rainbow path* if no two edges on this path share the same color. Let Γ be an edge colored l-connected graph, where l is a positive integer. For $1 \le k \le l$, the graph Γ is *rainbow k-connected* if any two distinct vertices of Γ are connected by k internally disjoint rainbow paths, while the coloring is called a *rainbow k-coloring*. The *rainbow k-connection number* of Γ, denoted by $rc_k(\Gamma)$, is the minimum number of colors required to color the edges of Γ to make the graph rainbow k-connected. For simplicity, we write $rc(\Gamma)$ for $rc_1(\Gamma)$ and call it *rainbow connection number*. A well-known theorem of Menger [14] shows that in every l-connected graph Γ with $l \ge 1$, there exist k internally disjoint paths connecting every two distinct vertices u and v for every integer k with $1 \le k \le l$. By coloring the edges of Γ with distinct colors, we know that every two distinct vertices of Γ are connected by k internally disjoint rainbow paths, and thus the function $rc_k(\Gamma)$ is well-defined for every $1 \le k \le l$. An easy observation is that $rc_k(\Gamma) \le rc_k(\Sigma)$ for each l-connected spanning subgraph Σ of the graph Γ. We note also the trivial fact that if C_n is a cycle with $n \ge 3$, then $rc_2(\mathsf{C}_n) = n$.

The concept of rainbow k-connection number was first introduced by Chartrand et al. ([3] for $k = 1$, and [4] for general k). Since then, a considerable amount of research has been carried out towards the function $rc_k(\Gamma)$, see [12] for a survey on this topic. Chartrand et al. [4] proved that for every integer $k \ge 2$, there exists an integer $f(k)$ such that if $n \ge f(k)$, then $rc_k(\mathsf{K}_n) = 2$. With a similar method, Li and Sun [11] obtained that for every integer $k \ge 2$, there exists an integer $g(k) = 2k\lceil \frac{k}{2} \rceil$ such that $rc_k(\mathsf{K}_{n,n}) = 3$ for any $n \ge g(k)$. Fujita et al. [6] and He et al. [8] investigated the rainbow k-connection number of random graphs. In particular, it was shown in [10] that if Γ is a 2-connected graph with n vertices, then $rc_2(\Gamma) \le n$ with equality if and only if Γ is a cycle of order n.

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Let G be a finite group with identity element 1. Let S be a subset of G such that $1 \notin S = S^{-1} := \{s^{-1} \mid s \in S\}$. The *Cayley graph* $\mathrm{Cay}(G, S)$ is defined on G such that two 'vertices' g and h are adjacent if and only if $g^{-1}h \in S$. Hence $\mathrm{Cay}(G, S)$ is a well-defined simple regular graph of valency $|S|$. It is well-known that $\mathrm{Cay}(G, S)$ is connected if and only if S is a generating set of G. In a Cayley graph $\mathrm{Cay}(G, S)$, an edge $\{g, h\}$ is called an *s-edge* if $g^{-1}h$ or $h^{-1}g$ equals some s in S.

Cayley graphs have been an active topic in algebraic graph theory for a long time. Actually, interconnection networks are often modeled by highly symmetric Cayley graphs [1]. The rainbow connection number of a graph can be applied to measure the safety of a network. Thus the object of the rainbow connection numbers of Cayley graphs should be meaningful. Li et al. [9], Lu and Ma [13] discussed the rainbow connection numbers of Cayley graphs. This motivates us to consider the rainbow 2-connection numbers of Cayley graphs. In this paper, we establish a lemma for bounding the rainbow 2-connection numbers of graphs satisfying certain structural properties. Using this lemma, we consider the rainbow 2-connection numbers of Cayley graphs, especially, those defined on abelian groups and on dihedral groups.

2. Rainbow 2-connection numbers of Cayley graphs

Let Γ be a graph. For $U, V \subseteq V(\Gamma)$, we denote by $\Gamma[U, V]$ the subgraph on $U \cup V$ with edge set $\{\{u, v\} \in E(\Gamma) \mid u \in U, v \in V\}$. For a partition $\mathcal{B} = \{U_0, U_1, \cdots, U_{m-1}\}$ of $V(\Gamma)$, define a graph $\Gamma_{\mathcal{B}}$ with vertex set $\mathcal{B}$ such that $U_i, U_j \in \mathcal{B}$ are adjacent in $\Gamma_{\mathcal{B}}$ if and only if some $u \in U_i$ is adjacent to some $v \in U_j$ in Γ. The graph $\Gamma_{\mathcal{B}}$ is called a *quotient graph* of Γ. The following technical lemma is very important.

