Erdős-Gallai-type results for colorful monochromatic connectivity of a graph*

Qingqiong Cai, Xueliang Li, Di Wu
Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, China
cqqnjnu620@163.com; lxl@nankai.edu.cn; wudiol@mail.nankai.edu.cn

Abstract

A path in an edge-colored graph is called a *monochromatic path* if all the edges on the path are with the same color. An edge-coloring of G is a *monochromatic connection coloring* (MC-coloring, for short) if there is a monochromatic path joining any two vertices in G. The *monochromatic connection number*, denoted by $mc(G)$, is defined to be the maximum number of colors used in an MC-coloring of a graph G. These concepts were introduced by Caro and Yuster, and they got some nice results. In this paper, we study two kinds of Erdős-Gallai-type problems for $mc(G)$, and completely solve them.

Keywords: monochromatic path, MC-coloring, monochromatic connection number, Erdős-Gallai-type problem.

AMS subject classification 2010: 05C15, 05C35, 05C38, 05C40.

1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [1]. For a graph G, we use $V(G)$, $E(G)$, $n(G)$, $m(G)$, $\Delta(G)$, $\delta(G)$, $diam(G)$ and $\overline{G}$ to denote the vertex set, the edge set, the number of vertices, the number of edges, the maximum degree, the minimum degree, the diameter and the complement of G, respectively. For $D \subseteq V(G)$, let $|D|$ be the number of vertices in D, and $G[D]$ be the subgraph of G induced by D.

Let G be a nontrivial connected graph with an edge-coloring $f : E(G) \rightarrow \{1, 2, \ldots, \ell\}$, $\ell \in \mathbb{N}$, where adjacent edges may be colored the same. A path of G is a *monochromatic*

*Supported by NSFC No.11371205, "973" program No.2013CB834204, and PCSIRT.

path if all the edges on the path are with the same colore. An edge-coloring of G is a *monochromatic connection coloring* (MC-coloring, for short) if there is a monochromatic path joining any two vertices in G. How colorful can an MC-coloring be? This question is the natural opposite of the recently well-studied problem on rainbow connection number [2, 4, 6, 9, 10] for which we seek to find an edge-coloring with minimum number of colors so that there is a rainbow path joining any two vertices.

The *monochromatic connection number* of G, denoted by $mc(G)$, is defined to be the maximum number of colors used in an MC-coloring of a graph G. An MC-coloring of G is called *extremal* if it uses $mc(G)$ colors.

Observation 1 ([3]). *In an extremal MC-coloring f of G, the subgraph of G induced by edges with one same color forms a tree.*

For a color i, the *color tree* T_i is the tree consisting of all the edges of G with color i. T_i is *nontrivial* if T_i has at least two edges; otherwise, T_i is *trivial*. A nontrivial color tree with t edges is said to *waste* $t-1$ colors. An extremal MC-coloring is called *simple* if any two nontrivial color trees T_i and T_j intersect in at most one vertex.

Observation 2 ([3]). *Every connected graph G has a simple extremal MC-coloring.*

These concepts were introduced by Caro and Yuster in [3]. A **general lower bound** for $mc(G)$ is $m(G)-n(G)+2$. Simply color the edges of a spanning tree with one color, and each of the remaining edges with a distinct fresh (namely, unused) color. Caro and Yuster gave some sufficient conditions for graphs attaining this lower bound.

Theorem 1 ([3]). *Let G be a connected graph with $n>3$ vertices and m edges. If G satisfies any of the following properties, then $mc(G)=m-n+2$.*
(a) *$\overline{G}$ is 4-connected.*
(b) *G is triangle-free.*
(c) *$\Delta(G)<n-\frac{2m-3(n-1)}{n-3}$. In particular, this holds if $\Delta(G)\leq(n+1)/2$ or $\Delta(G)\leq n-2m/n$.*
(d) *$Diam(G)\geq 3$.*
(e) *G has a cut vertex.*

Moreover, the authors proved some nontrivial upper bounds for $mc(G)$ in terms of the chromatic number, the connectivity and the minimum degree. Recall that a graph is called *s-perfectly-connected* if it can be partitioned into $s+1$ parts $\{v\},V_1,\dots,V_s$, such that each V_j induces a connected subgraph, any pair V_j,V_r induces a corresponding complete bipartite graph, and v has precisely one neighbor in each V_j. Notice that such a graph has minimum degree s, and v has degree s.

