Construction of Quantum Caps in Projective Space $PG(r, 4)$ and Quantum Codes of Distance 4

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Abstract

Constructions of quantum caps in projective space $PG(r, 4)$ by recursive methods and computer search are discussed. For each even n satisfying $n \geq 282$, each odd z satisfying $z \geq 275$, a quantum n-cap and a quantum z-cap in $PG(k-1, 4)$ with suitable k are constructed and $[[n, n-2k, 4]]$ and $[[z, z-2k, 4]]$ quantum codes are derived from the constructed quantum n-cap and z-cap respectively. For $n \geq 282$ and $n \neq 286$, 756 and 5040, or $z \geq 275$, the results on the sizes of quantum caps and quantum codes are new, and all the obtained quantum codes are optimal codes according to the quantum Hamming bound. While constructing quantum caps, we also obtain many large caps in $PG(r, 4)$ for $r \geq 11$, these results about large caps provide very good low bound on the maximal size of complete in $PG(r, 4)$ for $r \geq 11$.

Keywords: Projective space, Cap, Self-orthogonal code, Quantum code

1 Introduction

Let $\mathbb{F}_q$ be the finite field of q elements, $PG(r, q)$ be the r-dimensional projective space over $\mathbb{F}_q$. An n-cap in $PG(r, q)$ is a set of points no three of which are collinear [18, 4]. If we write the n points of an n-cap $\mathcal{K}$ in $PG(k-1, q)$ as columns of a matrix, we obtain a $k \times n$ matrix $\mathbf{G}_{k,n}$ such that any three columns of $\mathbf{G}_{k,n}$ are linearly independent, $\mathbf{G}_{k,n}$ is called a *representative matrix* of $\mathcal{K}$. If we use $\mathcal{C}$ to denote the q-ary linear code generated by $\mathbf{G}_{k,n}$, then $\mathcal{C}$ is called a *corresponding code* of $\mathcal{K}$ in [14] or a *cap code* of $\mathcal{K}$ in [25]. The matrix $\mathbf{G}_{k,n}$ is called a linear orthogonal array of strength 3 in [2], and it is a parity check matrix of a q-ary linear $[n, n-k, 4]_q$ code.

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An n-cap is *complete* if it is not contained in any $(n+1)$-cap [18]. An n-cap in $PG(r,q)$ with maximal size is called a *maximal cap* in $PG(r,q)$. One important problem on caps is to determine the maximal size of complete caps in $PG(r,q)$. Denote the size of a maximal cap in $PG(r,q)$ as $m_2(r,q)$, and the largest size of a known complete cap as $\overline{m}_2(r,q)$. Much work has been done on $m_2(r,q)$ and $\overline{m}_2(r,q)$, and on construction of caps, see [1,6-14] and references therein. In recent years, people find that quantum caps in $PG(r,4)$ can be used to construct quantum error-correcting codes (QECCs), see [2, 3, 25, 27]. If the cap code of an n-cap in $PG(k-1,4)$ is a self-orthogonal linear $[n,k]_4$ code (see the following for definition), it is called a *self-orthogonal cap* in [25] and a *quantum cap* in [2]. First, we introduce some basic facts on quantum error-correcting codes for later use.

Quantum error-correcting code is a well developed discipline in recent years [26, 12]. The most widely studied class of quantum codes are binary quantum stabilizer codes [5, 15, 16]. A thorough discussion on the principles of binary quantum stabilizer codes was given in [5]. In [5], Calderbank, Rain, Shor and Sloane change the problem of finding QECCs into the problem of finding trace self-orthogonal additive codes over the quaternary field $\mathbb{F}_4$.

Let $\mathbb{F}_4 = \{0, 1, \omega, \varpi\}$ be the finite field with four elements, where $\omega^2 = \varpi$, $\varpi = \omega + 1$, and $\omega^3 = 1$. The conjugate of an element $x \in \mathbb{F}_4$ is defined as $\overline{x} = x^2$. Let $\mathbb{F}_4^n$ be the n-dimensional vector space over $\mathbb{F}_4$. An additive subgroup of $\mathbb{F}_4^n$ of size 2^m is called an $(n, 2^m)$ additive code. A k-dimensional subspace of $\mathbb{F}_4^n$ is called a linear $[n,k]_4$ code. The *trace inner product* of $\mathbf{u}$, $\mathbf{v} \in \mathbb{F}_4^n$ is defined to be $(\mathbf{u},\mathbf{v})_T$ and the *Hermitian inner product* of $\mathbf{u}$, $\mathbf{v}$ is defined to be $(\mathbf{u},\mathbf{v})_h$, where

$$(\mathbf{u},\mathbf{v})_T = Tr(\mathbf{u}\overline{\mathbf{v}}^T) = \textstyle\sum_1^n (u_j\overline{v_j} + \overline{u_j}v_j) = \sum_1^n (u_j v_j^2 + u_j^2 v_j),$$

$$(\mathbf{u},\mathbf{v})_h = \textstyle\sum_1^n u_j\overline{v_j} = \sum_1^n u_j v_j^2.$$

If $\mathcal{C}$ is an $(n, 2^m)$ additive code, its *trace dual code* is an $(n, 2^{2n-m})$ additive code defined as $\mathcal{C}^{\perp_T} = \{\mathbf{u} \in \mathbb{F}_4^n \mid (\mathbf{u},\mathbf{v})_T = 0 \text{ for all } \mathbf{v} \in \mathcal{C}\}$. If $\mathcal{C}$ is a linear $[n,k]_4$ code, its *Hermitian dual code* is an $[n, n-k]_4$ linear code and defined as $\mathcal{C}^{\perp_h} = \{\mathbf{u} \in \mathbb{F}_4^n \mid (\mathbf{u},\mathbf{v})_h = 0 \text{ for all } \mathbf{v} \in \mathcal{C}\}$. An additive code $\mathcal{C}$ is *trace self-orthogonal* if $\mathcal{C} \subseteq \mathcal{C}^{\perp_T}$, and a linear code $\mathcal{C}$ is *self-orthogonal* if $\mathcal{C} \subseteq \mathcal{C}^{\perp_h}$. A linear code $\mathcal{C}$ is self-orthogonal if and only if all its codewords have even weights, and a self-orthogonal $[n,k]_4$ code must be a trace self-orthogonal $(n, 2^{2k})$ [5].

The following statement on constructing binary stabilizer code (or additive quantum code) from trace self-orthogonal code was proven in [5].

Theorem 1.1 (Theorem 2 of [5]) *Suppose $\mathcal{C}$ is an $(n, 2^{n-k})$ trace self-orthogonal additive* *code. If there is no vectors of weight* $< d$ *in* $\mathcal{C}^{\perp_T}\backslash\mathcal{C}$*, then there is a quantum* $[[n, k, d]]$ *code* $\mathcal{Q}$.

Since a linear self-orthogonal $[n, k]_4$ code must be an $(n, 2^{2k})$ trace self-orthogonal additive code [5], hence we have the following:

Proposition 1.1 *Suppose* $\mathcal{C}$ *is an* $[n, k]_4$ *self-orthogonal code. If there are no vectors of weight* $< d$ *in* $\mathcal{C}^{\perp_h}\backslash\mathcal{C}$*, then there is a quantum* $[[n, n-2k, d]]$ *code* $\mathcal{Q}$.

The quantum code $\mathcal{Q}$ obtained in Proposition 1.1 is called a linear quantum code stabilized by $\mathcal{C}$, see [5, 21]. If there is no vectors of weight $\leq d$ in $\mathcal{C}$, then $\mathcal{Q}$ is called a pure (or nondegenerate) quantum code. According to the above discussion and [2, 3], we have

Proposition 1.2 [5, 21]*Let* $\mathbf{G}_{k,n}$ *be a* $k \times n$ *matrix over* $\mathbb{F}_4$*. Then the following are equivalent:*

(1) A quantum n-cap $\mathcal{K}$ *in* $PG(k-1, 4)$ *with matrix* $\mathbf{G}_{k,n}$*.*

(2) The matrix $\mathbf{G}_{k,n}$ *generates a self-orthogonal code with dual distance 4.*

(3) A quantum $[[n, n-2k, 4]]$ *code.*

One central theme in quantum error-correction is the construction of quantum codes with good parameters [5]. Among them, optimal quantum codes received much attention. A quantum code $\mathcal{Q}$ =$[[n, k, d]]$ is *optimal* if there is no $[[n, k, d+1]]$ code. In the case of $d = 2$, all optimal stabilizer codes are known. In the simplest nontrivial case $d = 3$, it took people near 17 years to determine all optimal stabilizer codes, see [16, 5, 21, 22, 23, 29]. In the case of $d = 4$, a systematic construction for all lengths has not been achieved yet. Previous known results on optimal quantum codes with $d = 4$ are short length codes (for length $n \leq 128$ please see [17]), two moderate lengths codes $[[146, 132, 4]]$ and $[[152, 136, 4]]$ codes and two long lengths codes $[[756, 740, 4]]$, $[[5040, 5020, 4]]$, see [5, 21, 25, 27, 2, 3, 17], and many of these optimal quantum codes are constructed from quantum caps or derived from quantum caps. To construct good quantum codes from quantum caps, many methods have been used. Refs. [25, 2, 3] use different methods to discuss construction of quantum caps in $PG(r, 4)$ for $r \leq 6$. Ref. [3] has determined all quantum caps in $PG(r, 4)$ for $r \leq 4$, and gives a 756-quantum cap in $PG(7, 4)$, a 5040-quantum cap in $PG(9, 4)$ and 2(m+1)-quantum cap in $PG(m, 4)$ for m odd. Tonchev used non-constructive method in [27] to deduce the existence of some quantum caps from a 41-quantum cap in $PG(4, 4)$ and a 126-quantum cap in $PG(5, 4)$, and obtain many $[[n, k, 4]]$ codes. Recently, inspired by [27] and [13], we derived many quantum caps from a 286-quantum cap in $PG(6, 4)$ [28] by shortening construction that used in [5, 27].

In this work, for each even n satisfying $n \geq 282$, each odd z satisfying $z \geq 275$, we will construct a quantum n-cap and z-cap in $PG(k-1,4)$ with suitable k. Then we construct $[[n, n-2k, 4]]$ and $[[z, z-2k, 4]]$ quantum codes with good parameters and show all these obtained $[[n, n-2k, 4]]$, $[[z, z-2k, 4]]$ codes are optimal codes according to the quantum Hamming bound.

Proposition 1.3 (The quantum Hamming bound [15])*If* $\mathcal{Q}= [[n,k,d]]$ *is a pure quantum code,* $t = \lfloor \frac{d-1}{2} \rfloor$, *then* $2^{n-k} \geq \sum_{i=0}^{t} 3^i \binom{n}{i}$.

This paper is organized as follows: in section 2, some preliminary materials, known facts and notations are introduced. In section 3, some new facts on quantum caps in $PG(5,4)$ are presented and some new quantum caps in $PG(6,4)$ are constructed by recursive methods and computer search. In section 4 and 5, for even n, constructions of quantum n-caps in $PG(r,4)$ for both $7 \leq r \leq 12$ and $r \geq 13$ are presented. In section 6 and 7, for odd z, constructions of quantum z-caps in $PG(r,4)$ for both $6 \leq r \leq 13$ and $r > 13$ are presented. The last section gives conclusion and disussion.

Before introducing our work, we make some notations for later use.

Notations 1.1 (i) In the following sections, let both n and m be even number and ≥ 6, z be odd and ≥ 21, and all matrices and vectors are over $\mathbb{F}_4$. In each generator matrix of linear codes, we use 2 *and* 3 *to represent* ω *and* ϖ respectively. And, let $\mathbf{1}_m = (1, 1, \cdots, 1)$ and $\mathbf{0}_m = (0, 0, \cdots, 0)$ be the all one vector and the all zero vector of length m, respectively.
(ii) For a matrix (or vector) $\mathbf{P}$ and $x \in \mathbf{F}_4$, the multiplication of $\mathbf{P}$ by x is denoted as $\mathbf{P}(x)$, and the juxtaposition $(\mathbf{P}, \mathbf{P}, ..., \mathbf{P})$ of s-copies of $\mathbf{P}$ is denoted as $s\mathbf{P}$. The conjugate transpose of $\mathbf{P}$ is denoted as $\mathbf{P}^{\dagger}$.
(iii) For an m-cap $\mathcal{K}$ in $PG(k-1,4)$ with matrix $\mathbf{G}_{k,m}$ of size $k \times m$, we will identify $\mathcal{K}$ with $\mathbf{G}_{k,m}$ and call $\mathbf{G}_{k,m}$ as an m-cap in $PG(k-1,4)$. Let $\mathbf{G}_{k,m} = (\alpha_1, \ldots, \alpha_m)$ where α_i is the i-th column of $\mathbf{G}_{k,m}$, $N_m = \{1, \cdots, m\}$ be the *index set* of $\mathbf{G}_{k,m}$. If $I = \{i_1, \ldots, i_s\} \subset N_m$, then I uniquely determines a submatrix $\mathbf{G}_I = (\alpha_{i_1}, \ldots, \alpha_{i_s})$ of $\mathbf{G}_{k,m}$, it is clear $\mathbf{G}_I$ is a subcap of $\mathbf{G}_{k,m}$. Denote $\overline{I} = N_m \setminus I$, then $\mathbf{G}_{\overline{I}}$ is also a subcap of $\mathbf{G}_{k,m}$.
(iv) An $[n,k,d]_4$ code is denoted as $[n,k,d]$ for short.

2 Preliminaries

In this section, we recall some known facts on quantum caps and related knowledge on linear codes over $\mathbb{F}_4$, and make some preparations for the following sections. For more details on codes please see [20].

2.1 Known facts on quantum caps in low dimensional spaces

Let we recall known results on quantum caps in $PG(r,4)$ for $r \le 6$. There is a unique quantum cap in $PG(2,4)$ under projective equivalence, it is a 6-cap and a maximal cap; quantum a-caps exist for $a = 8, 10, 12, 14, 17$ in $PG(3,4)$; quantum b-caps exist for $10 \le b \le 41$ and $b \ne 11, 37, 39$ in $PG(4,4)$; for details see [2, 3, 25]. For $r = 5$ the known maximal 126-cap (Glynn cap [1, 14]) is also a quantum cap. Using non-constructive approach, Tonchev deduced that there are quantum c-caps for $c \in A_1 \cup B_1$ from the 126-cap [27], where $A_1 = \{n, 126 - n \mid 6 \le n \le 18\}$ and $B_1 = \{x \mid x \text{ is an integer}, 20 \le x \le 106\}$. For $r = 6$, we showed that there are quantum m-caps for $m \in A_2 \cup B_2$ from a 286-cap [28], where $A_2 = \{n, 286 - n \mid 6 \le n \le 12\}$ and $B_2 = \{x \mid x \text{ is an integer}, 13 \le x \le 273\}$.

Now, we list some of these known quantum caps in $PG(r,4)$ for $r \le 4$ in matrix form, as for known quantum caps in $PG(r,4)$ for $r = 5, 6$ please see the next section, these caps are useful ingredients for our constructions in this paper.