Lemma 2.1. *Let Γ be a 2-connected graph. Assume that $V(\Gamma)$ has a partition $\mathcal{B} = \{U_0, U_1, \cdots, U_{m-1}\}$ such that $\Gamma_{\mathcal{B}}$ is 2-connected, and for each i, the subgraph $\Gamma[U_i, U_i]$ is 2-connected.*

(i) *Suppose that for each pair of adjacent vertices U_i and U_j in $\Gamma_{\mathcal{B}}$, the subgraph $\Gamma[U_i, U_j]$ has no isolate vertices. Then*

$$rc_2(\Gamma) \le \max\{rc_2(\Gamma[U_i, U_i]) \mid 0 \le i < m\} + rc_2(\Gamma_{\mathcal{B}}).$$

(ii) *Suppose that $E(\Gamma[U_i, U_{i+1}]) \neq \emptyset$ for $0 \le i < m$, and every $u \in U_i$ is adjacent to some $v \in U_{i-1}$ or some $w \in U_{i+1}$ in Γ, reading the subscripts modulo m. Then*

$$rc_2(\Gamma) \le (\max\{rc_2(\Gamma[U_i, U_i]) \mid 0 \le i < m\} + 1)m.$$

Proof. Denote $\Gamma_i = \Gamma[U_i, U_i]$ and $c = \max\{rc_2(\Gamma_i) \mid 0 \le i < m\}$.

(i) Let C be a set of c colors and D be a set of $rc_2(\Gamma_{\mathcal{B}})$ colors with $C \cap D = \emptyset$. For $\Gamma_{\mathcal{B}}$, we choose a rainbow 2-coloring $\bar{\theta} : E(\Gamma_{\mathcal{B}}) \to D$. For each graph Γ_i, assign a rainbow 2-coloring $\theta_i : E(\Gamma_i) \to C$. Define an edge-coloring θ of Γ by

$$\theta(e) = \begin{cases} \theta_i(e) & \text{if } e \in E(\Gamma_i) \text{ for } 0 \le i < m; \\ \bar{\theta}(\{U_i, U_j\}) & \text{if } \{U_i, U_j\} \in E(\Gamma_{\mathcal{B}}) \text{ and} \\ & e \in E(\Gamma[U_i, U_j]). \end{cases}$$

Let u and v be any two distinct vertices of Γ. If u and v are contained in some Γ_i, then there exist two internally disjoint rainbow paths by means of the rainbow 2-coloring θ_i. Suppose $u \in V(\Gamma_i)$ and $v \in V(\Gamma_j)$ satisfying $i \neq j$. In the quotient graph $\Gamma_{\mathcal{B}}$, there exist two internally disjoint rainbow paths connecting U_i and U_j. Denote them by $U_i, U_{i_1}, U_{i_2}, \cdots, U_{i_s}, U_j$ and $U_i, U_{j_1}, U_{j_2}, \cdots, U_{j_t}, U_j$. Since U_{i_s} and U_j are adjacent in $\Gamma_{\mathcal{B}}$, by the assumptions, we know that the subgraph $\Gamma[U_{i_s}, U_j]$ has no isolate vertices. Then there exists a vertex $v_{i_s} \in U_{i_s}$ satisfying $v_{i_s}v \in E(\Gamma)$. Similarly, there exist some vertices $v_{i_r} \in U_{i_r}$ for $1 \le r \le s-1$ and $v_i \in U_i$ such that $v_{i_s}v_{i_{s-1}}, v_{i_{s-1}}v_{i_{s-2}}, \cdots, v_{i_2}v_{i_1}, v_{i_1}v_i \in E(\Gamma)$. Obviously, $\mathsf{P}' = u, \mathsf{P}^1, v_i, v_{i_1}, v_{i_2}, \cdots, v_{i_s}, v$ is a rainbow path connecting u and v, where P^1 is a rainbow path between u and v_i in Γ_i. Since U_i and U_{j_1} are adjacent in $\Gamma_{\mathcal{B}}$, by the assumptions, we have that the subgraph $\Gamma[U_i, U_{j_1}]$ has no isolate vertices. Thus there exists a vertex $v_{j_1} \in U_{j_1}$ satisfying $uv_{j_1} \in E(\Gamma)$. Similarly, there exist some vertices $v_{j_r} \in U_{j_r}$ for $2 \le r \le t$ and $v_j \in U_j$ such that $v_{j_1}v_{j_2}, v_{j_2}v_{j_3}, \cdots, v_{j_{t-1}}v_{j_t}, v_{j_t}v_j \in E(\Gamma)$. Obviously, $\mathsf{P}'' = u, v_{j_1}, v_{j_2}, \cdots, v_{j_t}, v_j, \mathsf{P}^2, v$ is also a rainbow path connecting u and v, where P^2 is a rainbow path between v_j and v in Γ_j. Note that P' and P'' are internally disjoint. Thus Γ is rainbow 2-connected with the edge-coloring θ, and so $rc_2(\Gamma) \le \max\{rc_2(\Gamma[U_i, U_i]) \mid 0 \le i < m\} + rc_2(\Gamma_{\mathcal{B}})$.

(ii) Consider the spanning subgraph Σ of Γ with edge set

$$E(\Sigma) = \left(\bigcup_{i=0}^{m-1} E(\Gamma_i)\right) \cup \left(\bigcup_{i=0}^{m-1} E(\Gamma[U_i, U_{i+1}])\right).$$

Since $E(\Gamma[U_i, U_{i+1}]) \neq \emptyset$ for $0 \le i < m$, we obtain that $\Sigma_{\mathcal{B}}$ is a cycle of length m. Let $C_0, C_1, \cdots, C_{m-1}$ be c-sets of colors such that $C_i \cap C_j = \emptyset$ if $0 \le i < j < m$. For each graph Γ_i, since $rc_2(\Gamma_i) \le c$, we assign a rainbow 2-coloring $\eta_i : E(\Gamma_i) \to C_i$. Choose m colors $c_1, c_2, \cdots, c_m$ which are not used above. Define an edge-coloring η of Σ as follows:

$$\eta(e) = \begin{cases} \eta_i(e) & \text{if } e \in E(\Gamma_i) \text{ for } 0 \le i < m; \\ c_i & \text{if } e \in E(\Gamma[U_{i-1}, U_i]) \text{ for } 1 \le i \le m. \end{cases}$$