Theorem 2 ([3]). (1) *Any connected graph G satisfies $mc(G)\leq m-n+\chi(G)$.*
(2) *If G is not k-connected, then $mc(G)\leq m-n+k$. This is sharp for any k.*

(3) *If* $\delta(G) = s$, *then* $mc(G) \leq m - n + s$, *unless* G *is* s*-perfectly-connected, in which case* $mc(G) = m - n + s + 1$.

Among many interesting problems in extremal graph theory is the Erdős-Gallai-type problem to determine the maximum or minimum value of a graph parameter with some given properties. In [5, 8], the authors considered the following Erdős-Gallai-type question for rainbow connection number $rc(G)$: given two integers k, n with $1 \leq k \leq n-1$, compute and minimize the function $h(n,k)$ with the property: if a connected graph G on n vertices has at least $h(n,k)$ edges, then $rc(G) \leq k$. Moreover, the authors in [7, 11, 12] investigated another Erdős-Gallai-type question for rainbow connection number $rc(G)$: given two integers k, n with $1 \leq k \leq n-1$, compute the minimum number $t(n,k)$ of edges in a connected graph G on n vertices such that $rc(G) \leq k$. Motivated by these, we study two kinds of Erdős-Gallai-type problems for $mc(G)$ in this paper.

Problem A. Given two positive integers n and k with $1 \leq k \leq \binom{n}{2}$, compute the minimum integer $f(n,k)$ such that if a connected graph G on n vertices has at least $f(n,k)$ edges, then $mc(G) \geq k$.

Problem B. Given two positive integers n and k with $1 \leq k \leq \binom{n}{2}$, compute the maximum integer $g(n,k)$ such that if a connected graph G on n vertices has at most $g(n,k)$ edges, then $mc(G) \leq k$.

It is worth mentioning that the two parameters $f(n,k)$ and $g(n,k)$ are equivalent to another two parameters. Let $t(n,k) = \min\{|E(G)| : |V(G)| = n, mc(G) \geq k\}$ and $s(n,k) = \max\{|E(G)| : |V(G)| = n, mc(G) \leq k\}$. It is easy to see that $t(n,k) = g(n,k-1)+1$ and $s(n,k) = f(n,k+1)-1$. This paper is devoted to determining the exact values of $f(n,k)$ and $g(n,k)$ for all integers n, k with $1 \leq k \leq \binom{n}{2}$.

Theorem 3. *Given two positive integers* n *and* k *with* $1 \leq k \leq \binom{n}{2}$,

$$f(n,k) = \begin{cases} n+k-2 & \text{if } 1 \leq k \leq \binom{n}{2} - 2n + 4 \quad (1) \\ \binom{n}{2} + \left\lceil \frac{k-\binom{n}{2}}{2} \right\rceil & \text{if } \binom{n}{2} - 2n + 5 \leq k \leq \binom{n}{2} \quad (2) \end{cases}$$

Theorem 4. *Given two positive integers* n *and* k *with* $1 \leq k \leq \binom{n}{2}$,

$$g(n,k) = \begin{cases} \binom{n}{2} & \text{if } k = \binom{n}{2} \quad (3) \\ k+t-1 & \text{if } \binom{n-t}{2} + t(n-t-1) + 1 \leq k \leq \binom{n-t}{2} + t(n-t) - 1 \quad (4) \\ k+t-2 & \text{if } k = \binom{n-t}{2} + t(n-t) \quad (5) \end{cases}$$

for $2 \leq t \leq n-1$.

2 Main results

2.1 The result for $f(n,k)$

We first give some useful lemmas.

Lemma 1. *Let H be a connected graph on n vertices, and G be a connected spanning subgraph of H. If $mc(H) = m(H) - n + 2$, then $mc(G) = m(G) - n + 2$.*

Proof. It suffices to prove that $mc(G) \le m(G) - n + 2$. At first, color the edges of G with $mc(G)$ colors such that there is a monochromatic path joining any two vertices. Then, give each edge in $E(H) - E(G)$ a different fresh color. Hereto we get an MC-coloring of H using $mc(G) + m(H) - m(G)$ colors, which implies that $mc(G) + m(H) - m(G) \le mc(H)$. Therefore, $mc(G) \le mc(H) - m(H) + m(G) = (m(H) - n + 2) - m(H) + m(G) = m(G) - n + 2$. □

Lemma 2. *Let n and p be two integers with $0 \le p \le \binom{n-1}{2}$. Then every connected graph G with n vertices and $m = \binom{n}{2} - p$ edges satisfies $mc(G) \ge \binom{n}{2} - 2p$.*

Proof. Proving that $mc(G) \ge \binom{n}{2} - 2p$ amounts to finding an MC-coloring of G which wastes at most p colors. We distinguish the following two cases.