In $PG(2,4)$ and $PG(3,4)$, the maximal caps are the 6-cap and the 17-cap, respectively. These two caps are quantum caps and their matrices are as follows:

$$\mathbf{G}_{3,6} = \begin{pmatrix} 111111 \\ 231213 \\ 112233 \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_6 \end{pmatrix},$$

$$\mathbf{G}_{4,17} = \begin{pmatrix} 11111111111111110 \\ 02223113121123331 \\ 00332130120322110 \\ 00012212022011131 \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 & 0 \\ \beta_1 & \cdots & \beta_{16} & \beta_{17} \end{pmatrix}.$$

In $PG(4,4)$, the maximal size of caps is 41, up to projective equivalence, there are two 41-caps and one of them is a quantum cap [7]. The quantum 41-cap has matrix $\mathbf{M}_{5,41}$ [7], where

$$\mathbf{M}_{5,41} = \begin{pmatrix} 10000112213322333222333020022100311310012 \\ 01000100200210110110130300230321231311222 \\ 00100012002001101101103302003312213311222 \\ 00010110011100011111111111111111111101011 \\ 00001001111122222211133333300022222200113 \end{pmatrix}.$$

The cap code of $\mathbf{M}_{5,41}$ has weight enumerator

$$Wt_{41}(z) = 1 + 9z^{24} + 12z^{26} + 105z^{28} + 660z^{30} + 90z^{32} + 36z^{34} + 51z^{36} + 60z^{38}.$$

According to this weight enumerator and $M_{5,41}$, we can give a new cap $G_{5,41}$ that is projective equivalent to $M_{5,41}$ as follows:

$$
\mathbf{G}_{5,41} = \begin{pmatrix}
11111111111111111111111111111111111111000 \\
03001000031033010230100110331213213213011 \\
00100320100330112201103003322222213213000 \\
00011302320003112212232321121123303032001 \\
00000012322111222231121200032231100131131
\end{pmatrix}
= \begin{pmatrix}
1 & \dots & 1 & 0 & 0 & 0 \\
\gamma_1 & \dots & \gamma_{38} & \gamma_{39} & \gamma_{40} & \gamma_{41}
\end{pmatrix}.
$$

A general recursive construction for caps is the following theorem of [8].

Theorem 2.1 [8]. *Assume the following exist:*

(1) A k-cap $K_1 \subset PG(r, q)$ and a hyperplane H of $PG(r, q)$ such that $|K_1 \backslash H| = t$, and

(2) an m-cap in $PG(l, q)$.

Then there is a $(tm + k - t)$-cap in $PG(r + l, q)$.

From Theorem 2.1, one can deduce the following corollary (parallel to corollary 1 of [8]), which can be used to construct quantum caps in higher dimensional projective space using matrix method.

Corollary 2.2 *Let $\mathbf{G}_{r+1,a}$ be an a-cap in $PG(r, 4)$ and $\mathbf{G}_{r+1,a}$ generates a cap code $\mathcal{C}_a$. If there is a codeword $\alpha \in \mathcal{C}_a$ with weight $wt(\alpha) = a$, then for any b-cap $\mathbf{G}_{l+1,b}$ in $PG(l, q)$, one can construct an ab-cap in $PG(r + l, 4)$.*

Proof. Without loss of generality, we can assume

$$
\mathbf{G}_{r+1,a} = \begin{pmatrix} 1 & \cdots & 1 \\ \mathbf{x}_1 & \cdots & \mathbf{x}_a \end{pmatrix} = \begin{pmatrix} \mathbf{1_a} \\ \mathbf{X} \end{pmatrix},
$$

and

$$
\mathbf{G}_{l+1,b} = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \mathbf{y}_1 & \cdots & \mathbf{y}_c & \mathbf{y}_{c+1} & \cdots & \mathbf{y}_b \end{pmatrix}.
$$

Construct

$$
\mathbf{G}_{r+l+1,ab} = \begin{pmatrix} \mathbf{1_a} & \cdots & \mathbf{1_a} & \mathbf{0_a} & \cdots & \mathbf{0_a} \\ a\mathbf{y}_1 & \cdots & a\mathbf{y}_c & a\mathbf{y}_{c+1} & \cdots & a\mathbf{y}_b \\ \mathbf{X} & \cdots & \mathbf{X} & \mathbf{X} & \cdots & \mathbf{X} \end{pmatrix}.
$$

Similar to the proof of Theorem 8 in [8], one can check any three columns of $\mathbf{G}_{r+l+1,ab}$ are linearly independent, hence the corollary holds.

Notations 2.1 If we use $\mathbf{G}_{r+1,a}$ and $\mathbf{G}_{l+1,b}$ to construct a $\mathbf{G}'_{r+l+1,ab}$ as

$$
\mathbf{G}'_{r+l+1,ab} = \begin{pmatrix} \mathbf{G}_{l+1,b} & \mathbf{G}_{l+1,b} & \cdots & \mathbf{G}_{l+1,b} \\ b\mathbf{x}_1 & b\mathbf{x}_2 & \cdots & b\mathbf{x}_a \end{pmatrix},
$$

it is easy to see $\mathbf{G}'_{r+l+1,ab}$ is also an ab-cap in $PG(r+l,4)$, and $\mathbf{G}'_{r+l+1,ab}$ can be obtained from $\mathbf{G}_{r+l+1,ab}$ by column permutation.

3 On quantum caps in $PG(5,4)$ and $PG(6,4)$

In this section, we will give some constructive results about a 126-cap $\mathbf{G}_{6,126}$ in $PG(5,4)$ and a 288-cap in $PG(6,4)$ and related results on quantum caps. These results are bases of our discussion in the following sections.

3.1 Some new facts about a 126-cap in $PG(5,4)$

In [14], Glynn gave a construction of a 126-cap in $PG(5,4)$ using geometric method. This 126-cap in $PG(5,4)$ is denoted as $\mathbf{S}_{126}$ and its matrix is denoted as $\mathbf{M}_{6,126}$ in [13]. $\mathbf{M}_{6,126}$ generates a $[126,6,88]$ code and its weight polynomial is $Wt_{126}(z) = 1+945z^{88}+3087z^{96}+63z^{120}$. From this weight polynomial and $\mathbf{M}_{6,126}$, one can give a new 126-cap with matrix $\mathbf{G}_{6,126}$, the cap $\mathbf{G}_{6,126}$ is projective equivalent to $\mathbf{M}_{6,126}$, for the matrix $\mathbf{G}_{6,126}$ please see the Appendix A. We denote $\mathbf{G}_{6,126}$ as

$$\mathbf{G}_{6,126} = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \zeta_1 & \cdots & \zeta_{120} & \zeta_{121} & \cdots & \zeta_{126} \end{pmatrix} = \begin{pmatrix} \mathbf{1_{120}} & \mathbf{0_6} \\ \mathbf{A} & \mathbf{B} \end{pmatrix}.$$

It is clear that A, B satisfy $AA^\dagger = \mathbf{0}$, $BB^\dagger = \mathbf{0}$, $A\mathbf{1}_{120}^\dagger = \mathbf{0}$ and$B\mathbf{1}_6^\dagger = \mathbf{0}$.

Make new observation on $\mathbf{G}_{6,126}$, we find that: $\mathbf{G}_{6,126}$ can be partitioned into $\mathbf{G}_{6,126}$ $= (\mathbf{Q}_1, \mathbf{Q}_2, \cdots, \mathbf{Q}_{15}, \mathbf{R})$. Each of $\mathbf{Q}_i$ is a 6×8 matrix and $\mathbf{R}$ is a 6×6 matrix. The matrix $(\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3)$ can be partitioned into four submatrices $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$, where the index sets of $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ are

$$I_1 = \{1,2,3,4,9,10\},\ I_2 = \{5,6,11,12,17,18\},$$
$$I_3 = \{7,8,21,22,23,24\},\ I_4 = \{13,14,15,16,19,20\},$$

respectively. It is easy to check that $\mathbf{Q}_i\mathbf{Q}_i^\dagger = \mathbf{0}$ for $1 \le i \le 15$, $\mathbf{R}\mathbf{R}^\dagger = \mathbf{0}$ and $\mathbf{P}_j\mathbf{P}_j^\dagger = \mathbf{0}$ for $1 \le j \le 4$. Let $\mathbf{G}_{10}$ be the submatrix of $\mathbf{G}_{6,126}$ with index set $\mathbf{J} = \{5,\cdots,8,11,\cdots,16\}$, we find that $\mathbf{G}_{10}\mathbf{G}\dagger_{10} = \mathbf{0}$ and $\mathbf{G}_{10}\mathbf{1}_{10}^\dagger = \mathbf{0}$. Using sub-matrices of $\mathbf{G}_{6,126}$, one can deduce the following results by constructive method.

Lemma 3.1 *Let* $6 \le n \le 120$ *or* $n = 126$. *Then* $\mathbf{G}_{6,126}$ *has a sub-matrix* $\mathbf{G}_n$ *of size* $6\times n$ *such that* $\mathbf{G}_n\mathbf{G}_n^\dagger = \mathbf{0}$ *and* $\mathbf{G}_n\mathbf{1}_n^\dagger = \mathbf{0}$.

Proof. If $n = 8i$, $1 \le i \le 15$, then $\mathbf{G}_n = (\mathbf{Q}_1, \cdots, \mathbf{Q}_i)$ is a desired sub-matrix. For $n = 8i + 2$ and $1 \le i \le 14$, then $\mathbf{G}_{10}$, $\mathbf{G}_{18} = (\mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4)$, and $\mathbf{G}_n = (\mathbf{G}_{18}, \mathbf{Q}_4, \cdots, \mathbf{Q}_{i+1})$ for $3 \le i \le 14$ are desired matrices.

Suppose $n = 8i + 4$ with $1 \le i \le 14$. Deleting the columns of $\mathbf{G}_{10}$ from $\mathbf{G}_{6,126}$, one obtains $\mathbf{G}_{116}$. Let $\mathbf{G}_{12} = (\mathbf{P}_1, \mathbf{R})$, $\mathbf{G}_n = (\mathbf{P}_1, \mathbf{Q}_4, \cdots, \mathbf{Q}_{i+2}, \mathbf{R})$ for $2 \le i \le 13$. Thus, the lemma holds for $n = 8i + 4$ with $1 \le i \le 14$.

Suppose $n = 8i+6$ with $1 \le i \le 15$, then $\mathbf{G}_n = (\mathbf{P}_1, \mathbf{Q}_3, \cdots, \mathbf{Q}_{i+2})$ for $1 \le i \le 13$ and $\mathbf{G}_{118} = (\mathbf{Q}_2, \mathbf{Q}_3, \cdots, \mathbf{Q}_{15}, \mathbf{R})$ are desired matrices. Hence, the lemma holds for $n = 8i+6$ with $1 \le i \le 15$.

Thus, summarizing the previous discussion, the lemma follows.

From Lemma 3.1, one can give $[[n, n-12, 4]]$ quantum codes for $42 \le n \le 120$ that is obtained in [27].

Corollary 3.2 *For $42 \le n \le 120$ or $n = 126$ then there is a quantum n-cap in $PG(5,4)$. Hence, there is an $[[n, n-12, 4]]$ quantum code.*

In order to construct odd size of quantum caps in section 6, we also need a 120-cap in $PG(5,4)$. Denote the quantum cap formed by the first 120 columns of $\mathbf{G}_{6,126}$ as $\mathbf{G}_{6,120}$. Add ω times of the first row of $\mathbf{G}_{6,120}$ to its third row, then add the first row of $\mathbf{G}_{6,120}$ to its fourth row, after such row transformations on $\mathbf{G}_{6,120}$, we can obtain a new matrix $\mathbf{G}'_{6,120}$, it has the form

$$\mathbf{G}'_{6,120} = \begin{pmatrix} 1 & \cdots & 1 \\ \zeta'_1 & \cdots & \zeta'_{120} \end{pmatrix}$$

where $\zeta'_1 = (0,0,0,0,0)^T$.

From the existence results on odd size quantum sub-cap of the 126-cap in [27], we can deduce the following lemma for later use.

Lemma 3.3 ([27]) *If odd number z satisfying $21 \le z \le 105$, then $\mathbf{G}_{6,126}$ has a quantum z sub-cap. Hence there exists a quantum z-cap in $PG(5,4)$.*

3.2 Quantum cap in $PG(6,4)$ and a 288-cap

In this subsection, we will discuss construction of quantum caps in $PG(6,4)$ derived from a 288-cap.

Recall

$$\mathbf{G}_{4,17} = \begin{pmatrix} 1 & \cdots & 1 & 0 \\ \beta_1 & \cdots & \beta_{16} & \beta_{17} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{16} & \mathbf{0} \\ \mathbf{D} & \beta_{17} \end{pmatrix}.$$

Construct

$$\mathbf{G}_{7,288}=\begin{pmatrix} \mathbf{1}_{16} & \mathbf{1}_{16} & \cdots & \mathbf{1}_{16} & \mathbf{0}_{16} & \mathbf{0}_{16} \\ \mathbf{D} & \mathbf{D} & \cdots & \mathbf{D} & 16\beta_{17} & \mathbf{D} \\ 16\beta_1 & 16\beta_2 & \cdots & 16\beta_{16} & \mathbf{D} & 16\beta_{17} \end{pmatrix}.$$

According to [7, 13], $\mathbf{G}_{7,288}$ forms a 288-cap and it is easy to check $\mathbf{G}_{7,288}$ is not a quantum cap. For the weight polynomial of its cap code, see Appendix B.

Deleting the columns $(0,\beta_{17}^T,\beta_1^T)^T$ and $(0,\beta_1^T,\beta_{17}^T)^T$ from $\mathbf{G}_{7,288}$, we get a quantum 286-cap $\mathbf{G}_{7,286}$. Thus, we can assume $\mathbf{G}_{7,286}=\begin{pmatrix} \delta_1 & \cdots & \delta_{286} \end{pmatrix}$. Then, we will give construction of quantum caps from sub-caps of $\mathbf{G}_{7,286}$.

In [28], using shortening method, we showed that $\mathbf{G}_{7,286}$ has $7\times n$ sub-matrix $G_{7,n}$ satisfying $G_{7,n}G_{7,n}^{\dagger}=\mathbf{0}$, where n=6, 8, 10, 12, 274, 276, 278, 280, or $13\leq n\leq 273$. In the following, we will show there is a quantum 282-cap and a 284-cap in $PG(6,4)$. Let

$$\mathbf{P}_{7,12}=\begin{pmatrix} 000000000000 \\ 111111111111 \\ 000000231213 \\ 111111112233 \\ 111111111111 \\ 231213000000 \\ 112233111111 \end{pmatrix},\ \mathbf{Z}_{7,20}=\begin{pmatrix} 00000000000000000000 \\ 11111001101111111110 \\ 01120112013000002010 \\ 00322333123101110131 \\ 21212232321233201032 \\ 00000000100201230300 \\ 21212232221030111232 \end{pmatrix}$$

$$\mathbf{Q}_{7,16}=\begin{pmatrix} 0000000000000000 \\ 1000110111111111 \\ 3111231200000000 \\ 0020113011111111 \\ 3133331320002202 \\ 0000000022332121 \\ 3133331321203130 \end{pmatrix},$$

and denote the first 6 columns of $\mathbf{P}_{7,12}$ as $\mathbf{PL}_{7,6}$.