Let u and v be any two distinct vertices of Γ. If u and v are contained in some U_i for $0 \le i \le m-1$, then there exist two internally disjoint rainbow paths connecting u and v by means of the rainbow 2-coloring η_i. Without loss of generality, we assume that $u \in U_i$ and $v \in U_j$ with $0 \le i \neq j \le m-1$. Then there also exist two internally disjoint rainbow paths connecting u and v since $\Sigma_{\mathcal{B}}$ is a cycle and the colors $c_1, c_2, \cdots, c_m$ are not used in Γ_i for $0 \le i \le m-1$. Hence Γ is rainbow 2-connected, and so part (ii) follows from enumerating the number of colors used for η. □

Let G be a group and N a normal subgroup of G. Then all (left) cosets of N in G form a group under the product

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$$(gN)(hN) = ghN,$$

which is denoted by G/N and called *the quotient group* of G with respect to N.

Theorem 2.2. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected Cayley graph with $1 \notin S = S^{-1}$. Suppose that $X \subseteq S$ such that $N := \langle S \setminus (X \cup X^{-1})\rangle \neq G$ satisfying $|G/N| \geq 3$ and $|N| \geq 3$. Set $Y = S \setminus (X \cup X^{-1})$ and $\Sigma = \mathrm{Cay}(N, Y)$. If N is normal in G, then*

$$rc_2(\Gamma) \leq rc_2(\Sigma) + rc_2\big(\mathrm{Cay}(\bar{G}, \bar{X})\big),$$

where $\bar{G} = G/N$ and $\bar{X} = \{xN \mid x \in S \setminus N\}$.

Proof. Since N is normal in G, we have $G = \langle X, Y\rangle \leq \langle X, N\rangle = \langle X\rangle N$, and thus $G = \langle X\rangle N$. Let m be the index of N in G. Then $m = \frac{|G|}{|N|}$. Let $g_0N = N, g_1N, \cdots, g_iN, \cdots, g_{m-1}N$ be all distinct left cosets of N in G. Denote $U_i = g_iN$ for $0 \leq i < m$. Then $\mathcal{B} = \{U_i \mid 0 \leq i < m\}$ is a partition of $V(\Gamma)$. Define a mapping ϕ_i from U_0 to U_i as follows: $g \mapsto g_ig$ for each $g \in U_0 = N$. It is easy to check that the mapping ϕ_i is an isomorphism between $\Gamma[U_0, U_0]$ and $\Gamma[U_i, U_i]$. Thus each subgraph $\Gamma[U_i, U_i]$ contains a spanning subgraph isomorphic to the connected Cayley graph $\Sigma = \mathrm{Cay}(N, Y)$, and so $rc_2(\Gamma[U_i, U_i]) \leq rc_2(\Sigma)$.

Note that $gNh = ghN$ for any $g, h \in G$. Suppose that $E(\Gamma[U_i, U_j]) \neq \emptyset$, where $i \neq j$. Then there exist some $g, h \in N$ and $x \in S \setminus N$ such that $g_igx = g_jh$. Thus

$$g_iNx = g_igNx = g_igx\big(x^{-1}Nx\big) = g_igxN = g_jhN = g_jN.$$

It follows that $\Gamma[U_i, U_j]$ contains a perfect matching, that is, $\Gamma[U_i, U_j]$ has no isolate vertices. By Lemma 2.1(i), $rc_2(\Gamma) \leq rc_2(\Sigma) + rc_2(\Gamma_{\mathcal{B}})$. Consider the quotient graph $\Gamma_{\mathcal{B}}$. Thus U_i and U_j are adjacent if and only if $g_jN = g_iNx = (g_iN)(xN)$ for some $x \in S \setminus N$. It follows that $\Gamma_{\mathcal{B}} = \mathrm{Cay}(\bar{G}, \bar{X})$, and hence the result follows. □

A graph is called *vertex transitive* if for any two vertices there is an automorphism of the graph mapping one vertex to the other one. By Theorem 2.2, we obtain the following result.

Corollary 2.3. *Let Γ, G and N be as in Theorem 2.2. Then $rc_2(\Gamma) \leq |N| + \frac{|G|}{|N|}$.*

Proof. Applying [7, Theorem 3.4.2], we know that a connected vertex transitive graph of order no less than three must be 2-connected. Thus, by [10, Theorem 1.4], if Γ is a connected vertex transitive graph of order no less than three, then $rc_2(\Gamma) \leq |V(\Gamma)|$. Notice that a Cayley graph must be vertex transitive. Hence

$$rc_2(\Gamma) \leq rc_2(\Sigma) + rc_2\big(\mathrm{Cay}(\bar{G}, \bar{X})\big) \leq |N| + \frac{|G|}{|N|}. \qquad \square$$

Let G be a finite group. For an element $x \in G$, denote by $|x|$ the order of x in G. A subset X of G is a *minimal generating set* if G is generated by X but not by any proper subset of X. Now we consider the rainbow 2-connection numbers of Cayley graphs on abelian groups.