Case 1. $n - 2 \le p \le \binom{n-1}{2}$.

By the general lower bound, we have $mc(G) \ge m - n + 2 \ge m - p = \binom{n}{2} - 2p$.

Case 2. $0 \le p \le n - 3$.

Let $\widetilde{G}$ be the graph obtained from $\overline{G}$ by deleting all the isolated vertices. If $n(\widetilde{G}) \le p+1 (\le n-2)$, then we can find at least two vertices v_1, v_2 of degree $n-1$ in G. Take a star S with $E(S) = \{v_1v : v \in V(\widetilde{G})\}$. We give all the edges in S one color, and every other edge in G a different fresh color. Obviously, it is an MC-coloring of G, which wastes at most p colors. If $n(\widetilde{G}) \ge p+2$, say $n(\widetilde{G}) = p+t$ $(t \ge 2)$, then $\widetilde{G}$ has at least t components, since $m(\widetilde{G}) = p$. First assume that $\widetilde{G}$ has exactly two components C_1 and C_2. Then we get that $t = 2$, $n(C_j) \ge 2$, and all the missing edges of G lie in C_j for $j \in \{1, 2\}$. Take a double star S' in G as follows: one vertex from C_1 is adjacent to all the vertices in C_2, and one vertex from C_2 is adjacent to all the vertices in C_1. Give all the edges in S' one color, and every other edge in G a different fresh color. Clearly, this is an MC-coloring of G, which wastes p colors, since S' has exactly $p+1$ edges. Now assume that $\widetilde{G}$ has $\ell \ge 3$ components $C_1, C_2, \ldots, C_\ell$. Then we get that $\ell \ge t$, $n(C_j) \ge 2$, and all the missing edges of G lie in C_j. For each $j \in \{1, 2, \ldots, \ell\}$, select a vertex v_j from C_j, and give the star in G induced by the edges $E_j = \{v_ju : u \in V(C_{j+1})\}$ one fresh color (cyclically, $\ell + 1 = 1$). Each other edge in G receives a different fresh color. Obviously, it is an MC-coloring of G, and the number of wasted colors is $\sum_{j=1}^{\ell}(n(C_j) - 1) = p + t - \ell \le p$. □

As an immediate consequence, we obtain the following corollary. Note that the condition $p < \binom{n}{2}/2$ is presented here to ensure that $\binom{n}{2} - 2p > 0$.

Corollary 1. *Let n and p be two integers with $0 \leq p < \binom{n}{2}/2$. Then $f(n, \binom{n}{2} - 2p) \leq \binom{n}{2} - p$.*

Lemma 3 ([3])**.** *If G is a complete t-partite graph with n vertices and m edges, then $mc(G) = m - n + t$.*

Given two positive integers n and t with $3 \leq t \leq n$, let G_n^t be the graph defined as follows: partition the vertex set of the complete graph K_n into t vertex classes $V_1, V_2, \ldots, V_t$, where $||V_j| - |V_r|| \leq 1$ for $1 \leq j \neq r \leq t$; for each $j \in \{1, \ldots t\}$, select a vertex v_j^* from V_j, and delete all the edges joining v_j^* to other vertices in V_j. The remaining edges in V_j ($1 \leq j \leq t$) are called *internal edges*. Clearly, G_n^t contains a spanning subgraph isomorphic to a complete t-partite graph. It follows from Lemma 3 that $mc(G_n^t) \geq m(G_n^t) - n + t = \left(\binom{n}{2} - n + t\right) - n + t = \binom{n}{2} - 2n + 2t$. Next we will show that $mc(G_n^t) = \binom{n}{2} - 2n + 2t$. The proof is similar to that of Lemma 3. We begin with an easy observation.