Delete the last 30 columns of $\mathbf{G}_{7,286}$, we can obtain a quantum 256-cap $\mathbf{G}_{7,256}$. Firstly, extending $\mathbf{G}_{7,256}$ by $\mathbf{PL}_{7,6}$, we get a quantum 262-cap $\mathbf{G}_{7,262}$. Then extending $\mathbf{G}_{7,262}$ by $\mathbf{Z}_{7,20}$, one can get a 282-cap. Thirdly, extend the 256-cap $\mathbf{G}_{7,256}$ by $\mathbf{P}_{7,12}$, we obtain a quantum cap $\mathbf{G}_{7,268}$. Continue to extend $\mathbf{G}_{7,268}$ by $\mathbf{Q}_{7,16}$, we get a 284-cap. For weight polynomials of the cap codes of quantum caps constructed in this subsection and weight polynomials of their dual codes, please see Appendix B.

Summarizing the results of [28] and pervious discussion, we have

Lemma 3.3 *(1) If n is even, $122 \leq n \leq 286$ and $n \neq 126$, then there is a quantum n-cap in $PG(6,4)$. Hence, there is an $[[n, n-14, 4]]$ quantum code.*
(2) If z is odd and $107 \leq z \leq 273$, then there is a quantum z-cap in $PG(6,4)$. Hence, there is a $[[z, z-14, 4]]$ quantum code.

We also need a 288-cap $\widetilde{\mathbf{G}}'_{7,288}$ in $PG(6,4)$ to construct large quantum cap in $PG(11,4)$. The process of deriving $\widetilde{\mathbf{G}}'_{7,288}$ is as follows: according to the weight polynomial of the cap code generated by $\mathbf{G}_{7,288}$, we give a new generator matrix $G'_{7,288}$, see Appendix B, then adding ϖ times of the first row to the second row of $\mathbf{G}'_{7,288}$, we get a matrix $\widetilde{\mathbf{G}}'_{7,288}$, it has the form

$$\widetilde{\mathbf{G}}'_{7,288} = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \widetilde{\eta}_1 & \cdots & \widetilde{\eta}_{271} & \widetilde{\eta}_{272} & \cdots & \widetilde{\eta}_{288} \end{pmatrix},$$

where $\widetilde{\eta}_1 = \mathbf{0}_6^T$.

4 Even size quantum caps in $PG(r,4)$ for $7 \leq r \leq 12$

In this section, we discuss constructions of even size quantum caps in $PG(r,4)$ for $7 \leq r \leq 12$, from the five large caps constructed in [13] and a new large cap constructed below. We will give these six large caps in the following, for weight polynomials of their cap codes, please see Appendix D and E. Our constructions are inductive and based on some sub-caps of the 126-cap and 288-cap given in section 3. The results are presented in six subsections.

4.1 Even size quantum caps in $PG(7,4)$

Using the 6-cap $\mathbf{G}_{3,6}$ in $PG(2,4)$ and the 126-cap $\mathbf{G}_{6,126}$ in $PG(5,4)$ given above, according to [13], one can construct a quantum 756-cap $\mathbf{G}_{8,756}$ in $PG(7,4)$ and it satisfies $\mathbf{G}_{8,756}\mathbf{1}_{756}^{\dagger} = \mathbf{0}$, where

$$\mathbf{G}_{8,756} = \begin{pmatrix} \mathbf{1}_{120} & \mathbf{1}_{120} & \cdots & \mathbf{1}_{120} & \mathbf{0}_6 & \mathbf{0}_6 & \cdots & \mathbf{0}_6 \\ A & A & \cdots & A & B & B & \cdots & B \\ 120\alpha_1 & 120\alpha_2 & \cdots & 120\alpha_6 & 6\alpha_1 & 6\alpha_2 & \cdots & 6\alpha_6 \end{pmatrix}.$$

This 756-cap in $PG(7,4)$ is not equivalent to the 756-cap given by [7].

Lemma 4.1. *Let $6 \leq n \leq 750$ or $n = 756$. Then $G_{8,756}$ has a sub-matrix $\mathbf{G}_n = \mathbf{G}_{8,n}$ of size $8 \times n$ such that $\mathbf{G}_n\mathbf{G}_n^{\dagger} = \mathbf{0}$ and $\mathbf{G}_n\mathbf{1}_n^{\dagger} = \mathbf{0}$.*

Proof. For $6 \leq n \leq 114$, let $\mathbf{G}_{6,n}$ be a sub-matrix of $\mathbf{G}_{6,126}$ given in Lemma 3.1. Construct

$$\mathbf{G}_{8,n} = \begin{pmatrix} \mathbf{G}_{6,n} \\ n\alpha_1 \end{pmatrix}.$$

If $116 \le n \le 224$, let $m = n - 110$. Construct

$$\mathbf{G}_{8,n} = \begin{pmatrix} \mathbf{G}_{6,110} & \mathbf{G}_{6,m} \\ 110\alpha_1 & m\alpha_2 \end{pmatrix}.$$

If $226 \le n \le 334$, let $m = n - 220$. Construct

$$\mathbf{G}_{8,n} = \begin{pmatrix} \mathbf{G}_{6,110} & \mathbf{G}_{6,110} & \mathbf{G}_{6,m} \\ 110\alpha_1 & 110\alpha_2 & m\alpha_3 \end{pmatrix}.$$

For $336 \le n \le 378$, let $m = n - 330$. Construct

$$\mathbf{G}_{8,n} = \begin{pmatrix} \mathbf{G}_{6,110} & \mathbf{G}_{6,110} & \mathbf{G}_{6,110} & \mathbf{G}_{6,m} \\ 110\alpha_1 & 110\alpha_2 & 110\alpha_3 & m\alpha_4 \end{pmatrix}.$$

It is easy to check all the matrices $\mathbf{G}_{8,n}\mathbf{G}_{8,n}^{\dagger} = \mathbf{0}$ and $\mathbf{G}_{8,n}\mathbf{1}_n^{\dagger} = \mathbf{0}$ for $6 \le n \le 378$. Delete the columns of $\mathbf{G}_{8,n}$ from $\mathbf{G}_{8,756}$ for $6 \le n \le 378$, and denote the result matrix as $\mathbf{G}_{8,m}$ with $m = 756 - n$ and m satisfies $378 \le m \le 750$. Then $\mathbf{G}_{8,m}$ is a desired matrix for $378 \le m \le 750$. Hence the lemma holds.

From Lemma 4.1, we can deduce

Corollary 4.2 *If $288 \le n \le 750$ or $n = 756$. Then there is a quantum n-cap in $PG(7,4)$. Hence, there is an $[[n, n-16, 4]]$ quantum code.*

Notation 4.1. From the construction of $\mathbf{G}_{8,756}$, we know that the weight of both of the last two rows of $\mathbf{G}_{8,756}$ are 756. Thus, multiplying a permutational matrix on the left of $\mathbf{G}_{8,756}$ and multiplying an invertible diagonal matrix on the right of $\mathbf{G}_{8,756}$, one can obtain a quantum cap $\mathbf{G}'_{8,756}$ as

$$\mathbf{G}'_{8,756} = \begin{pmatrix} \mathbf{1}_{126} & \mathbf{1}_{126} & \mathbf{1}_{126} & \mathbf{1}_{126} & \mathbf{1}_{126} & \mathbf{1}_{126} \\ \mathbf{G}_{6,126} & \mathbf{G}_{6,126} & \mathbf{G}_{6,126}(3) & \mathbf{G}_{6,126}(3) & \mathbf{G}_{6,126}(2) & \mathbf{G}_{6,126}(2) \\ \mathbf{1}_{126}(2) & \mathbf{1}_{126}(3) & \mathbf{1}_{126}(3) & \mathbf{1}_{126} & \mathbf{1}_{126}(2) & \mathbf{1}_{126} \end{pmatrix}.$$

From relation of $\mathbf{G}_{8,756}$ and $\mathbf{G}'_{8,756}$, one can deduce that Lemma 4.1 also holds for $\mathbf{G}'_{8,756}$.

Lemma 4.1$'$ *Let $6 \le n \le 750$ or $n = 756$. Then $\mathbf{G}'_{8,756}$ has a sub-matrix $\mathbf{G}'_n = \mathbf{G}'_{8,n}$ of size $8 \times n$ such that $\mathbf{G}'_n\mathbf{G}'^{\dagger}_n = \mathbf{0}$ and $\mathbf{G}'_n\mathbf{1}_n^{\dagger} = \mathbf{0}$.*

4.2 Even size quantum caps in $PG(8,4)$

The present known largest cap in $PG(8,4)$ is a quantum 2136-cap $\mathbf{G}_{9,2136}$ that is given in [13]. This cap is also larger than the 2110-cap of [7]. The matrix $\mathbf{G}_{9,2136}$ is

$$\mathbf{G}_{9,2136} = \begin{pmatrix} \mathbf{1}_{120} & \mathbf{1}_{120} & \cdots & \mathbf{1}_{120} & \mathbf{0}_{120} & \mathbf{0}_6 & \mathbf{0}_6 & \cdots & \mathbf{0}_6 \\ \mathbf{A} & \mathbf{A} & \cdots & \mathbf{A} & \mathbf{A} & \mathbf{B} & \mathbf{B} & \cdots & \mathbf{B} \\ 120\beta_1 & 120\beta_2 & \cdots & 120\beta_{16} & 120\beta_{17} & 6\beta_1 & 6\beta_2 & \cdots & 6\beta_{16} \end{pmatrix}.$$

Lemma 4.3. *Let* $6 \leq n \leq 2130$ *or* $n = 2136$*. Then* $\mathbf{G}_{9,2136}$ *has a sub-matrix* $\mathbf{G}_n = \mathbf{G}_{9,n}$ *of size* $9 \times n$ *such that* $\mathbf{G}_n\mathbf{G}_n^{\dagger} = 0$ *and* $\mathbf{G}_n\mathbf{1}_n^{\dagger} = 0$.

Proof. For $6 \leq n \leq 334$, with similar to the discussion of Lemma 4.1, one can prove the lemma holds. For $110i + 6 \leq n \leq 110i + 114$ with $3 \leq i \leq 9$, let $m = n - 110i$, $\mathbf{G}_{6,110}$ and $\mathbf{G}_{6,m}$ be as in Lemma 3.1. Construct

$$\mathbf{G}_{9,n} = \begin{pmatrix} \mathbf{G}_{6,110} & \mathbf{G}_{6,110} & \cdots & \mathbf{G}_{6,110} & \mathbf{G}_{6,m} \\ 110\beta_1 & 110\beta_2 & \cdots & 110\beta_i & m\beta_{i+1} \end{pmatrix}.$$

It is obvious that $\mathbf{G}_{9,n}$ is a desired sub-matrix of $\mathbf{G}_{9,2136}$. Thus the lemma holds for $346 \leq n \leq 1104$.

Delete the columns of $\mathbf{G}_{9,n}$ from $\mathbf{G}_{9,2136}$ for $6 \leq n \leq 1068$, and denote the result matrix as $\mathbf{G}_{9,m}$ with $m = 2136 - n$ and $1068 \leq m \leq 2130$. Then $G_{9,m}$ is a desired matrix for $1068 \leq m \leq 2130$. Hence the lemma follows.

From Lemma 4.3, we can deduce

Corollary 4.4. *If* $758 \leq n \leq 2130$ *or* $n = 752, 754, 2136$*. Then there is a quantum* n*-cap in* $PG(8,4)$*. Hence, there is an* $[[n, n-18, 4]]$ *quantum code.*

4.3 Even size quantum caps in $PG(9,4)$

Recall that the cap $\mathbf{G}_{5,41}$ given in section 2 is denoted as

$$\mathbf{G}_{5,41} = \begin{pmatrix} 1 & \dots & 1 & 0 & 0 & 0 \\ \gamma_1 & \dots & \gamma_{38} & \gamma_{39} & \gamma_{40} & \gamma_{41} \end{pmatrix}.$$

Deleting the 2nd, 14th, 15th and 24th columns of $(\gamma_1, \gamma_2, \cdots, \gamma_{38})$, one obtains a matrix $\mathbf{E}$ of size 4×34, then denote $\mathbf{E} = (\gamma_1', \gamma_2', \cdots, \gamma_{34}')$. In [13], using $\mathbf{G}_{6,126}$, $\mathbf{G}_{5,41}$ and E, we construct a 5124-cap in $PG(9,4)$ which is larger than the 5040-cap of [7]. This 5124-cap is a quantum cap and has matrix $\mathbf{G}_{10,5124}$ as follows:

$$\mathbf{G}_{10,5124} = \begin{pmatrix} \mathbf{1}_{120} & \mathbf{1}_{120} & \cdots & \mathbf{1}_{120} & \mathbf{0}_{120} & \mathbf{0}_{120} & \mathbf{0}_{120} & \mathbf{0}_6 & \cdots & \mathbf{0}_6 \\ \mathbf{A} & \mathbf{A} & \cdots & \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{B} & \cdots & \mathbf{B} \\ 120\gamma_1 & 120\gamma_2 & \cdots & 120\gamma_{38} & 120\gamma_{39} & 120\gamma_{40} & 120\gamma_{41} & 6\gamma_1' & \cdots & 6\gamma_{34}' \end{pmatrix}.$$

Similar to the discussion of Lemma 4.3, one can obtain the following lemma and corollary.

Lemma 4.5. *Let $6 \le n \le 5118$ or $n = 5124$. Then $\mathbf{G}_{10,5124}$ has a sub-matrix $\mathbf{G}_n = \mathbf{G}_{10,n}$ of size $10 \times n$ such that $\mathbf{G}_n\mathbf{G}_n^{\dagger} = \mathbf{0}$ and $\mathbf{G}_n\mathbf{1}_n^{\dagger} = \mathbf{0}$.*

Corollary 4.6. *If $2138 \le n \le 5118$ or $n = 2132, 2134, 5124$. Then there is a quantum n-cap in $PG(9,4)$. Hence, there is an $[[n, n-20, 4]]$ quantum code.*

4.4 Even size quantum caps in $PG(10,4)$

Using the 126-cap in $PG(5,4)$ given above, we can construct a quantum 15840-cap in $PG(10,4)$ [13] with matrix $\mathbf{G}_{11,15840}$ as follow:

$$\mathbf{G}_{11,15840} = \begin{pmatrix} \mathbf{1}_{120} & \mathbf{1}_{120} & \cdots & \mathbf{1}_{120} & \mathbf{0}_{120} & \cdots & \mathbf{0}_{120} & \mathbf{0}_6 & \cdots & \mathbf{0}_6 \\ \mathbf{A} & \mathbf{A} & \cdots & \mathbf{A} & \mathbf{A} & \cdots & \mathbf{A} & \mathbf{B} & \cdots & \mathbf{B} \\ 120\zeta_1 & 120\zeta_2 & \cdots & 120\zeta_{120} & 120\zeta_{121} & \cdots & 120\zeta_{126} & 6\zeta_1 & \cdots & 6\zeta_{120} \end{pmatrix}.$$

Similar to the discussion of Lemma 4.3, one can obtain the following lemma and corollary.