Theorem 2.4. *Let G be a finite abelian group and S a generating set of G such that $1 \notin S = S^{-1}$. Set $\Gamma = \mathrm{Cay}(G, S)$. Then the following statements hold.*

(i) *$rc_2(\Gamma) \leq \sum_{x\in X} |x|$, where X is an arbitrary minimal generating set of G contained in S.*
(ii) *Either G is cyclic and S consists of generators of G; or there are two proper divisors m and n of $|G|$ such that $|G| = mn$ and $rc_2(\Gamma) \leq m + n$.*

Proof. (i) We prove part (i) by induction on the orders of groups. Let X be an arbitrary minimal generating set of G with $X \subseteq S$. Take $x \in X$, set $Y = X \setminus \{x\}$ and $N = \langle Y\rangle$. Thus $G = \langle X\rangle = \langle x\rangle N$, and $|G/N| \leq |x|$.

Suppose $|G/N| = 2$. Denote $V_0 = N$ and $V_1 = xN$. Let $\Sigma = \mathrm{Cay}(G, X \cup X^{-1})$. Then Σ is a connected spanning subgraph of Γ. Clearly,

$$\Sigma[V_0] \cong \Sigma[V_1] \cong \mathrm{Cay}\big(N, Y \cup Y^{-1}\big).$$

If $|N| = 2$, then $Y \cup Y^{-1}$ only contains an element, denoted by y. Thus $X = \{x, y\}$, and Σ is a cycle of length 4. Hence $rc_2(\Gamma) \leq rc_2(\Sigma) = 4 = |x| + |y|$. Now we may assume that $|N| \geq 3$. Note that $rc_2(\Gamma[V_i]) \leq rc_2(\Sigma[V_i]) = rc_2(\mathrm{Cay}(N, Y \cup Y^{-1}))$ for $0 \leq i \leq 1$. Let C be a set of $rc_2(\mathrm{Cay}(N, Y \cup Y^{-1}))$ colors. We choose a rainbow 2-coloring $\theta_i : E(\Sigma[V_i]) \to C$ for $0 \leq i \leq 1$ such that for any two elements $u_0, v_0 \in V_0 = N$ with $u_0v_0 \in E(\Sigma[V_0])$, we have $\theta_0(u_0v_0) \neq \theta_1(u_1v_1)$, where $u_1 = xu_0$ and $v_1 = xv_0$. In addition, we assign a new color to every edge $uv \in E(\Sigma[V_0, V_1])$. Let u and v be any two distinct vertices of Σ. If $u, v \in V_i$ for $0 \leq i \leq 1$, then there exist two internally disjoint rainbow paths connecting u and v in $\Sigma[V_i]$ by means of the rainbow 2-coloring θ_i. Without loss of generality, now we assume that $u \in V_0$ and $v \in V_1$. Suppose u and v are adjacent in Σ. Then $v = xu = ux$. Take an element $y \in Y$. Obviously, uv and u, u', v', v are two internally disjoint rainbow paths connecting u and v in Σ, where $u' = uy \in V_0$ and $v' = uyx \in V_1$. Suppose u and v are not adjacent in Σ. Then u, P^1, u_1, v and u, v_1, P^2, v are two internally disjoint rainbow paths connecting u and v in Σ, where $u_1 \in V_0, v_1 \in V_1$, P^1 is a rainbow path between u and u_1 in $\Sigma[V_0]$, and P^2 is a rainbow path between v_1 and v in $\Sigma[V_1]$. Thus Σ is rainbow 2-connected with the above edge-coloring. Since Σ is a connected spanning subgraph of Γ, we have that Γ is rainbow 2-connected. Part (i) follows by induction.

Suppose $|G/N| \geq 3$. Assume that $|N| = 2$. Then

$$rc_2(\Gamma) \leq rc_2\big(\mathrm{Cay}\big(G, X \cup X^{-1}\big)\big) \leq 1 + |x| < 2 + |x|.$$

Assume that $|N| \geq 3$, by Theorem 2.2,

$$\begin{aligned} rc_2(\Gamma) &\leq rc_2\big(\mathrm{Cay}\big(G, X \cup X^{-1}\big)\big) \\ &\leq rc_2\big(\mathrm{Cay}\big(N, Y \cup Y^{-1}\big)\big) + |x|. \end{aligned}$$

Since $|N| < |G|$, and Y is also a minimal generating set of N, by induction, we have that $rc_2(\mathrm{Cay}(N, Y \cup Y^{-1})) \leq \sum_{y\in Y} |y|$, and so $rc_2(\Gamma) \leq rc_2(\mathrm{Cay}(G, X \cup X^{-1})) \leq \sum_{x\in X} |x|$.