Observation 3. *Let f be an extremal MC-coloring of a connected graph G. Then every nontrivial color tree in f contains at least one pair of nonadjacent vertices.*

Proof. Suppose by contradiction that T_i is a nontrivial color tree, in which all the pairs of vertices are adjacent in G. Then we can adjust the coloring of T_i. Color one edge of T_i with color i, and each other edge of T_i with a different fresh color. Obviously, the new coloring is still an MC-coloring, but uses more colors than f, a contradiction. □

Lemma 4. *Let n and t be two integers with $3 \leq t \leq n$. Then $mc(G_n^t) = \binom{n}{2} - 2n + 2t$.*

Proof. From the arguments above, it suffices to prove that $mc(G_n^t) \leq \binom{n}{2} - 2n + 2t$. To see that, we need the following three claims.

Claim 1. In any simple extremal MC-coloring f of G_n^t, each nontrivial color tree intersects exactly two vertex classes.

Proof of Claim 1. Suppose that a nontrivial color tree T_i intersects $s \geq 3$ vertex classes, say $V_1, V_2, \ldots, V_s$. Let $P_j = V(T_i) \cap V_j$ and $|P_j| = p_j$ for $1 \leq j \leq s$. Denote by x the number of internal edges in $G_n^t[\bigcup_{j=1}^s P_j]$. Then $G_n^t[\bigcup_{j=1}^s P_j]$ has $\sum_{1 \leq j < r \leq s} p_j p_r + x$ edges in total. Observe that T_i has $\sum_{j=1}^s p_j - 1$ edges, and since the coloring f is simple, each other edge in $G_n^t[\bigcup_{j=1}^s P_j]$ forms a trivial color tree. Thus we get that $G_n^t[\bigcup_{j=1}^s P_j]$ is colored using $\sum_{1 \leq j < r \leq s} p_j p_r - \sum_{j=1}^s p_j + x + 2$ colors. Now we adjust the coloring of $G_n^t[\bigcup_{j=1}^s P_j]$. For each $j \in \{1, 2, \ldots, s\}$, select one vertex $u_j \in P_j$, and color the star induced by the edges $E_j = \{u_j u : u \in P_{j+1}\}$ with one fresh color (cyclically, $s + 1 = 1$). Each other edge in $G_n^t[\bigcup_{j=1}^s P_j]$ receives a different fresh color. Obviously, the new coloring is still an MC-coloring, but now it uses $\sum_{1\le j<r\le s} p_jp_r - \sum_{j=1}^{s} p_j + x + s$ colors, contradicting the fact that f is extremal. Now suppose that a nontrivial color tree T_i intersects only one vertex class, say V_1. Since v_1^* is an isolated vertex in $G_n^t[V_1]$, we get that $v_1^* \notin V(T_i)$. Then T_i contains no pairs of nonadjacent vertices, a contradiction. Thus each nontrivial color tree intersects exactly two vertex classes. □

Claim 2. There exists a simple extremal MC-coloring of G_n^t such that each nontrivial color tree is a star or a double star, which does not contain any internal edges.

Proof of Claim 2. Suppose that f is a simple extremal MC-coloring of G_n^t, and T_i is a nontrivial color tree in f. Let P_j, p_j and x be the same as in Claim 1. By Claim 1, we may assume that T_i intersects V_1 and V_2 with $1 \le p_1 \le p_2$. Since f is simple, any edge in $G_n^t[P_1 \bigcup P_2]$ but not in T_i must be a trivial color tree. Thus $G_n^t[P_1 \bigcup P_2]$ is colored using $p_1p_2 - p_1 - p_2 + x + 2$ colors. We distinguish the following two cases (the case $p_1 = p_2 = 1$ is excluded, since then T_i is a trivial color tree, a contradiction).

Case 1. $p_1 = 1$ and $p_2 \ge 2$

If T_i is the star joining the only vertex in P_1 to all the vertices in P_2, then we are done. Otherwise, we adjust the coloring as follows: color the star with color i , and each other edge in $G_n^t[P_1 \bigcup P_2]$ with a different fresh color. Clearly, the new coloring is still a simple extremal MC-coloring. Moreover, now the nontrivial color tree in $G_n^t[P_1 \bigcup P_2]$ is a star containing no internal edges.

Case 2. $2 \le p_1 \le p_2$.