Lemma 4.7. *Let $6 \le n \le 15834$ or $n = 15840$. Then $\mathbf{G}_{11,15840}$ has a sub-matrix $\mathbf{G}_n = \mathbf{G}_{11,n}$ of size $11 \times n$ such that $\mathbf{G}_n\mathbf{G}_n^{\dagger} = \mathbf{0}$ and $\mathbf{G}_n\mathbf{1}_n^{\dagger} = \mathbf{0}$.*

Corollary 4.8. *If $5126 \le n \le 15834$ or $n = 5120, 5122, 15840$. Then there is a quantum n-cap in $PG(10,4)$. Hence, there is an $[[n, n-22, 4]]$ quantum code.*

4.5 Even size quantum caps in $PG(11,4)$

In section 3.2, we give a 288-cap with matrix $\widetilde{\mathbf{G}}'_{7,288}$ in $PG(6,4)$, where

$$\widetilde{\mathbf{G}}'_{7,288} = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \widetilde{\eta}_1 & \cdots & \widetilde{\eta}_{271} & \widetilde{\eta}_{272} & \cdots & \widetilde{\eta}_{288} \end{pmatrix}.$$

Deleting the 46th, 91st, 136th, 181st, 226th and 271st columns of $(\widetilde{\eta}_1, \widetilde{\eta}_1, \cdots, \widetilde{\eta}_{271})$, denote the resulting matrix as $\mathbf{F} = (\widetilde{\eta}'_1, \cdots, \widetilde{\eta}'_{265})$. Using $\mathbf{G}_{6,126}$, $\mathbf{G}'_{7,288}$ and $\mathbf{F}$, we can construct a 36150-cap $\mathbf{G}_{12,36150}$ in $PG(11,4)$, where

$$\mathbf{G}_{12,36150} = \begin{pmatrix} \mathbf{1}_{120} & \mathbf{1}_{120} & \cdots & \mathbf{1}_{120} & \mathbf{0}_{120} & \cdots & \mathbf{0}_{120} & \mathbf{0}_6 & \cdots & \mathbf{0}_6 \\ \mathbf{A} & \mathbf{A} & \cdots & \mathbf{A} & \mathbf{A} & \cdots & \mathbf{A} & \mathbf{B} & \cdots & \mathbf{B} \\ 120\widetilde{\eta}_1 & 120\widetilde{\eta}_2 & \cdots & 120\widetilde{\eta}_{271} & 120\widetilde{\eta}_{272} & \cdots & 120\widetilde{\eta}_{288} & 6\widetilde{\eta}'_1 & \cdots & 6\widetilde{\eta}'_{265} \end{pmatrix}.$$

This is a quantum cap and is larger than the 36084 cap given in [13]. For weight polynomials of this cap code and its dual code, please see Appendix D. Similar to the discussion of Lemma 4.3, one can derive the following lemma and corollary.

Lemma 4.9 *Let $6 \le n \le 36144$ or $n = 36150$. Then $\mathbf{G}_{12,36150}$ has a sub-matrix $\mathbf{G}_n = \mathbf{G}_{12,n}$ of size $12 \times n$ such that $\mathbf{G}_n\mathbf{G}_n^{\dagger} = \mathbf{0}$ and $\mathbf{G}_n\mathbf{1}_n^{\dagger} = \mathbf{0}$.*

Corollary 4.10 *Let $15842 \le n \le 36144$ or $n = 15836, 15838, 36150$. Then there is a quantum n-cap in $PG(11,4)$. Hence, there is an $[[n, n-24, 4]]$ quantum code.*

4.6 Even size quantum caps in $PG(12, 4)$

Denote the 756-cap $\mathbf{G}'_{8,756}$ given in subsection 4.1 as

$$\mathbf{G}'_{8,756} = \begin{pmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{756} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{756} \\ \mathbf{Y} \end{pmatrix}.$$

According to Corollary 2.2, using $\mathbf{G}'_{8,756}$ and $\mathbf{G}_{6,126}$, one can construct a 95256-cap in $PG(12, 4)$ as in [13]. This is a quantum 95256-cap and it has matrix $\mathbf{G}_{13,95256}$ as follows:

$$\mathbf{G}_{13,95256} = \begin{pmatrix} \mathbf{1}_{756} & \mathbf{1}_{756} & \cdots & \mathbf{1}_{756} & \mathbf{0}_{756} & \cdots & \mathbf{0}_{756} \\ \mathbf{Y} & \mathbf{Y} & \cdots & \mathbf{Y} & \mathbf{Y} & \cdots & \mathbf{Y} \\ 756\zeta_1 & 756\zeta_2 & \cdots & 756\zeta_{120} & 756\zeta_{121} & \cdots & 756\zeta_{126} \end{pmatrix}.$$

Using Lemma $4.1'$, similar to the discussion of Lemma 4.3, one can deduce the following lemma and corollary

Lemma 4.11 *Let* $6 \le n \le 95250$ *or* $n = 95256$. *Then* $\mathbf{G}_{13,95256}$ *has a sub-matrix* $\mathbf{G}_n = \mathbf{G}_{13,n}$ *of size* $13 \times n$ *such that* $\mathbf{G}_n\mathbf{G}_n^{\dagger} = \mathbf{0}$ *and* $\mathbf{G}_n\mathbf{1}_n^{\dagger} = \mathbf{0}$.

Corollary 4.12 *If* $36152 \le n \le 95250$ *or* $n = 36146, 36148, 95256$. *Then there is a quantum* n*-cap in* $PG(12, 4)$. *Hence, there is an* $[[n, n-26, 4]]$ *quantum code.*

5 Even size quantum caps in $PG(r, 4)$ for $r \ge 13$

In this section, we discuss constructions of quantum caps in $PG(r, 4)$ for $r \ge 13$, using two incomplete quantum caps and known large caps in previous sections as main ingredients.

5.1 A sub-cap of $\mathbf{G}_{6,126}$ and a sub-cap of $\mathbf{G}_{11,15840}$

Recall $\mathbf{G}_{6,126} = (\mathbf{Q}_1, \mathbf{Q}_2, \cdots, \mathbf{Q}_{15}, \mathbf{R})$ and $(\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3)$ can be partitioned into four sub-matrices $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$. Deleting the columns of $\mathbf{P}_1$ from $\mathbf{G}_{6,126}$, one can obtain a matrix $\widetilde{\mathbf{G}}_{6,120} = (\mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4, \mathbf{Q}_4, \cdots, \mathbf{Q}_{15}, \mathbf{R})$ and the third row of $\widetilde{\mathbf{G}}_{6,120}$ has weight 120. From $\mathbf{G}_{6,126}$, we know $\widetilde{\mathbf{G}}_{6,120}$ can be denoted as

$$\mathbf{G}_{6,120} = \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \zeta_5 & \cdots & \zeta_8 & \zeta_{11} & \cdots & \zeta_{120} & \zeta_{121} & \cdots & \zeta_{126} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{114} & \mathbf{0}_6 \\ \mathbf{A}_1 & \mathbf{B} \end{pmatrix}.$$

Since $\mathbf{P}_1\mathbf{P}_1^{\dagger} = \mathbf{0}$ and $\mathbf{P}_1\mathbf{1}^{\dagger}{}_6 = \mathbf{0}$, hence $\widetilde{\mathbf{G}}_{6,120}$ is also a quantum cap, $\mathbf{A}_1\mathbf{1}^{\dagger}{}_{114} = \mathbf{0}$, $\mathbf{B}\mathbf{1}_6^{\dagger} = \mathbf{0}$ and $\widetilde{\mathbf{G}}_{6,120}\mathbf{1}_{120}^{\dagger} = \mathbf{0}$.

Denote the index set of $(\mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4, \mathbf{R})$ as $N_{24} = \{1, 2, \cdots, 24\}$, let S_1, S_2 and S_3 be $S_1 = \{1, \cdots, 6, 13, 14\}$, $S_2 = \{7, \cdots, 10, 15, 16, 22, 24\}$, $S_3 = \{11, 12, 17, \cdots, 21, 23\}$.

Denote the sub-matrix with index set S_i as $\widetilde{\mathbf{Q}}_i$ for $1 \leq i \leq 3$. Then the sub-matrix $(\mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4, \mathbf{R})$ of $\widetilde{\mathbf{G}}_{6,120}$ can be partitioned into three sub-matrices $\widetilde{\mathbf{Q}}_1, \widetilde{\mathbf{Q}}_2$ and $\widetilde{\mathbf{Q}}_3$, all these three matrices are of size 6×8 and satisfy $\widetilde{\mathbf{Q}}_i \widetilde{\mathbf{Q}}_i^{\dagger} = 0$.

Similar to the discussion of Lemma 3.1, one can derive:

Lemma 5.1 *Let* $6 \leq n \leq 114$ *or* $n = 120$. *Then* $\widetilde{\mathbf{G}}_{6,120}$ *has a sub-matrix* $\mathbf{G}_n$ *of size* $6 \times n$ *such that* $\mathbf{G}_n \mathbf{G}_n^{\dagger} = \mathbf{0}$ *and* $\mathbf{G}_n \mathbf{1}_n^{\dagger} = \mathbf{0}$.

Using $\widetilde{\mathbf{G}}_{6,120}$ and $\mathbf{G}_{6,126}$, we can construct a $\mathbf{G}_{11,15084}$ as follows

$$\mathbf{G}_{11,15084} = \begin{pmatrix} \mathbf{1}_{114} & \mathbf{1}_{114} & \cdots & \mathbf{1}_{114} & \mathbf{0}_{114} & \cdots & \mathbf{0}_{114} & \mathbf{0}_6 & \cdots & \mathbf{0}_6 \\ \mathbf{A}_1 & \mathbf{A}_1 & \cdots & \mathbf{A}_1 & \mathbf{A}_1 & \cdots & \mathbf{A}_1 & \mathbf{B} & \cdots & \mathbf{B} \\ 114\zeta_1 & 114\zeta_2 & \cdots & 114\zeta_{120} & 114\zeta_{121} & \cdots & 114\zeta_{126} & 6\zeta_1 & \cdots & 6\zeta_{120} \end{pmatrix}.$$

It is easy to see that $\mathbf{G}_{11,15084}$ is a quantum sub-cap of $\mathbf{G}_{11,15840}$ and the weight of the third row of $\mathbf{G}_{11,15084}$ is 15084.

Using a similar method of proving Lemma 4.3, one can deduce

Lemma 5.2 *Let* $6 \leq n \leq 15078$ *or* $n = 15084$, *then* $\mathbf{G}_{11,15084}$ *has a sub-matrix* $\mathbf{G}_n$ *of size* $11 \times n$ *such that* $\mathbf{G}_n \mathbf{G}_n^{\dagger} = \mathbf{0}$ *and* $\mathbf{G}_n \mathbf{1}_n^{\dagger} = \mathbf{0}$.

Exchanging the first row and the third row of $\mathbf{G}_{11,15084}$ and then multiplying an invertible diagonal matrix on the right of $\mathbf{G}_{11,15084}$, one can obtain a matrix $\mathbf{G}'_{11,15084}$ and the first row of $\mathbf{G}'_{11,15084}$ is $\mathbf{1}_{15084}$. Hence, we can assume

$$\mathbf{G}'_{11,15084} = \begin{pmatrix} 1 & \cdots & 1 \\ \psi_1 & \cdots & \psi_{15084} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{15084} \\ \mathbf{W} \end{pmatrix}.$$

From Lemma 5.2, we can also derive

Lemma 5.3 *Let* $6 \leq n \leq 15078$ *or* $n = 15084$. *Then* $\mathbf{G}'_{11,15084}$ *has a sub-matrix* $\mathbf{G}_n$ *of size* $11 \times n$ *such that* $\mathbf{G}_n \mathbf{G}_n^{\dagger} = \mathbf{0}$ *and* $\mathbf{G}_n \mathbf{1}_n^{\dagger} = \mathbf{0}$.

Using $\mathbf{G}'_{11,15084}$, we can derive another quantum cap $\widetilde{\mathbf{G}}'_{11,15084}$, which will be used in section 7. First, add ω times of the first row in $\mathbf{G}'_{11,15084}$ to the second and the seventh row, respectively. Then add the first row to the fourth row and the ninth row, respectively. Now a new quantum cap with matrix $\widetilde{\mathbf{G}}'_{11,15084}$ can be obtained as

$$\widetilde{\mathbf{G}}'_{11,15084} = \begin{pmatrix} 1 & \cdots & 1 \\ \widetilde{\psi}_1 & \cdots & \widetilde{\psi}_{15084} \end{pmatrix}$$

where $\widetilde{\psi}_1 = \mathbf{0}_{10}^T$. From the construction of $\widetilde{\mathbf{G}}'_{11,15084}$, one can see Lemma 5.3 also hold for $\widetilde{\mathbf{G}}'_{11,15084}$.

5.2 Even size quantum caps in $PG(r,4)$ for $r \geq 13$

For $3 \leq r_1 \leq 12$, denote the size of the largest known cap (given in the above sections) of $PG(r_1,4)$ as M_{r_1}, and suppose the corresponding caps as

$$\mathbf{G}_{r_1+1,M_{r_1}} = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \varphi_1 & \cdots & \varphi_{a_{r_1}} & \varphi_{a_{r_1}+1} & \cdots & \varphi_{M_{r_1}} \end{pmatrix}.$$

Let $N = 15084$. If $3 \leq r_1 \leq 12$ and $r_1 \neq 7$, using $\mathbf{G}'_{11,15084}$ and $\mathbf{G}_{r_1+1,M_{r_1}}$, one can construct an NM_{r_1}-cap $\mathbf{G}_{10+r_1+1,NM_{r_1}}$ in $PG(10+r_1,4)$ as

$$\mathbf{G}_{10+r_1+1,NM_{r_1}} = \begin{pmatrix} \mathbf{G}_{r_1+1,M_{r_1}} & \mathbf{G}_{r_1+1,M_{r_1}} & \cdots & \mathbf{G}_{r_1+1,M_{r_1}} \\ M_{r_1}\psi_1 & M_{r_1}\psi_2 & \cdots & M_{r_1}\psi_{15084} \end{pmatrix}.$$

Using $\mathbf{G}'_{8,756}$ and $\mathbf{G}_{11,M_{10}}$, one can construct an $756M_{10}$-cap $\mathbf{G}_{18,756M_{10}}$ in $PG(17,4)$ as

$$\mathbf{G}_{18,756M_{10}} = \begin{pmatrix} \mathbf{G}_{11,M_{10}} & \mathbf{G}_{11,M_{10}} & \cdots & \mathbf{G}_{11,M_{10}} \\ M_{10}\theta_1 & M_{10}\theta_2 & \cdots & M_{10}\theta_{756} \end{pmatrix}.$$

It is easy to see that $\mathbf{G}_{10+r_1+1,NM_{r_1}}$ and $\mathbf{G}_{18,756M_{10}}$ are quantum caps.