(ii) If $\langle x\rangle = G$ for each $x \in S$, then G is cyclic and S consists of generators of G. Hence we assume that there

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are $x \in S$ and $Y \subseteq S$ such that $|Y| \geq 1$ and $G = \langle x, Y\rangle$ but $\langle Y\rangle \neq G$. Denote $N = \langle Y\rangle$. By the proof of part (i) and Theorem 2.2, part (ii) follows by setting $|N| = m$ and $|G/N| = n$. □

For an integer $n \geq 3$, the *ladder* L_n of order $2n$ is a cubic graph constructed by taking two copies of the cycle C_n on disjoint vertex sets $(u_1, u_2, \cdots, u_n)$ and $(v_1, v_2, \cdots, v_n)$, then joining the corresponding vertices $u_i v_i$ for $1 \leq i \leq n$. The *Möbius ladder* M_n of order $2n$ is obtained from the ladder by deleting the edges u_1u_n and v_1v_n, and then inserting edges u_1v_n and u_nv_1.

Lemma 2.5. *Let n be an integer with $n \geq 3$. Then*

(i) $rc_2(L_n) \leq n$.
(ii) $rc_2(M_n) \leq n$.

Proof. Let $U = \{u_i | 1 \leq i \leq n\}$ and $V = \{v_i | 1 \leq i \leq n\}$. Denote $u_{n+1} = u_1$ and $v_{n+1} = v_1$.

(i) Define an edge-coloring θ of the graph L_n as follows.

$$\theta(e) = \begin{cases} i & \text{if } e = u_iu_{i+1} \text{ and } e = u_iv_i \text{ for } 1 \leq i \leq n; \\ i-1 & \text{if } e = v_iv_{i+1} \text{ for } 2 \leq i \leq n; \\ n & \text{if } e = v_1v_2. \end{cases}$$

Let x and y be any two distinct vertices of L_n. Suppose $x, y \in U$ or $x, y \in V$, clearly, there exist two internally disjoint rainbow paths between x and y contained in the cycle $\mathsf{C}^1 = (u_1, u_2, \cdots, u_n, u_1)$ or $\mathsf{C}^2 = (v_1, v_2, \cdots, v_n, v_1)$.

Suppose $x = u_i \in U$ and $y = v_j \in V$. If $j = i$, then u_iv_i and $u_i, u_{i+1}, v_{i+1}, v_i$ are two internally disjoint rainbow paths connecting x and y. If $j = i+1$, then u_i, v_i, v_{i+1} and u_i, u_{i+1}, v_{i+1} are two internally disjoint rainbow paths between x and y. If $i+2 \leq j \leq n$, then $u_i, u_{i+1}, \cdots, u_j, v_j$ and $u_i, v_i, v_{i-1}, \cdots, v_1, v_n, v_j$ are two internally disjoint rainbow paths connecting x and y. If $1 \leq j \leq i-1$, then $u_i, u_{i+1}, \cdots, u_n, u_1, \cdots, u_j, v_j$ and $u_i, v_i, v_{i-1}, \cdots, v_j$ are two internally disjoint rainbow paths between x and y.

Combining the above arguments, L_n is rainbow 2-connected by the edge-coloring θ, and so $rc_2(L_n) \leq n$.

(ii) Define an edge-coloring η of the graph M_n as follows.

$$\eta(e) = \begin{cases} i & \text{if } e = u_iu_{i+1} \text{ and } e = v_iv_{i+1} \text{ for} \\ & 1 \leq i \leq n-1; \\ n & \text{if } e = u_1v_n \text{ and } e = v_1u_n; \\ i & \text{if } e = u_iv_i \text{ for } 1 \leq i \leq n. \end{cases}$$

Let $u_i \in U$ and $v_j \in V$. Without loss of generality, assume that $i \leq j$. Obviously, $u_i, u_{i+1}, \cdots, u_j, v_j$ and $u_i, u_{i-1}, \cdots, u_1, v_n, \cdots, v_j$ are two internally disjoint rainbow paths connecting u_i and v_j. For any distinct vertices $u_i, u_j \in U$ with $i \leq j$, there exist two internally disjoint rainbow paths $u_i, u_{i+1}, \cdots, u_j$ and $u_i, v_i, \cdots, v_1, u_n, u_{n-1}, \cdots, u_j$. For any distinct vertices $v_i, v_j \in V$ with $i \leq j$, there exist two internally disjoint rainbow paths $v_i, v_{i+1}, \cdots, v_j$ and $v_i, u_i, \cdots, u_1, v_n, v_{n-1}, \cdots, v_j$. Therefore, M_n is rainbow 2-connected by the edge-coloring η, that is, $rc_2(M_n) \leq n$. □

Note that $L_3 \cong \mathsf{K}_3 \Box \mathsf{K}_2$ and $M_3 \cong \mathsf{K}_{3,3}$. Applying [4, Figure 2], we have that $rc_2(L_3) = rc_2(\mathsf{K}_3 \Box \mathsf{K}_2) = 3$. It was proved in [4] that for each integer $r \geq 2$, $rc_2(\mathsf{K}_{r,r}) = 4$ if $r = 2$, and $rc_2(\mathsf{K}_{r,r}) = 3$ if $r \geq 3$. Thus $rc_2(M_3) = rc_2(\mathsf{K}_{3,3}) = 3$. In [5], the following results were proved: (i) $rc_2(Q_3) = 4$. (ii) $rc_1(M_4) = 2, rc_2(M_4) = 4$ and $rc_3(M_4) = 5$. Note that the 3-dimensional cube Q_3 is isomorphic to L_4. Hence $rc_2(L_4) = rc_2(M_4) = 4$.