If T_i is a double star joining a certain vertex $u_i \in P_1$ to all the vertices in P_2, and joining a certain vertex $v_i \in P_2$ to all the vertices in P_1, then we are done. Otherwise, we adjust the coloring as follows: select one double star as stated above, and color it with color i, and each other edge in $G_n^t[P_1 \bigcup P_2]$ with a different fresh color. Clearly, the new coloring is still a simple extremal MC-coloring. Moreover, now the nontrivial color tree in $G_n^t[P_1 \bigcup P_2]$ is a double star containing no internal edges. □

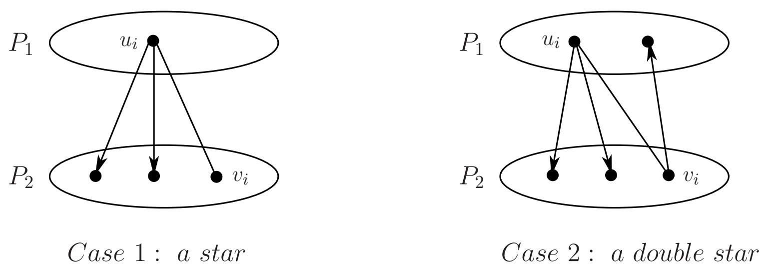

Figure 1: The illustration of Claim 2.

Now we may assume that every nontrivial color tree T_i in f is a star or a double star containing no internal edges. In fact, the stars can be viewed as degenerated double stars, by letting an arbitrary leaf perform the role of the other center of a double star. So we assume that all nontrivial color trees in f are double stars (some are possibly degenerated).

For a nontrivial color tree T_i, let u_i and v_i denote the two centers. Orient all the edges of T_i incident with u_i other than u_iv_i (if there are any) as going from u_i toward the leaves. Similarly, orient all the edges of T_i incident with v_i other than u_iv_i (if there are any) as going from v_i toward the leaves. Keep u_iv_i as unoriented. Since T_i contains no internal edges, all the oriented edges incident with u_i (if there are any) are oriented from u_i to the same vertex class (the vertex class of v_i), and all the oriented edges incident with v_i (if there are any) are oriented from v_i to the same vertex class (the vertex class of u_i). It is easily seen that the number of wasted colors of T_i is equal to the number of oriented edges in T_i.

Claim 3. For each $j \in \{1, \dots, t\}$, the number of oriented edges entering V_j is at least $|V_j| - 1$.

Proof of Claim 3. Assume that there are double stars $T_1, T_2, \dots, T_\ell$ (some are possibly degenerated) to monochromatically connect $|V_j| - 1$ pairs of nonadjacent vertices in V_j. Let e_i $(1 \leq i \leq \ell)$ denote the number of oriented edges entering V_j in T_i. Since T_i is used to monochromatically connect pairs of nonadjacent vertices in V_j, and all the pairs of nonadjacent vertices in V_j contain v_j^*, we get that v_j^* appears in each T_i $(1 \leq i \leq \ell)$. So T_i $(2 \leq i \leq \ell)$ covers at most e_i vertices in V_j but not in $\bigcup_{q=1}^{i-1} T_q$. Thus we have $(e_1 + 1) + \sum_{i=2}^{\ell} e_i \geq |V_j|$, that is, $\sum_{i=1}^{\ell} e_i \geq |V_j| - 1$. □

Note that the total number of wasted colors in f is equal to the number of oriented edges in G_n^t. It follows from Claim 3 that this number is at least $\sum_{j=1}^{t}(|V_j| - 1) = n - t$. Thus $mc(G_n^t) \leq m(G_n^t) - (n-t) = \binom{n}{2} - 2n + 2t$. We complete the proof of Lemma 4. □

We are now ready to prove Theorem 3.

Proof of Theorem **3.** Clearly, $f(n, 1) = n - 1$, so the assertion holds for $k = 1$. For $2 \leq k \leq \binom{n}{2} - 2n + 4$, it follows from the general lower bound that if a connected graph G on n vertices satisfies $m(G) \geq n + k - 2$, then $mc(G) \geq k$, implying $f(n, k) \leq n + k - 2$. To prove $f(n, k) \geq n + k - 2$, it suffices to find a connected graph G_k on n vertices such that $m(G_k) = n + k - 3$ and $mc(G_k) \leq k - 1$. Now we construct a graph H as follows: first take a copy of K_{n-2}, then add two vertices u, v, and join u to some vertices in K_{n-2}, and join v to all the other vertices in K_{n-2}. Obviously, $m(H) = \binom{n}{2} - n + 1$ and $diam(H) = 3$. By Theorem 1(d), we have $mc(H) = m(H) - n + 2 = \binom{n}{2} - 2n + 3$. So H is just the graph G_k we want for $k = \binom{n}{2} - 2n + 4$. For $2 \leq k \leq \binom{n}{2} - 2n + 3$, we take G_k as a connected spanning subgraph of H with $m(G_k) = n + k - 3$ edges. It follows from Lemma 1 that $mc(G_k) = m(G_k) - n + 2 = k - 1$. This completes the proof of (1).