For $r = 10i + r' \geq 23$ with $0 \leq r' \leq 9$, using the above recursive construction step by step, one can construct a quantum cap of size M_r as follows: an $M_r = N^{i-1}M_{10+r'}$ quantum cap in $PG(r,4)$ for $0 \leq r' \leq 2$ or $r' = 7$, and an $M_r = N^i M_{r'}$ quantum cap in $PG(r,4)$ for $3 \leq r' \leq 9$ and $r' \neq 7$. Denote the matrix of this quantum cap of size M_r as G_{r+1,M_r}. Using Lemma 5.3 and inductive method, one can prove the following lemma and corollary.

Lemma 5.4 *Let $r = 10i + r' \geq 13$, $N = 15084$, $M_r = N^{i-1}M_{10+r'}$ for $0 \leq r' \leq 2$ or $r' = 7$, $M_r = N^i M_{r'}$ for $3 \leq r' \leq 9$ and $r' \neq 7$, $\mathbf{G}_{r+1,M_r}$ be the quantum caps constructed by previous method. If $6 \leq n \leq M_r - 6$ or $n = M_r$, then $\mathbf{G}_{r+1,M_r}$ has a sub-matrix $\mathbf{G}_n$ of size $(r+1) \times n$ such that $\mathbf{G}_n\mathbf{G}_n^\dagger = \mathbf{0}$ and $\mathbf{G}_n\mathbf{1}_n^\dagger = \mathbf{0}$.*

Corollary 5.5 *Let $r \geq 13$, M_r as be given in Lemma 5.4 and $M_{12} = 95256$. If $M_{r-1}+2 \leq n \leq M_r - 6$ or $n = M_{r-1} - 4, M_{r-1} - 2, M_r$, then there is a quantum n-cap in $PG(r,4)$. Hence, there is an $[[n, n-2(r+1), 4]]$ quantum code.*

6 Odd size quantum cap in $PG(r,4)$ for $7 \leq r \leq 13$

In the following two sections, we discuss construction of odd size quantum z-cap for $z \geq 275$ from a new quantum cap $\mathbf{G}'_{r+1,M'_r}$ in $PG(r,4)$ for $r \geq 7$. Each new quantum cap $\mathbf{G}'_{r+1,M'_r}$ has the form of

$$\mathbf{G}_{r+1,M'_r} = (B_1, B_2, \cdots, B_u),$$

where B_1 can be used to deduce odd size quantum caps, and the other B_i can be used to contribute even size quantum n-cap constructed in previous sections. First, we discuss construction of $\mathbf{G'}_{r+1,M'_r}$ for $7 \le r \le 13$ in this section.

6.1 Odd size quantum cap in $PG(7,4)$

To deduce odd size quantum cap in $PG(7,4)$, we need to construct a new 756-cap in $PG(7,4)$ which is not equivalent to the 756-cap in section 4. Using $\mathbf{G}'_{3,6}$ in the following and $\mathbf{G}_{6,126}$, we construct a matrix $\mathbf{G}''_{8,756}$ as follows:

$$\mathbf{G}'_{3,6} = \begin{pmatrix} 111111 \\ 030131 \\ 013301 \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ \alpha'_1 & \cdots & \alpha'_6 \end{pmatrix},$$

$$\mathbf{G}''_{8,756} = \begin{pmatrix} \mathbf{G}_{6,126} & \mathbf{G}_{6,126} & \cdots & \mathbf{G}_{6,126} \\ 126\alpha'_1 & 126\alpha'_2 & \cdots & 126\alpha'_6 \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{120} & \mathbf{0}_6 & \cdots & \mathbf{1}_{120} & \mathbf{0}_6 \\ \mathbf{A} & \mathbf{B} & \cdots & \mathbf{A} & \mathbf{B} \\ 120\alpha'_1 & 6\alpha'_1 & \cdots & 120\alpha'_6 & 6\alpha'_6 \end{pmatrix}.$$

Due to $\alpha'_1 = (0,0)^T$, from the first 126 columns of $\mathbf{G}''_{8,756}$, we can deduce an odd quantum s-cap exist for $21 \le s \le 105$. According to section 3, for $6 \le n \le 624$ or $n = 630$, the last 630 columns of $\mathbf{G}''_{8,756}$ can give an quantum n-cap. Hence there is an odd quantum $z = (n+s)$-cap in $PG(7,4)$ for $21 \le z \le 735$. This proves

Lemma 6.1 *For odd z satisfying $275 \le z \le 735$, there exists a quantum z-cap in* $PG(7,4)$. Thus there is an $[[z, z-16, 4]]$ quantum code.

6.2 Odd size quantum cap in $PG(8,4)$

A 2136-cap $\mathbf{G}_{9,2136}$ in $PG(8,4)$ is given in section 4. We rearrange its columns by column permutation, then a 2136-cap $\mathbf{G}'_{9,2136}$ can be obtained as follow:

$$\mathbf{G}'_{9,2136} = \begin{pmatrix} \mathbf{1}_{120} & \mathbf{0}_6 & \cdots & \mathbf{1}_{120} & \mathbf{0}_6 & \mathbf{0}_{120} \\ \mathbf{A} & \mathbf{B} & \cdots & \mathbf{A} & \mathbf{B} & \mathbf{A} \\ 120\beta_1 & 6\beta_1 & \cdots & 120\beta_{16} & 6\beta_{16} & 120\beta_{17} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{G}_{6,126} & \mathbf{G}_{6,126} & \cdots & \mathbf{G}_{6,126} & \mathbf{A}'_{120} \\ 126\beta_1 & 126\beta_2 & \cdots & 126\beta_{16} & 120\beta_{17} \end{pmatrix},$$

where $\mathbf{A}'_{120} = \begin{pmatrix} \mathbf{0}_{120} \\ \mathbf{A} \end{pmatrix}$.

From $\beta_1 = (0,0,0)^T$, similar to the discussion of Lemma 6.1, one can derive $\mathbf{G}'_{9,2136}$ has quantum sub-cap of odd size z for $21 \leq z \leq 2115$. Thus the following lemma holds.

Lemma 6.2 *For odd z satisfying $737 \leq z \leq 2115$, there exists a quantum z-cap in* $PG(8,4)$. Hence there is an $[[z, z-18, 4]]$ quantum code.

6.3 Odd size quantum cap in $PG(9,4)$

In section 4.3, we obtain a 5124-cap $\mathbf{G}_{10,5124}$ as follows:

$$\mathbf{G}_{10,5124} = \begin{pmatrix} \mathbf{1}_{120} & \mathbf{1}_{120} & \cdots & \mathbf{1}_{120} & \mathbf{0}_{120} & \mathbf{0}_{120} & \mathbf{0}_{120} & \mathbf{0}_6 & \cdots & \mathbf{0}_6 \\ \mathbf{A} & \mathbf{A} & \cdots & \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{A} & \mathbf{B} & \cdots & \mathbf{B} \\ 120\gamma_1 & 120\gamma_2 & \cdots & 120\gamma_{38} & 120\gamma_{39} & 120\gamma_{40} & 120\gamma_{41} & 6\delta_1 & \cdots & 6\delta_{34} \end{pmatrix}.$$

Recall $\gamma_1 = \delta_1 = (0,0,0,0)^T$, hence $\mathbf{G}_{10,5124}$ has a sub-matrix $\mathbf{G}_{10,126} = \begin{pmatrix} \mathbf{G}_{6,126} \\ \mathbf{0}_{4,126} \end{pmatrix}$, this sub-matrix can give odd size quantum z-cap for $21 \leq z \leq 105$. The remain columns of $\mathbf{G}_{10,5124}$ can provide even size quantum n-cap for $6 \leq n \leq 4992$ or $n = 4998$ according to section 4. Thus we have

Lemma 6.3 *For odd z satisfying $2117 \leq z \leq 5103$, there exists a quantum z-cap in $PG(9,4)$. Hence there is an $[[z, z-20, 4]]$ quantum code.*

6.4 Odd size quantum cap in $PG(10,4)$

To get odd size quantum cap in $PG(10,4)$, we need construct a new large quantum cap in $PG(10,4)$, this new cap can be constructed from two caps in $PG(5,4)$, the 120-cap $\mathbf{G}'_{6,120}$ and $\mathbf{G}_{6,126}$.

Delete the 2nd, the 6th, the 20th and the 22nd columns from $(\zeta'_1, \cdots, \zeta'_{120})$, and denote the matrix formed by remain columns as $(\zeta''_1, \cdots, \zeta''_{116})$.

Using $\mathbf{G}'_{6,120}$, $\mathbf{G}_{6,126}$ and $(\zeta''_1, \cdots, \zeta''_{116})$, a new matrix can be constructed as

$$\mathbf{G}'_{11,15816} = \begin{pmatrix} \mathbf{1}_{120} & \mathbf{1}_{120} & \cdots & \mathbf{1}_{120} & \mathbf{0}_{120} & \cdots & \mathbf{0}_{120} & \mathbf{0}_6 & \cdots & \mathbf{0}_6 \\ \mathbf{A} & \mathbf{A} & \cdots & \mathbf{A} & \mathbf{A} & \cdots & \mathbf{A} & \mathbf{B} & \cdots & \mathbf{B} \\ 120\zeta'_1 & 120\zeta'_2 & \cdots & 120\zeta'_{120} & 120\zeta_{121} & \cdots & 120\zeta_{126} & 6\zeta''_1 & \cdots & 6\zeta''_{116} \end{pmatrix}.$$

The matrix $\mathbf{G}'_{11,15816}$ is a quantum 15816-cap in $PG(10,4)$, it generates a cap code $[15816, 11, 10992]$. For weight enumerators of this code and its dual code, please see Appendix E.

From $\zeta'_1 = \zeta''_1 = (0,0,0,0,0)^T$, one can see that $\mathbf{G}_{11,15816}$ has a sub-matrix $\mathbf{G}_{11,126} = \begin{pmatrix} \mathbf{G}_{6,126} \\ \mathbf{0}_{5,126} \end{pmatrix}$. Similar to the discussion of lemma 6.1, we have

Lemma 6.4 *For odd z satisfying $5105 \leq z \leq 15795$, there exists a quantum z-cap in $PG(10,4)$. Hence there is an $[[z, z-22, 4]]$ quantum code.*

6.5 Odd size quantum cap in $PG(11,4)$

The 36150-cap constructed in subsection 4.5 has matrix $\mathbf{G}_{12,36150}$, where

$$\mathbf{G}_{12,36150} = \begin{pmatrix} \mathbf{1}_{120} & \mathbf{1}_{120} & \cdots & \mathbf{1}_{120} & \mathbf{0}_{120} & \cdots & \mathbf{0}_{120} & \mathbf{0}_6 & \cdots & \mathbf{0}_6 \\ \mathbf{A} & \mathbf{A} & \cdots & \mathbf{A} & \mathbf{A} & \cdots & \mathbf{A} & \mathbf{B} & \cdots & \mathbf{B} \\ 120\widetilde{\eta}_1 & 120\widetilde{\eta}_2 & \cdots & 120\widetilde{\eta}_{271} & 120\widetilde{\eta}_{272} & \cdots & 120\widetilde{\eta}_{288} & 6\widetilde{\eta}'_1 & \cdots & 6\widetilde{\eta}'_{265} \end{pmatrix}.$$

From $\widetilde{\eta}_1 = \widetilde{\eta}'_1 = (0,0,\cdots,0)^T$, one can see $\mathbf{G}_{12,36150}$ has a sub-matrix $\mathbf{G}_{12,126} = \begin{pmatrix} \mathbf{G}_{6,126} \\ \mathbf{0}_{6,126} \end{pmatrix}$. This sub-matrix can give odd size quantum s-cap in $PG(11,4)$ for $21 \leq s \leq 105$, the other columns of $\mathbf{G}_{12,36150}$ form a $\mathbf{G}_{12,36024}$ matrix and $\mathbf{G}_{12,36024}$ can provide even size quantum n-cap in $PG(11,4)$. Similar to the discussion of Lemma 6.1, we have

Lemma 6.5 *For odd z satisfying $15233 \leq z \leq 36129$, there exists a quantum z-cap in $PG(11,4)$. Hence there is an $[[z, z-24, 4]]$ quantum code.*

6.6 Odd size quantum cap in $PG(12,4)$

Perform elementary row transformation on $\mathbf{G}'_{8,756} = \begin{pmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{756} \end{pmatrix}$, one can change it into $\widetilde{\mathbf{G}}'_{8,756} = \begin{pmatrix} 1 & \cdots & 1 \\ \theta'_1 & \cdots & \theta'_{756} \end{pmatrix}$, where $\theta'_1 = \mathbf{0}_7^T$. It is clear that $\widetilde{\mathbf{G}}'_{8,756}$ is also a quantum cap and is equivalent to $\mathbf{G}'_{8,756}$.

A new 95256-cap $\mathbf{G}'_{13,95256}$ in $PG(12,4)$ can be constructed from $\widetilde{\mathbf{G}}'_{8,756}$ and $\mathbf{G}_{6,126}$ as follows:

$$\mathbf{G}'_{13,95256} = \begin{pmatrix} \mathbf{G}_{6,126} & \cdots & \mathbf{G}_{6,126} \\ 126\theta'_1 & \cdots & 126\theta'_{756} \end{pmatrix}$$

From the construction of $\mathbf{G}'_{13,95256}$, one can deduce

Lemma 6.6 *For odd z satisfying $34131 \leq z \leq 95235$, there exists a quantum z-cap in $PG(12,4)$. Hence there is an $[[z, z-26, 4]]$ quantum code.*

6.7 Odd size quantum cap in $PG(13,4)$

Recall a 2136-cap $\mathbf{G}_{9,2136}$ in $PG(8,4)$ is constructed in subsection 4.2, the weight of its second row is 2034. Thus, exchanging the first and the second row of $\mathbf{G}_{9,2136}$ gives a new

2136-cap $\mathbf{G}'_{9,2136}$. Multiply an invertible diagonal matrix on the right of $\mathbf{G}'_{9,2136}$ and then permutate the columns of the resulting matrix, we can change $\mathbf{G}'_{9,2136}$ into $\mathbf{G}''_{9,2136}$, whose first row is $(\mathbf{1}_{2034}, \mathbf{0}_{102})$. Perform elementary row transformation on $\mathbf{G}''_{9,2136}$, we can obtain a 2034-cap $\mathbf{G}''_{9,2034}$ in $PG(8,4)$, $\mathbf{G}''_{9,2034}$ has the form

$$\mathbf{G}'_{9,2034} = \begin{pmatrix} 1 & \cdots & 1 \\ \varepsilon_1 & \cdots & \varepsilon_{2034} \end{pmatrix}, \text{ where } \varepsilon_1 = \mathbf{0}_8.$$

Similar to previous subsection, using $\mathbf{G}''_{9,2034}$ and $\mathbf{G}_{6,126}$, we can construct a 256284-cap in $PG(13,4)$ as follow:

$$\mathbf{G}_{14,256284} = \begin{pmatrix} \mathbf{G}_{6,126} & \cdots & \mathbf{G}_{6,126} \\ 126\varepsilon_1 & \cdots & 126\varepsilon_{2034} \end{pmatrix}$$

From the construction of $\mathbf{G}_{14,256284}$, one can deduce the following lemma holds.