Let $n \geq 1$ be an integer. We use D_{2n} to denote *the dihedral group* generated by two elements, say a and b, such that $|a| = n$, $|b| = 2$, $b^{-1}ab = a^{-1}$. (Note that $\mathrm{D}_2 = \mathbb{Z}_2$ and $\mathrm{D}_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$.) Then $\mathrm{D}_{2n} = \langle a\rangle \cup \langle a\rangle b = \{a^i \mid 0 \leq i < n\} \cup \{a^ib \mid 0 \leq i < n\}$.

Let C_n be a cycle with vertex set $U = \{u_1, u_2, \cdots, u_n\}$, reading the subscripts modulo n, and let P_m be a path with vertex set $V = \{v_1, v_2, \cdots, v_m\}$, reading the subscripts modulo m. The *brick product* of C_n and P_m, denoted by $\mathsf{C}_n^{[m]}$, is the graph defined on $U \times V$ such that (u_i, v_j) and $(u_{i'}, v_{j'})$ are adjacent if and only if either

(1) $i - i' \equiv \pm 1 \pmod n$ and $j = j'$, or
(2) $i = i'$, $i + j \equiv 0 \pmod 2$, $j' = j+1$ and $j = 1, 2, \cdots, m-1$.

For convenience, denote by $\mathsf{C}_{n,j}$ the n-cycle in $\mathsf{C}_n^{[m]}$ on the vertex sets $\{(u_i, v_j) : i = 1, 2, \cdots, n\}$.

To prove the following results, we state two useful lemmas as follows.

Lemma 2.6. *(See [13].)*

(i) *For $0 \leq i \leq n-1$, each a^ib is an involution.*
(ii) *If n is odd, then D_{2n} has a unique conjugacy class of involutions, which is $\{a^ib \mid 0 \leq i \leq n-1\}$.*
(iii) *If n is even, then D_{2n} has exactly three conjugacy classes of involutions, which are $\{a^{\frac{n}{2}}\}$, $\{a^{2i}b \mid 0 \leq i < \frac{n}{2}\}$ and $\{a^{2i+1}b \mid 0 \leq i < \frac{n}{2}\}$.*
(iv) *If m is a divisor of n then $\langle a\rangle$ has a unique subgroup of order m, which is $\langle a^{\frac{n}{m}}\rangle$. If $N \leq \langle a\rangle$, then N is normal in D_{2n} and the quotient group D_{2n}/N is a dihedral group generated by $\{aN, bN\}$.*
(v) *If X is a (minimal) generating set of D_{2n}, then X contains some involution a^sb, and $(X \cap \langle a\rangle) \cup \{xa^sb \mid a^sb \neq x \in X \setminus \langle a\rangle\}$ is a (minimal) generating set of $\langle a\rangle$.*
(vi) *Set $n = p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$ for distinct primes p_i. If Y is a minimal generating set of $\langle a\rangle$, then $|Y| \leq r$. If X is a minimal generating set of D_{2n}, then $|X| \leq r+1$.*

Lemma 2.7. *(See [13].) Let $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$ be a connected cubic Cayley graph. Then one of the following cases occurs.*

(i) *$rc(\Gamma) = \lceil\frac{n+1}{2}\rceil$, and Γ is the ladder graph of order 2n.*
(ii) *$rc(\Gamma) = \lceil\frac{n}{2}\rceil$, and Γ is the Möbius ladder of order 2n.*
(iii) *$\Gamma \cong \mathrm{Cay}(\mathrm{D}_{2n}, \{b, a^sb, a^tb\})$ for some integers s and t, and either*
  (iii.1) *$rc(\Gamma) \leq (l+1)\lceil\frac{m}{2}\rceil$, where $l \in \{|a^s|, |a^t|, |a^{s-t}|\}$ and $m = \frac{n}{l} \geq 2$; or*
  (iii.2) *n is odd, and s, t and s − t are coprime to n.*

Similar to Lemma 2.7, we investigate the rainbow 2-connection numbers of cubic Cayley graphs on dihedral groups.

Theorem 2.8. *Let $\Gamma = \mathrm{Cay}(\mathrm{D}_{2n}, S)$ be a connected cubic Cayley graph. Thus one of the following cases occurs.*

(i) $rc_2(\Gamma) \le n$, *and Γ is the ladder graph of order $2n$.*
(ii) $rc_2(\Gamma) \le n$, *and Γ is the Möbius ladder of order $2n$.*
(iii) $\Gamma \cong \mathrm{Cay}(\mathrm{D}_{2n}, \{b, a^s b, a^t b\})$ *for some integers s and t, and either*
  (iii.1) *n is odd, and s, t and $s-t$ are coprime to n; or*
  (iii.2) $rc_2(\Gamma) \le 2l\lceil \frac{m}{2} \rceil + m$, *where $l \in \{|a^s|, |a^t|, |a^{s-t}|\}$ and $m = \frac{n}{l} \ge 2$.*

Proof. Suppose $S \cap \langle a \rangle \ne \emptyset$, by Lemma 2.5 and Lemma 2.7, one of the case (i) and the case (ii) must occur.