Lemma 6.7 *For odd z satisfying $95237 \leq z \leq 256263$, there exists a quantum z-cap in $PG(12,4)$. Hence there is a $[[z, z-28, 4]]$ quantum code.*

7 Quantum z-cap in $PG(r,4)$ for $r > 13$

In section 6, for $7 \leq r_1 \leq 13$, a new large cap in $PG(r_1,4)$ and its quantum sub-caps with odd size are constructed. Let the maximal size of cap in $PG(r_1,4)$ constructed in section 6 be M'_{r_1}, denote the M'_{r_1}-cap in $PG(r_1,4)$ as $\mathbf{G}'_{r_1+1,M'_{r_1}}$. Based on these results, in this section, we will construct large caps $\mathbf{G}'_{r+1,M'_r}$ and their odd size sub-caps in $PG(r,4)$ for $r > 13$. First, we discuss construction of $\mathbf{G}'_{r+1,M'_r}$ for $14 \leq r \leq 23$.

For $r = 14$, using $\mathbf{G}'_{6,120}$ and $\mathbf{G}_{10,M'_9}$, one can construct an $M'_{14} = 120M'_9$ cap as

$$\mathbf{G}'_{15,M'_{14}} = \begin{pmatrix} \mathbf{G}'_{10,M'_9} & \mathbf{G}'_{10,M'_9} & \cdots & \mathbf{G}'_{10,M'_9} \\ M'_9\zeta'_1 & M'_9\zeta'_2 & \cdots & M'_9\zeta'_{120} \end{pmatrix}.$$

For $r = 16$, using $\mathbf{G}''_{9,2034}$ and $\mathbf{G}'_{9,M'_8}$, one can construct an $M'_{16} = 2034M'_8$-cap $\mathbf{G}'_{17,M'_{16}}$ in $PG(16,4)$ as

$$\mathbf{G}'_{17,M'_{16}} = \begin{pmatrix} \mathbf{G}'_{9,M'_8} & \mathbf{G}'_{9,M'_8} & \cdots & \mathbf{G}'_{9,M'_8} \\ M'_8\varepsilon_1 & M'_8\varepsilon_2 & \cdots & M'_8\varepsilon_{2034} \end{pmatrix}.$$

For $r = 17$, using $\widetilde{\mathbf{G}}'_{8,756}$ given in subsection 4.6 and $\mathbf{G}'_{11,M'_{10}}$, one can construct an $M'_{17} = 756M'_{10}$-cap $\mathbf{G}'_{18,756M'_{10}}$ in $PG(17,4)$ as

$$\mathbf{G}'_{18,M'_{17}} = \begin{pmatrix} \mathbf{G}'_{11,M'_{10}} & \mathbf{G}'_{11,M'_{10}} & \cdots & \mathbf{G}'_{11,M'_{10}} \\ M'_{10}\theta'_1 & M'_{10}\theta'_2 & \cdots & M'_{10}\theta'_{756} \end{pmatrix}.$$

For $r = 10 + r'$ with $r' = 5$ or $8 \le r' \le 13$, using $\widetilde{\mathbf{G}}'_{11,15084}$ given in subsection 5.1 and $\mathbf{G}'_{r'+1,M'_{r'}}$, one can construct an $M'_r = NM'_{r'}$-cap $\mathbf{G'}_{r+1,M'_r}$ in $PG(r, 4)$ as

$$\mathbf{G}'_{r+1,M'_r} = \begin{pmatrix} \mathbf{G}'_{r'+1,M'_{r'}} & \mathbf{G}'_{r'+1,M'_{r'}} & \cdots & \mathbf{G}'_{r'+1,M'_{r'}} \\ M'_{r'}\widetilde{\psi_1} & M'_{r'}\widetilde{\psi_2} & \cdots & M'_{r'}\widetilde{\psi}_{15084} \end{pmatrix}.$$

It is easy to see that each of these $\mathbf{G}_{r+1,M'_r}$ (for $14 \le r \le 23$) is a quantum cap, and it has a quantum sub-cap of odd size z for $21 \le z \le M'_r - 21$.

Let $N = 15084$ as above. For $r = 10i + r' \ge 23$ with $0 \le r' \le 9$, using the previous recursive construction step by step, one can construct a quantum cap of size M'_r as follows: an $M'_r = N^{i-1}M'_{10+r'}$ quantum cap in $PG(r, 4)$ for $0 \le r' \le 7$ and $r' \ne 5$, and an $M'_r = N^i M'_{r'}$ quantum cap in $PG(r, 4)$ for $r' = 5, 8, 9$. Denote the matrix of the quantum cap of size M'_r as G'_{r+1,M'_r}.

Using inductive method, one can prove the following lemma and corollary.

Lemma 7.1 Let $M'_{12} = 256284$ and $r > 13$, $M'_r = N^{i-1}M'_{10+r'}$ for $0 \le r' \le 7$ and $r' \ne 5$, $M'_r = N^i M'_{r'}$ for $r' = 5, 8, 9$, $\mathbf{G}'_{r+1,M'_r}$ be the quantum caps constructed by previous method. If $M'_{r-1} - 19 \le z \le M'_r - 21$, then $\mathbf{G}'_{r+1,M'_r}$ has a sub-matrix $\mathbf{G}_z$ of size $(r+1) \times z$ such that $\mathbf{G}_z\mathbf{G}_z^\dagger = \mathbf{0}$.

Corollary 7.2 *Let Let $M'_{12} = 256284$, $r > 13$ and M'_r as be given in Lemma 7.1. If z is odd and satisfies $M'_{r-1} - 19 \le z \le M'_r - 21$, then there exists a quantum z-cap in* $PG(r, 4)$. Hence there is a $[[z, z - 2r - 2, 4]]$ quantum code.

8 Conclusion

For each even n satisfying $n \ge 282$, each odd z satisfying $z \ge 275$, we constructed a quantum n-cap and a quantum z-cap in $PG(k-1, 4)$ with suitable k by recursive method and computer search. For $n \ge 282$ and $n \ne 286, 756, 5040$, $z \ge 275$, these quantum n-caps, quantum z-caps and related $[[n, n-2k, 4]]$, $[[z, z-2k, 4]]$ quantum codes are new ones. It is clear that the two quantum codes $[[282, 268, 4]]$ and $[[284, 270, 4]]$ are pure codes, using the quantum Hamming bound, one can check they are also optimal codes. We will show all the quantum codes constructed in this paper are pure and optimal codes. Combining the results of [5, 25, 27, 2, 3, 17, 28] on quantum caps, for each $n \ge 6$ and $n \ne 11$, one can give an optimal linear $[[n, k, 4]]$ quantum code.

We collect all these quantum n-caps with $n \ge 282$, quantum z-caps with $z \ge 275$ and their related quantum $[[n, n-2k, 4]]$, $[[z, z-2k, 4]]$ codes in Table 8.1 and Table 8.2, respectively.

Table 8.1 Quantum n-caps in $PG(r,4)$ and $[[n,k,4]]$ QECCs.

r	size of quantum n-cap in $PG(r,4)$	QECCs
7	$288 \le n \le 750$ or $n=756$	$[[n,n-16,4]]$
8	$758 \le n \le 2130$ or $n=752$, 754, 2136	$[[n,n-18,4]]$
9	$2138 \le n \le 5118$ or $n=2132$, 2134, 5124	$[[n,n-20,4]]$
10	$5126 \le n \le 15834$ or $n=5120$, 5122, 15840	$[[n,n-22,4]]$
11	$15842 \le n \le 36144$ or $n=15836$, 15838, 36150	$[[n,n-24,4]]$
12	$36152 \le n \le 95250$ or $n=36146$, 36148, 95256	$[[n,n-26,4]]$
$r=10i+r'$ $0 \le r' \le 2$	$M_{r-1}+2 \le n \le M_r-6$ or $n=M_{r-1}-4, M_{r-1}-2, M_r=15084^{i-1}M_{10+r'}$	$[[n,n-2r-2,4]]$
$r=10i+r'$ $3 \le r' \le 9$	$M_{r-1}+2 \le n \le M_r-6$ or $n=M_{r-1}-4, M_{r-1}-2,\ M_r=15084^{i}M_{r'}$	$[[n,n-2r-2,4]]$

Table 8.2 Quantum z-caps in $PG(r,4)$ and $[[z,k,4]]$ QECCs.

r	size of quantum z-cap in $PG(r,4)$	QECCs
7	$275 \le z \le 735$	$[[z,z-16,4]]$
8	$737 \le z \le 2115$	$[[z,z-18,4]]$
9	$2117 \le z \le 5103$	$[[z,z-20,4]]$
10	$5105 \le z \le 15795$	$[[z,z-22,4]]$
11	$15797 \le z \le 36129$	$[[z,z-24,4]]$
12	$36131 \le z \le 95235$	$[[z,z-26,4]]$
13	$95237 \le z \le 256263$	$[[z,z-28,4]]$
14	$256265 \le z \le 120 \times 5124 - 21$	$[[z,z-30,4]]$
16	$15084 \times 126 - 19 \le z \le 2034 \times 2136 - 21$	$[[z,z-34,4]]$
17	$2034 \times 2136 - 19 \le z \le 756 \times 15816 - 21$	$[[z,z-36,4]]$
$r=10i+r'$ $0 \le r' \le 7$, $r' \ne 5$	$M'_{r-1}-19 \le z \le M'_r-21$ or $M'_r=15084^{i-1}M'_{10+r'}$	$[[z,z-2r-2,4]]$
$r=10i+r'$ $r'=5,8,9$	$M'_{r-1}-19 \le z \le M'_r-21$ or $M'_r=15084^{i}M'_{r'}$	$[[z,z-2r-2,4]]$

Theorem 8.1 *Let $r \ge 6$, M_r be as given in Lemma 5.4. Then all the $[[n,n-2r-2,4]]$ quantum codes given in Table 8.1 are pure and optimal codes.*

Proof. (1) For $288 \le n \le 750$ or $n=756$, from the construction of $\mathbf{G}_{8,n}$, we can deduce the cap code of $\mathbf{G}_{8,n}$ has minimum distance $d>4$. Hence, $\mathbf{G}_{8,n}$ gives a pure $[[n,n-2r-2,4]]=[[n,n-16,4]]$ quantum code. It is easy to check $\sum_{i=0}^{2} 3^i\binom{n}{i} > 2^{16}$ for $n \ge 288$, one can derive a pure $[[n,n-16,5]]$ code does not exist by the quantum Hamming bound, thus $[[n,n-16,4]]$ is an optimal code.

(2) Similar to the discussion of (1), we can check the $[[n,n-2r-2,4]]$ codes are pure optimal quantum codes for $8 \le r \le 12$ and n given in Table 8.1.

(3) Using inductive method, we can check that: for $r \ge 13$, the cap codes of the quantum n-caps with n given in Table 8.1 have minimum distance $d>4$. The checking

process is much lengthy, we omit it here.

Next, we show that there are no pure $[[n, n-2r-2, 5]]$ codes for $r \geq 13$ and $M_{r-1}-4 \leq n \leq M_r$. Recall that $M_4 = 41$, $M_5 = 126$, $M_6 = 288$, $M_7 = 756$, $\cdots$, $M_{12} = 95256$, $M_{13} = 17 \times 15084$, $M_{14} = 41 \times 15084$. For $5 \leq r \leq 14$, it is easy to check $\frac{3}{\sqrt{2}}(M_{r-1}-5) > 2^{r+1}$, which implies $\sum_{i=0}^{2} 3^i \binom{M_{r-1}-4}{i} > 2^{2r+2}$ and an $[[n, n-2r-2, 5]]$ code does not exist for $n \geq M_{r-1}-4$.

From $M_{10+r} = 15084 M_r$ for $r \geq 4$ and $15084 \geq 2^{11}$, one can deduce

$$\frac{3}{\sqrt{2}}(M_{10+r-1}-5) \geq 2^{10+r+1} \text{ and } \sum_{i=0}^{2} 3^i \binom{M_{10+r-1}-4}{i} > 2^{2(10+r+1)}$$

for $r \geq 4$. Hence an $[[n, n-2r-2, 5]]$ code does not exist for $n \geq M_{10+r-1}-4$. Using induction on i, one can derive there is no pure $[[n, n-2r-2, 5]]$ code for $r = 10i + r' \geq 13$ and $n \geq M_{r-1}-4$.

Summarizing the previous discussion, the theorem follows.

Theorem 8.2 *Let $r \geq 6$, M'_r be as given in section 7. Then all the $[[z, z-2r-2, 4]]$ quantum codes given in Table 8.2 are pure and optimal codes.*

Remark 8.1 The 756-cap in $PG(7,4)$, 2136-cap in $PG(8,4)$, 5124-cap in $PG(9,4)$, 15840-cap in $PG(10,4)$ are complete caps according to [13]. The 36150-cap in $PG(11,4)$ is also a complete cap. We do not know whether the 95256-cap in $PG(12,4)$, the M_r-cap and M'_r-cap in $PG(r,4)$ $(r \geq 13)$ are complete caps or not. This needs further study.

According to [11], from each quantum n-cap in $PG(r,4)$, one can construct an $[[n, n-2r-2, 4/4]]_4$ asymmetric quantum code. In [24], we constructed many maximal entanglement-assisted quantum codes from sub-caps of known large caps in $PG(r,4)$ for $r \leq 4$. We will consider construction of entanglement-assisted quantum codes from sub-caps of quantum caps in this paper in the future.

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Appendix A

The 126-cap $\mathbf{G}_{6,126}$ mentioned in Section 2 is as follows:

$$\mathbf{G}_{6,126} = (\mathbf{Q}_1, \mathbf{Q}_2, \cdots, \mathbf{Q}_{15}, \mathbf{R}) = (\mathbf{M}_1 \quad \mathbf{M}_2 \quad \mathbf{M}_3),$$

where

$$\mathbf{M}_1 = (\mathbf{Q}_1, \mathbf{Q}_2, \cdots, \mathbf{Q}_6)$$

$$= \begin{pmatrix} 111111111111111111111111111111111111111111111111 \\ 231223121312121313232313002312310023123100322113 \\ 000011330022112233332211232323231212121213131313 \\ 112211223322223333111133001122330011223300112233 \\ 000000000000000000000000111111112222222233333333 \\ 000011330011222211223333000000000000000000000000 \end{pmatrix},$$

$$\mathbf{M}_2 = (\mathbf{Q}_7, \mathbf{Q}_8, \cdots, \mathbf{Q}_{12}) =$$

$$\begin{pmatrix} 111111111111111111111111111111111111111111111111 \\ 112321132232213133232131333212131123211322231213 \\ 232323231212121213131313232323231212121213131313 \\ 001122330011223300112233001122330011223300112233 \\ 111111112222222233333333111111112222222233333333 \\ 111111111111111111111111222222222222222222222222 \end{pmatrix},$$

$$\mathbf{M}_3 = (\mathbf{Q}_{13}, \mathbf{Q}_{14}, \mathbf{Q}_{15}, \mathbf{R}) = \begin{pmatrix} 111111111111111111111111000000 \\ 223221313332121311321231111111 \\ 232323231212121213131313231213 \\ 001122330011223300112233000000 \\ 111111112222222233333333000000 \\ 333333333333333333333333112233 \end{pmatrix}.$$

Appendix B

The weight polynomial of code generated by 288-cap $\mathbf{G}_{7,288}$ in $PG(6,4)$ is

$Wt_{288}(z) = 1 + 1089z^{202} + 270z^{203} + 120z^{204} + 990z^{206} + 18z^{207} + 225z^{210} + 5400z^{215} + 900z^{216} + 3267z^{218} + 360z^{219} + 2970z^{222} + 675z^{226} + 3z^{256} + 90z^{267} + 6z^{271}$.

The matrix $\mathbf{G}'_{7,288}$ given in section 3 is as follows:

$$\mathbf{G}'_{7,288} = (\mathbf{S}'_1 \; \mathbf{S}'_2 \; \mathbf{S}'_3 \; \mathbf{S}'_4 \; \mathbf{S}'_5) = \begin{pmatrix} 1 & \cdots & 1 & & 0 & \cdots & 0 \\ \widetilde{\eta}_1 & \cdots & \widetilde{\eta}_{271} & & \widetilde{\eta}_{272} & \cdots & \widetilde{\eta}_{288} \end{pmatrix},$$

where

$$\mathbf{S}'_1 = \begin{pmatrix} 111111111111111111111111111111111111111111111111111111111111 \\ 333333333333333333333333333333333333333333333333222222222222 \\ 000000000000000022222222222222222222222222222222333333333333 \\ 000000000000000000000000000000003333333333333333333333333333 \\ 011123323133122201112332313312220111233231331222033312212322 \\ 002213203102113300221320310211330022132031021133001132102301 \\ 000311310110333200031131011033320003113101103332000233230330 \end{pmatrix},$$

$$\mathbf{S}'_2 = \begin{pmatrix} 111111111111111111111111111111111111111111111111111111111111 \\ 222211111111111111111111111111111111222222222222222211111111 \\ 333311111111111111113333333333333333000000000000000011111111 \\ 333322222222222222221111111111111111333333333333333300000000 \\ 311102223113121123330222311312112333033312212322311102223113 \\ 332200332130120322110033213012032211001132102301332200332130 \\ 222100012212022011130001221202201113000233230330222100012212 \end{pmatrix},$$

$$\mathbf{S}'_3 = \begin{pmatrix} 111111111111111111111111111111111111111111111111111111111111 \\ 111111113333333333333333111111111111111111111111111111113333 \\ 111111111111111111111111000000000000000033333333333333331111 \\ 000000001111111111111111222222222222222200000000000000003333 \\ 121123330111233231331222022231131211233302223113121123330111 \\ 120322110022132031021133003321301203221100332130120322110022 \\ 022011130003113101103332000122120220111300012212022011130003 \end{pmatrix},$$

$$\mathbf{S}'_4 = \begin{pmatrix} 111111111111111111111111111111111111111111111111111111111111 \\ 333333333333222222222222222222222222222222222222222222222222 \\ 111111111111333333333333333322222222222222222222222222222222 \\ 333333333333222222222222222222222222222222221111111111111111 \\ 233231331222033312212322311103331221232231110333122123223111 \\ 132031021133001132102301332200113210230133220011321023013322 \\ 113101103332000233230330222100023323033022210002332303302221 \end{pmatrix},$$

$$\mathbf{S}_5' = \begin{pmatrix} 111111111111111111111111111111110000000000000000 \\ 000000000000000000000000000000001111111111111111 \\ 000000000000000022313011031322000000000000000000 \\ 111111111111111112230210312330000000000000000000 \\ 022231131211233322231131211233310222311312112333 \\ 003321301203221100000000000000000033213012032211 \\ 000122120220111322231131211233310001221202201113 \end{pmatrix}.$$

The weight polynomials of the codes and their dual generated by the quantum caps constructed in subsection 3.2 are as follows:

$Wt_{256}(z) = 1 + 288z^{176} + 2304z^{180} + 6780z^{192} + 6912z^{196} + 96z^{240} + 3z^{256}$,

$Wt_{256}^{\perp}(y) = 1 + 1220160y^4 + 116951040y^5 + \cdots$.

$Wt_{262}(z) = 1 + 156z^{180} + 180z^{182} + 1584z^{184} + 672z^{186} + 60z^{192} + 5184z^{196} + 1680z^{198} + 4752z^{200} + 2016z^{202} + 36z^{244} + 60z^{246} + 3z^{256}$,

$Wt_{262}^{\perp}(y) = 1 + 1316685y^4 + 132678720y^5 + \cdots$.

$Wt_{268}(z) = 1 + 3z^{180} + 54z^{184} + 252z^{186} + 1179z^{188} + 900z^{190} + 204z^{192} + 81z^{196} + 48z^{198} + 3942z^{200} + 2556z^{202} + 3753z^{204} + 2700z^{206} + 612z^{208} + 72z^{250} + 24z^{252} + 3z^{256}$,

$Wt_{268}^{\perp}(y) = 1 + 1421013y^4 + 149837328y^5 + \cdots$.

$Wt_{282}(z) = 1 + 6z^{194} + 150z^{196} + 1062z^{198} + 897z^{200} + 453z^{202} + 117z^{204} + 27z^{206} + 408z^{208} + 2754z^{210} + 2571z^{212} + 3609z^{214} + 2706z^{216} + 1200z^{218} + 279z^{220} + 45z^{222} + 3z^{256} + 3z^{260} + 57z^{262} + 33z^{264} + 3z^{266}$,

$Wt_{282}^{\perp}(y) = 1 + 1698990y^4 + 195873432y^5 + \cdots$.

$Wt_{284}(z) = 1 + 12z^{196} + 264z^{198} + 1332z^{200} + 648z^{202} + 324z^{204} + 120z^{206} + 48z^{208} + 720z^{210} + 3252z^{212} + 2472z^{214} + 4044z^{216} + 1680z^{218} + 1116z^{220} + 216z^{222} + 36z^{224} + 3z^{256} + 72z^{264} + 24z^{266}$,

$Wt_{284}^{\perp}(y) = 1 + 1742325y^4 + 203275920y^5 + \cdots$.

$Wt_{286}(z) = 1 + 1089z^{200} + 270z^{202} + 1110z^{204} + 18z^{206} + 225z^{208} + 5400z^{214} + 4167z^{216} + 360z^{218} + 2970z^{220} + 675z^{224} + 3z^{256} + 90z^{266} + 6z^{270}$,

$Wt_{286}^{\perp}(y) = 1 + 1791435y^4 + 210841560y^5 + \cdots$.

Appendix C

At the beginning of section 4, we have pointed out that [13] constructed six large caps in $PG(r, 4)$ for $7 \leq r \leq 12$. These six large caps are: 756-cap in $PG(7, 4)$, 2136-cap in $PG(8, 4)$, 5124-cap in $PG(9, 4)$, 15840-cap in $PG(10, 4)$, 36084-cap in $PG(11, 4)$ and

95256-cap in $PG(12,4)$. The weight polynomials of cap codes of these six caps and their dual codes are as follows:

$Wt_{756}(z) = 1+27z^{504}+450z^{516}+432z^{520}+981z^{528}+135z^{552}+6480z^{560}+32346z^{564}+7317z^{568}+5679z^{576}+10206z^{580}+1296z^{584}+108z^{592}+63z^{720}+9z^{744}+6z^{756}$,

$Wt^{\perp}_{756}(y) = 1+25841403y^4+\cdots$.

$Wt_{2136}(z) = 1+135z^{1490}+585z^{1492}+2160z^{1500}+45z^{1504}+135z^{1506}+\cdots+45z^{1986}+45z^{2002}+3z^{2010}+15z^{2034}$.

$Wt^{\perp}_{2136}(z) = 1+393124770y^4+307120772160y^5+\cdots$,

$Wt_{5124}(z) = 1+3z^{2988}+6z^{3030}+3z^{3240}+6z^{3270}+3z^{3276}+3z^{3486}+27z^{3498}+6z^{3504}+\cdots+24z^{4710}+3z^{4746}+3z^{4758}+45z^{4782}+15z^{4878}$.

$Wt^{\perp}_{5124}(y) = 1+3378253707y^4+\cdots$,

$Wt_{15840}(z) = 1+441z^{11016}+270z^{11052}+360z^{11064}+1512z^{11080}+1080z^{11112}+\cdots+27z^{12424}+3z^{14400}+90z^{14892}+30z^{15084}$,

$Wt^{\perp}_{15840}(y) = 1+83251482696y^4+417595251328704y^5+\cdots$.

$Wt_{36084}(z) = 1+135z^{25140}+6z^{25204}+36z^{25206}+180z^{25208}+3z^{25210}+\cdots+9z^{33072}+57z^{33450}+45z^{33812}+3z^{33954}+15z^{34356}$,

$Wt^{\perp}_{36084}(y) = 1+564963703299y^4+6475053526943088y^5+\cdots$,

$Wt_{95256}(z) = 1+18z^{63576}+378z^{65016}+144z^{65400}+864z^{65880}+3969z^{66240}+1890z^{66528}+\cdots+1728z^{73560}+72z^{74136}+144z^{74520}+126z^{90720}+9z^{92376}+3z^{95256}$,

$Wt^{\perp}_{95256}(y) = 1+8379131756670y^4+\cdots$.

Appendix D

The weight polynomials of the code generated by 36150-cap $\mathbf{G}_{12,36150}$ and its dual code in $PG(11,4)$ are

$Wt_{36150}(z) = 1+135z^{25206}+3z^{25246}+219z^{25252}+3z^{25258}+24z^{25320}+6z^{25326}+36z^{25332}+12z^{25338}+39z^{25344}+42z^{25348}+81z^{25350}+54z^{25354}+159z^{25356}+60z^{25360}+57z^{25362}+30z^{25366}+24z^{25368}+18z^{25374}+123z^{25380}+42z^{25386}+576z^{25388}+39z^{25392}+24z^{25398}+9z^{25404}+3z^{25406}+12z^{25410}+39z^{25412}+6z^{25416}+3z^{25418}+30z^{25422}+63z^{25428}+54z^{25434}+54z^{25440}+81z^{25446}+111z^{25452}+48z^{25458}+60z^{25464}+108z^{25470}+99z^{25476}+144z^{25478}+333z^{25480}+549z^{25482}+72z^{25484}+33z^{25488}+27z^{25494}+12z^{25500}+24z^{25502}+18z^{25506}+48z^{25508}+144z^{25510}+216z^{25512}+486z^{25514}+72z^{25516}+144z^{25518}+18z^{25520}+18z^{25526}+18z^{25532}+12z^{25534}+24z^{25538}+24z^{25540}+12z^{25544}+9z^{25546}+9z^{25548}+9z^{25552}+18z^{25554}+9z^{25558}+27z^{25560}+9z^{25564}+15z^{25566}+12z^{25570}+36z^{25572}+6z^{25576}+21z^{25578}+9z^{25584}+27z^{25590}+6z^{25596}+12z^{25602}+9z^{25608}+3z^{25620}+144z^{25638}+144z^{25640}+432z^{25642}+$

$72z^{25644} + 288z^{25646} + 3z^{25668} + 72z^{25670} + 72z^{25672} + 222z^{25674} + 36z^{25676} + 144z^{25678} + 6z^{25680} + 6z^{25686} + 27z^{25692} + 12z^{25698} + 12z^{25724} + 36z^{25726} + 42z^{25732} + 9z^{25734} + 6z^{25738} + 45z^{25750} + 6z^{25756} + 48z^{25758} + 36z^{25764} + 3z^{25770} + 12z^{25800} + 3z^{25806} + 18z^{25812} + 6z^{25818} + 72z^{25822} + 6z^{25824} + 6z^{25830} + 45z^{25848} + 144z^{25854} + 81z^{25860} + 81z^{25866} + 36z^{25868} + 48z^{25872} + 18z^{25878} + 60z^{25886} + 30z^{25892} + 51z^{25902} + 81z^{25908} + 51z^{25914} + 30z^{25918} + 99z^{25920} + 15z^{25924} + 93z^{25926} + 48z^{25932} + 51z^{25938} + 51z^{25944} + 90z^{25950} + 42z^{25956} + 3z^{25962} + 9z^{25968} + 72z^{25990} + 144z^{25992} + 252z^{25994} + 36z^{25996} + 6z^{26018} + 3z^{26040} + 3z^{26046} + 3z^{26050} + 3z^{26052} + 6z^{26236} + 18z^{26238} + 21z^{26244} + 3z^{26250} + 36z^{26334} + 6z^{26352} + 33z^{26358} + 33z^{26364} + 3z^{26370} + 21z^{26382} + 27z^{26388} + 18z^{26394} + 21z^{26400} + 39z^{26406} + 15z^{26412} + 9z^{26418} + 3z^{26530} + 42z^{26576} + 105z^{26582} + 93z^{26588} + 180z^{26592} + 54z^{26594} + 90z^{26598} + 15z^{26600} + 585z^{26604} + 6z^{26606} + 270z^{26610} + 180z^{26616} + 216z^{26640} + 144z^{26642} + 324z^{26644} + 396z^{26646} + 54z^{26654} + 144z^{26656} + 576z^{26658} + 807z^{26660} + 432z^{26662} + 396z^{26664} + 366z^{26666} + 216z^{26668} + 99z^{26672} + 147z^{26678} + 108z^{26684} + 144z^{26686} + 288z^{26688} + 489z^{26690} + 468z^{26692} + 252z^{26694} + 9z^{26696} + 147z^{26704} + 417z^{26710} + 477z^{26716} + 96z^{26720} + 279z^{26722} + 123z^{26726} + 153z^{26728} + 90z^{26732} + 117z^{26734} + 42z^{26736} + 42z^{26738} + 87z^{26740} + 111z^{26742} + 9z^{26744} + 42z^{26746} + 105z^{26748} + 24z^{26752} + 54z^{26754} + 39z^{26758} + 87z^{26760} + 57z^{26764} + 111z^{26766} + 432z^{26768} + 336z^{26770} + 726z^{26772} + 792z^{26774} + 711z^{26776} + 2466z^{26778} + 3276z^{26780} + 1956z^{26782} + 696z^{26784} + 1764z^{26786} + 2961z^{26788} + 1698z^{26790} + 2268z^{26792} + 5655z^{26794} + 3492z^{26796} + 324z^{26798} + 1221z^{26800} + 1656z^{26802} + 1476z^{26804} + 1110z^{26806} + 432z^{26808} + 720z^{26810} + 1515z^{26812} + 1329z^{26814} + 1908z^{26816} + 3264z^{26818} + 3420z^{26820} + 1908z^{26822} + 1104z^{26824} + 834z^{26826} + 540z^{26828} + 822z^{26830} + 387z^{26832} + 609z^{26836} + 894z^{26838} + 252z^{26840} + 399z^{26842} + 1902z^{26844} + 360z^{26846} + 1200z^{26848} + 987z^{26850} + 504z^{26852} + 498z^{26854} + 558z^{26856} + 180z^{26860} + 762z^{26862} + 63z^{26864} + 1884z^{26866} + 4257z^{26868} + 219z^{26870} + 18z^{26872} + 4131z^{26874} + 7587z^{26876} + 108z^{26878} + 372z^{26880} + 423z^{26882} + 216z^{26884} + 348z^{26886} + 174z^{26888} + 435z^{26892} + 234z^{26894} + 219z^{26896} + 1398z^{26898} + 2340z^{26900} + 540z^{26902} + 1923z^{26904} + 9858z^{26906} + 16860z^{26908} + 6060z^{26910} + 3408z^{26912} + 8499z^{26914} + 12606z^{26916} + 9693z^{26918} + 8547z^{26920} + 17154z^{26922} + 10989z^{26924} + 3456z^{26926} + 5004z^{26928} + 5808z^{26930} + 2646z^{26932} + 2817z^{26934} + 2691z^{26936} + 5724z^{26938} + 11673z^{26940} + 4692z^{26942} + 4311z^{26944} + 7143z^{26946} + 6204z^{26948} + 3609z^{26950} + 3528z^{26952} + 3798z^{26954} + 2313z^{26956} + 2823z^{26958} + 1134z^{26960} + 1404z^{26962} + 2886z^{26964} + 1950z^{26966} + 3888z^{26968} + 6723z^{26970} + 9801z^{26972} + 8460z^{26974} + 12468z^{26976} + 29799z^{26978} + 34686z^{26980} + 26046z^{26982} + 12531z^{26984} + 40473z^{26986} + 52311z^{26988} + 33336z^{26990} + 25467z^{26992} + 49965z^{26994} + 42960z^{26996} + 24624z^{26998} + 35613z^{27000} + 50139z^{27002} + 35523z^{27004} + 10953z^{27006} + 11805z^{27008} + 21216z^{27010} + 29892z^{27012} + 23826z^{27014} + 35838z^{27016} + 84306z^{27018} + 90804z^{27020} + 58332z^{27022} + 52728z^{27024} + 90960z^{27026} + 90555z^{27028} + 77877z^{27030} + 52029z^{27032} + 69576z^{27034} + 59769z^{27036} + 18564z^{27038} +$