Suppose $S \cap \langle a \rangle = \emptyset$. Set $S = \{x, y, z\}$. Using Lemma 2.6, one of x, y and z is conjugate to b. Without loss of generality, we assume that $b = g^{-1}zg$ for some $g \in \mathrm{D}_{2n}$. Write $g^{-1}xg = a^s b$ and $g^{-1}yg = a^t b$. Set $T = \{b, a^s b, a^t b\}$ and $\Sigma = \mathrm{Cay}(\mathrm{D}_{2n}, T)$. It is easily shown that $V(\Gamma) \to V(\Sigma), h \mapsto g^{-1}hg$ is an isomorphism from Γ to Σ. Thus $\Gamma \cong \mathrm{Cay}(\mathrm{D}_{2n}, \{b, a^s b, a^t b\})$ for some integers s and t. Without loss of generality, we can denote $T = \{b, a^s b, a^t b\} = \{x, y, z\}$. Assume that D_{2n} can be generated by any 2-subset of T. Then the case (iii.1) easily follows.

Now we may assume that there exist two elements $x, y \in T$ such that $|xy| < n$. Let $m = \frac{n}{|xy|}$ and $l = |xy|$. The cycle $(x, xy, xyx, \cdots, (xy)^{l-1}, (xy)^{l-1}x, x)$ in Γ will be called the (x, y)-cycle of Γ. Obviously, there exist m vertex-disjoint (x, y)-cycles of length $2l$. Since left multiplication by the element a of D_{2n} is an automorphism of Γ, and Γ is connected, there must exist a perfect matching of z-edges from the $\langle a \rangle$-vertices of one (x, y)-cycle to the $b\langle a \rangle$-vertices of another (x, y)-cycle. Hence there must exist a perfect matching from the $\langle a \rangle$-vertices of the second (x, y)-cycle to the $b\langle a \rangle$-vertices of the first (x, y)-cycle when $m = 2$ or another (x, y)-cycle when $m > 2$. Continuing this way, we obtain that Γ consists of $\mathrm{C}_{2l}^{[m]}$ together with a perfect matching joining the vertices of valency 2 in $\mathrm{C}_{2l,1}$ with the vertices of valency 2 in $\mathrm{C}_{2l,m}$.

Let F denote the perfect matching joining the vertices of valency 2 in $\mathrm{C}_{2l,1}$ with the vertices of valency 2 in $\mathrm{C}_{2l,m}$. Define an edge-coloring θ of the graph Γ by

$$\theta(e) = \begin{cases} (i, j) & e = (u_i, v_j)(u_{i+1}, v_j) \text{ for } \\ & 1 \le i \le 2l \text{ and } 1 \le j \le \lceil \frac{m}{2} \rceil; \\ (i, j - \lceil \frac{m}{2} \rceil) & e = (u_i, v_j)(u_{i+1}, v_j) \text{ for } \\ & 1 \le i \le 2l \text{ and } \lceil \frac{m}{2} \rceil + 1 \le j \le m; \\ (0, j) & e = (u_i, v_j)(u_i, v_{j+1}) \text{ for } \\ & 1 \le j \le m-1 \text{ and } \\ & i + j \equiv 0 \pmod 2; \\ (0, m) & e \in F. \end{cases}$$

It is easy to verify that Γ is rainbow 2-connected with the above coloring. Then the case (iii.2) follows from enumerating the number of colors used for θ. □

In the end of this section, we discuss the rainbow 2-connection numbers of Cayley graphs on D_{2p^k} or D_{2pq}, where $k \ge 1$ is an integer, p and q are distinct primes.

Theorem 2.9. *Let X be a minimal generating set of D_{2p^k}, where $k \ge 1$ is an integer. Set $S = X \cup X^{-1}$ and $\Gamma = \mathrm{Cay}(\mathrm{D}_{2p^k}, S)$. Then one of the following cases holds.*

(i) $rc_2(\Gamma) = 2p^k$, *and Γ is a cycle of order $2p^k$.*
(ii) $rc_2(\Gamma) \le p^k$, *and Γ is a ladder graph of order $2p^k$.*

Proof. Since X is a minimal generating set of D_{2p^k}, by Lemma 2.6, we have $|X| = 2$. Hence $X = \{a^i b, a^j b\}$ or $X = \{a^i, a^j b\}$ for some integers i and j. Then $S = X$ or $S = \{a^i, a^{-i}, a^j b\}$. It is easy to check that Γ is either a cycle or a ladder graph. Thus the theorem follows by Lemma 2.5. □

Theorem 2.10. *Let $G = \mathrm{D}_{2pq}$, where p and q are distinct odd primes. Let X be a minimal generating set of G. Set $S = X \cup X^{-1}$ and $\Gamma = \mathrm{Cay}(G, S)$. Then one of the following statements holds.*

(i) $|X| = 2$ *and Γ is either a cycle or a ladder graph.*
(ii) $|X| = 3$ *and either*
  (ii.1) $|\langle a \rangle \cap X| = 2$ *and* $rc_2(\Gamma) \le p + q + 1$; *or*
  (ii.2) $|\langle a \rangle \cap X| = 1$ *and* $rc_2(\Gamma) \le 2l + m$ *with* $\{l, m\} = \{p, q\}$.

Proof. Since X is a minimal generating set of D_{2p^k}, we obtain $2 \le |X| \le 3$ by Lemma 2.6. Suppose $|X| = 2$. It follows that Γ is either a cycle or a ladder graph by the same proof of Theorem 2.9.

Suppose $|X| = 3$. We claim that $\langle a \rangle \cap X \ne \emptyset$. To the contrary, assume that $X = \{a^r b, a^t b, a^s b\}$. Then

$$\left|a^{r-t}\right| \ne pq, \quad \left|a^{r-s}\right| \ne pq \quad \text{and} \quad \left|a^{t-s}\right| \ne pq.$$

Without loss of generality, we may assume $r - t = kp$ and $r - s = lp$, where $(k, q) = 1$ and $(l, q) = 1$. Thus $\langle a^p \rangle = \langle a^{r-s} \rangle \le \langle a^r b, a^s b \rangle$, that is $a^p \in \langle a^r b, a^s b \rangle$. An easy observation is that

$$a^{-kp} a^r b = a^{r-kp} b = a^t b \in \langle a^r b, a^s b \rangle.$$

This contradicts that X is a minimal generating set. Hence $\langle a \rangle \cap X \ne \emptyset$. Thus one of the following cases must occur.

(1) $S = \{a^r, a^{pq-r}, a^t b, a^s b\}$, where $(r, pq) = q$ and $(t - s, pq) = p$.
(2) $S = \{a^r, a^{pq-r}, a^t b, a^s b\}$, where $(r, pq) = p$ and $(t - s, pq) = q$.
(3) $S = \{a^r, a^{pq-r}, a^t, a^{pq-t}, a^s b\}$, where $\{|a^r|, |a^t|\} = \{p, q\}$.

If (1) holds, then we can check that $\Gamma \cong \mathrm{C}_{2q} \Box \mathrm{C}_p$, and hence $rc_2(\Gamma) \le 2q + p$ by Lemma 2.1. If (2) holds, then $\Gamma \cong \mathrm{C}_{2p} \Box \mathrm{C}_q$, and so $rc_2(\Gamma) \le 2p + q$ by Lemma 2.1.

Let $\Gamma_1 = \Gamma[\langle a \rangle]$ and $\Gamma_2 = \Gamma[b\langle a \rangle]$. If (3) holds, then $\Gamma_1 \cong \mathrm{Cay}(\mathbb{Z}_{pq}, S \cap \langle a \rangle)$. By Theorem 2.4, we have $rc_2(\Gamma_1) \le p + q$. Since left multiplication by a group element from D_{2pq} is a graph automorphism, we have $\Gamma_1 \cong \Gamma_2$. Assign a same edge-coloring to Γ_1 and Γ_2 with $p + q$ colors such that Γ_1 and Γ_2 are rainbow 2-connected. In addition, we

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give a new color to all a^sb-edges. Let u and v be any two distinct vertices of Γ. If $u, v \in \Gamma_i$ for $1 \le i \le 2$, then there exist two internally disjoint rainbow paths connecting u and v in Γ_i by means of the rainbow 2-coloring of Γ_i. Without loss of generality, now we assume that $u \in V(\Gamma_1)$ and $v \in V(\Gamma_2)$. Suppose u and v are not adjacent in Γ. Then u, P^1, u_1, v and u, v_1, P^2, v are two internally disjoint rainbow paths connecting u and v in Γ, where $u_1 \in V(\Gamma_1), v_1 \in V(\Gamma_2)$, P^1 is a rainbow path between u and u_1 in Γ_1, and P^2 is a rainbow path between v_1 and v in Γ_2. Suppose u and v are adjacent in Σ. Then uv and $u, \mathsf{P}^3, u_1, v_1, v$ are two internally disjoint rainbow paths connecting u and v in Γ, where $u_1 \in V(\Gamma_1), v_1 \in V(\Gamma_2)$, and P^3 is a rainbow path between u and u_1 in Γ_1. Therefore, Γ is rainbow 2-connected with the above edge-coloring, and so $rc_2(\Gamma) \le p + q + 1$. □

Let $\Gamma = \mathrm{Cay}(\mathrm{D}_{4p}, S)$ be a connected Cayley graph, where $S = \{a^p, a^tb, a^sb\}$ with $(t - s, 2p) = 2$. Then it is easy to verify that Γ is isomorphic to the ladder graph L_{2p}. By a similar proof of Theorem 2.10, the following result holds.

Theorem 2.11. *Let $G = \mathrm{D}_{4p}$, where p is an odd prime. Let X be a minimal generating set of G. Set $S = X \cup X^{-1}$ and $\Gamma = \mathrm{Cay}(G, S)$. Then one of the following statements holds.*

(i) *Γ is either a cycle or a ladder graph.*
(ii) *$S = \{a^r, a^{2p-r}, a^tb, a^sb\}$ with $(r, 2p) = 2$ and $(t - s, 2p) = p$, and $rc_2(\Gamma) \le p + 4$.*
(iii) *$S = \{a^p, a^r, a^{2p-r}, a^tb\}$ with $(r, 2p) = 2$, and $rc_2(\Gamma) \le p + 1$.*

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Highlights

- Establish a lemma for bounding the rainbow 2-connection numbers of some special graphs.
- Provide an upper bound for the rainbow 2-connection numbers of abelian Cayley graphs.
- Characterize the rainbow 2-connection numbers of cubic dihedrants.

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