$16521z^{27040}+37767z^{27042}+42246z^{27044}+35976z^{27046}+45645z^{27048}+62826z^{27050}+67551z^{27052}+61803z^{27054}+46197z^{27056}+78516z^{27058}+85173z^{27060}+53997z^{27062}+37485z^{27064}+62445z^{27066}+66198z^{27068}+36996z^{27070}+42264z^{27072}+75888z^{27074}+78588z^{27076}+63609z^{27078}+57777z^{27080}+99816z^{27082}+116370z^{27084}+69906z^{27086}+69270z^{27088}+145860z^{27090}+182178z^{27092}+137934z^{27094}+127602z^{27096}+299940z^{27098}+394902z^{27100}+351789z^{27102}+361329z^{27104}+531045z^{27106}+627573z^{27108}+670626z^{27110}+931413z^{27112}+1167318z^{27114}+666327z^{27116}+252957z^{27118}+396276z^{27120}+542769z^{27122}+432627z^{27124}+289716z^{27126}+195300z^{27128}+255552z^{27130}+302040z^{27132}+156429z^{27134}+164088z^{27136}+284487z^{27138}+230307z^{27140}+108099z^{27142}+133434z^{27144}+257226z^{27146}+244050z^{27148}+194319z^{27150}+104685z^{27152}+178569z^{27154}+195930z^{27156}+118290z^{27158}+102033z^{27160}+154602z^{27162}+132105z^{27164}+51705z^{27166}+45687z^{27168}+59070z^{27170}+53436z^{27172}+46008z^{27174}+44466z^{27176}+106194z^{27178}+128205z^{27180}+75735z^{27182}+52125z^{27184}+92301z^{27186}+73428z^{27188}+39276z^{27190}+37107z^{27192}+47838z^{27194}+29838z^{27196}+7206z^{27198}+11448z^{27200}+16272z^{27202}+19356z^{27204}+14925z^{27206}+7203z^{27208}+13557z^{27210}+16551z^{27212}+11829z^{27214}+19467z^{27216}+37839z^{27218}+35889z^{27220}+15300z^{27222}+8028z^{27224}+8325z^{27226}+3975z^{27228}+1485z^{27230}+930z^{27232}+336z^{27234}+855z^{27236}+1410z^{27238}+336z^{27240}+180z^{27242}+1821z^{27244}+1128z^{27246}+3285z^{27248}+6294z^{27250}+6480z^{27252}+2925z^{27254}+1071z^{27256}+720z^{27258}+666z^{27260}+2646z^{27262}+588z^{27264}+2286z^{27266}+3357z^{27268}+981z^{27270}+1047z^{27272}+2310z^{27274}+2103z^{27276}+1485z^{27278}+2217z^{27280}+2220z^{27282}+459z^{27284}+870z^{27286}+228z^{27288}+633z^{27290}+1362z^{27292}+1059z^{27294}+732z^{27296}+2268z^{27298}+3399z^{27300}+1518z^{27302}+1770z^{27304}+4122z^{27306}+2952z^{27308}+813z^{27310}+1368z^{27312}+2160z^{27314}+1494z^{27316}+1167z^{27318}+432z^{27320}+750z^{27322}+1839z^{27324}+747z^{27326}+468z^{27328}+957z^{27330}+693z^{27332}+756z^{27334}+582z^{27336}+837z^{27338}+324z^{27340}+474z^{27342}+210z^{27344}+420z^{27348}+453z^{27350}+252z^{27352}+471z^{27354}+1257z^{27356}+216z^{27358}+429z^{27360}+207z^{27362}+42z^{27364}+372z^{27366}+771z^{27368}+54z^{27370}+378z^{27372}+675z^{27374}+60z^{27376}+147z^{27378}+588z^{27380}+30z^{27382}+183z^{27384}+1281z^{27386}+1980z^{27388}+2748z^{27390}+210z^{27392}+2070z^{27394}+5739z^{27396}+96z^{27398}+90z^{27400}+312z^{27402}+186z^{27404}+168z^{27406}+468z^{27408}+144z^{27410}+411z^{27412}+597z^{27414}+396z^{27416}+2274z^{27418}+4596z^{27420}+2688z^{27422}+2292z^{27424}+4842z^{27426}+8445z^{27428}+5727z^{27430}+4152z^{27432}+7605z^{27434}+4143z^{27436}+48z^{27438}+840z^{27440}+90z^{27444}+1452z^{27446}+1548z^{27448}+1944z^{27450}+4857z^{27452}+1962z^{27454}+1413z^{27456}+2910z^{27458}+2112z^{27460}+846z^{27462}+1134z^{27464}+1680z^{27466}+327z^{27468}+1113z^{27470}+315z^{27472}+3z^{27474}+813z^{27476}+828z^{27478}+399z^{27480}+837z^{27482}+1257z^{27484}+651z^{27486}+531z^{27488}+807z^{27490}+237z^{27492}+429z^{27494}+606z^{27496}+633z^{27498}+162z^{27500}+558z^{27502}+819z^{27504}+582z^{27506}+630z^{27508}+2550z^{27510}+2133z^{27512}+2718z^{27514}+$

$2913z^{27516}+1233z^{27518}+1377z^{27520}+3420z^{27522}+1854z^{27524}+1035z^{27526}+987z^{27528}+1431z^{27530}+180z^{27532}+12z^{27534}+507z^{27536}+288z^{27538}+648z^{27540}+1365z^{27542}+1548z^{27544}+2736z^{27546}+4416z^{27548}+3786z^{27550}+3852z^{27552}+5406z^{27554}+7068z^{27556}+5040z^{27558}+4860z^{27560}+4962z^{27562}+792z^{27564}+60z^{27566}+540z^{27568}+144z^{27570}+414z^{27572}+825z^{27574}+288z^{27576}+990z^{27578}+1575z^{27580}+1347z^{27582}+1950z^{27584}+3444z^{27586}+3648z^{27588}+1584z^{27590}+1266z^{27592}+1887z^{27594}+504z^{27596}+1197z^{27598}+186z^{27600}+492z^{27604}+222z^{27606}+114z^{27610}+192z^{27612}+321z^{27614}+903z^{27616}+1143z^{27618}+1137z^{27620}+642z^{27622}+252z^{27624}+243z^{27626}+216z^{27628}+444z^{27630}+18z^{27632}+84z^{27634}+237z^{27636}+18z^{27638}+18z^{27640}+57z^{27642}+18z^{27644}+123z^{27648}+24z^{27650}+123z^{27654}+12z^{27656}+90z^{27660}+42z^{27666}+9z^{27672}+30z^{27710}+15z^{27716}+288z^{27718}+288z^{27720}+864z^{27722}+144z^{27724}+576z^{27726}+36z^{27742}+9z^{27744}+42z^{27748}+234z^{27750}+144z^{27752}+438z^{27754}+183z^{27756}+288z^{27758}+87z^{27762}+69z^{27768}+198z^{27774}+165z^{27780}+69z^{27786}+15z^{27792}+27z^{27798}+138z^{27804}+36z^{27806}+54z^{27810}+42z^{27812}+66z^{27816}+6z^{27818}+177z^{27822}+240z^{27828}+132z^{27834}+120z^{27838}+195z^{27840}+60z^{27844}+264z^{27846}+261z^{27852}+99z^{27858}+156z^{27864}+351z^{27870}+231z^{27876}+84z^{27882}+36z^{27884}+33z^{27888}+15z^{27894}+108z^{27902}+42z^{27908}+54z^{27914}+72z^{27916}+60z^{27920}+30z^{27926}+120z^{27966}+60z^{27972}+60z^{27998}+30z^{28004}+72z^{28006}+144z^{28008}+252z^{28010}+36z^{28012}+144z^{28038}+288z^{28040}+504z^{28042}+72z^{28044}+9z^{28098}+21z^{28224}+39z^{28230}+39z^{28236}+24z^{28242}+12z^{28248}+18z^{28254}+21z^{28260}+3z^{28266}+3z^{28278}+21z^{28284}+36z^{28286}+30z^{28290}+42z^{28292}+27z^{28296}+6z^{28298}+45z^{28302}+57z^{28308}+54z^{28314}+36z^{28320}+57z^{28326}+48z^{28332}+12z^{28338}+15z^{28344}+18z^{28350}+27z^{28356}+33z^{28368}+27z^{28374}+12z^{28380}+18z^{28386}+3z^{31848}+3z^{32520}+12z^{33120}+6z^{33126}+39z^{33132}+18z^{33138}+12z^{33144}+3z^{33510}+45z^{33878}+3z^{34014}+15z^{34422}$,

$Wt^{\perp}_{36150}(y)=1+553592502981y^4+6606240021787176y^5+\cdots.$

Appendix E

The $[15816, 11, 10992]$ cap code given in subsection 6.4 has weight polynomial

$Wt_{15816}(z)=1+72z^{10992}+63z^{10998}+36z^{11004}+36z^{11016}+171z^{11028}+27z^{11034}+180z^{11040}+72z^{11046}+180z^{11048}+72z^{11052}+108z^{11056}+432z^{11060}+297z^{11062}+432z^{11064}+108z^{11068}+252z^{11088}+216z^{11094}+540z^{11096}+72z^{11100}+1296z^{11124}+324z^{11126}+54z^{11136}+84z^{11184}+9z^{11190}+12z^{11208}+45z^{11220}+9z^{11226}+108z^{11248}+27z^{11254}+144z^{11256}+12z^{11376}+387z^{11604}+135z^{11610}+540z^{11616}+432z^{11620}+36z^{11652}+216z^{11708}+648z^{11714}+2160z^{11716}+648z^{11720}+2592z^{11744}+12636z^{11746}+27216z^{11748}+1512z^{11756}+2592z^{11760}+6156z^{11762}+15552z^{11764}+1512z^{11768}+2808z^{11772}+1944z^{11778}+2160z^{11780}+1944z^{11784}+69z^{11796}+2160z^{11800}+45z^{11802}+90720z^{11804}+116640z^{11806}+125208z^{11808}+37908z^{11810}+$

$48096z^{11812}+11880z^{11814}+11124z^{11816}+97632z^{11820}+77760z^{11822}+57456z^{11824}+28836z^{11826}+38016z^{11828}+15912z^{11832}+12960z^{11836}+12960z^{11838}+4320z^{11840}+6060z^{11844}+6480z^{11848}+5040z^{11850}+163260z^{11856}+347328z^{11858}+414729z^{11860}+86328z^{11862}+46188z^{11864}+405z^{11866}+330696z^{11868}+349920z^{11870}+358668z^{11872}+146880z^{11874}+203472z^{11876}+23112z^{11878}+37116z^{11880}+209736z^{11884}+233280z^{11886}+129600z^{11888}+576z^{11892}+38880z^{11900}+38880z^{11902}+13005z^{11904}+108z^{11908}+2160z^{11912}+3780z^{11920}+3240z^{11926}+5940z^{11928}+1080z^{11932}+72z^{11964}+135z^{11968}+216z^{11970}+720z^{11972}+216z^{11976}+3672z^{12000}+4212z^{12002}+9072z^{12004}+648z^{12006}+2268z^{12008}+720z^{12012}+1080z^{12016}+2052z^{12018}+5184z^{12020}+189z^{12022}+648z^{12024}+1044z^{12028}+648z^{12034}+1776z^{12036}+756z^{12040}+336z^{12042}+1092z^{12048}+207z^{12052}+648z^{12054}+1620z^{12056}+135z^{12058}+216z^{12060}+9072z^{12064}+12636z^{12066}+14688z^{12068}+3672z^{12076}+5076z^{12080}+6156z^{12082}+8208z^{12084}+891z^{12086}+2808z^{12088}+324z^{12092}+18z^{12096}+36z^{12100}+3888z^{12148}+972z^{12150}+252z^{12208}+27z^{12214}+36z^{12232}+324z^{12272}+81z^{12278}+432z^{12280}+36z^{12400}+3z^{14400}+57z^{14868}+9z^{14874}+36z^{14880}+15z^{15060}+3z^{15066}$.

The weight polynomial of its dual is

$Wt^{\perp}_{15816}(y) = 1 + 82032878250y^4 + \cdots$.

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