Proving (2) amounts to showing that if $k = \binom{n}{2} - 2n + 2t + 1$ or $k = \binom{n}{2} - 2n + 2t + 2$ $(2 \leq t \leq n - 1)$, then $f(n, k) = \binom{n}{2} - n + t + 1$. Let $k_1 = \binom{n}{2} - 2n + 2t + 1$, and $k_2 = \binom{n}{2} - 2n + 2t + 2$. It follows from Corollary 1 that $f(n, k_2) \leq \binom{n}{2} - n + t + 1$. Since $f(n, k_1) \leq f(n, k_2)$, if we prove $f(n, k_1) \geq \binom{n}{2} - n + t + 1$, then $f(n, k_1) = f(n, k_2) = \binom{n}{2} - n + t + 1$, and we are done. So it remains to prove $f(n, k_1) \geq \binom{n}{2} - n + t + 1$, that is to find a connected graph G_t on n vertices such that $m(G_t) = \binom{n}{2} - n + t$ and $mc(G_t) \leq k_1 - 1 = \binom{n}{2} - 2n + 2t$. If $3 \leq t \leq n-1$, then by Lemma 4 we can take $G_t = G_n^t$. If $t = 2$ (thus $n \geq 3$), then we can take $G_2 = P_3, C_4$ for $n = 3, 4$, respectively; for $n \geq 5$, we take G_2 as the graph obtained from a copy of K_{n-2} by adding two adjacent vertices u, v and joining u to exactly one vertex in K_{n-2} and joining v to all the other vertices in K_{n-2}. It is easy to see that $m(G_2) = \binom{n}{2} - n + 2$, $\delta(G_2) = 2$, and u is the only vertex of degree 2. Clearly, G_2 is not 2-perfectly-connected. It follows from Theorem 2(3) that $mc(G) \leq \binom{n}{2} - 2n + 4$. □

2.2 The result for $g(n, k)$

We start with a useful lemma. Recall that $\binom{1}{2} = 0$.

Lemma 5. *Let G be a connected graph with n vertices and m edges. If $\binom{n-t}{2} + t(n-t) \leq m \leq \binom{n-t}{2} + t(n-t) + (t-2)$ for some $t \in \{2, \ldots, n-1\}$, then $mc(G) \leq m - t + 1$. Moreover, the bound is sharp.*

Proof. Let f be a simple extremal MC-coloring of G. Since $2 \leq t \leq n-1$, we have $m \leq \binom{n}{2} - 1$, that is, G is not a complete graph. So there is at least one nontrivial color tree. Suppose that $T_1, \ldots, T_\ell$ are all the nontrivial color trees in f. Let $t_i = |V(T_i)|$ for $1 \leq i \leq \ell$. As T_i has $t_i - 1$ edges, it wastes $t_i - 2$ colors. To prove $mc(G) \leq m - t + 1$, it suffices to show that f wastes at least $t - 1$ colors, that is, $\sum_{i=1}^{\ell}(t_i - 2) \geq t - 1$. Since each T_i can monochromatically connect at most $\binom{t_i - 1}{2}$ pairs of nonadjacent vertices in G, we have

$$\sum_{i=1}^{\ell} \binom{t_i - 1}{2} \geq \binom{n}{2} - m.$$

Assume by contradiction that $\sum_{i=1}^{\ell}(t_i - 2) < t - 1$, namely, $\sum_{i=1}^{\ell}(t_i - 1) < t - 1 + \ell$. As T_i is nontrivial, we have $t_i - 1 \geq 2$. Thus $1 \leq \ell \leq t - 2$. Since $\binom{x}{2} + \binom{y}{2} \leq \binom{x-1}{2} + \binom{y+1}{2}$ for $x \leq y + 1$, the expression $\sum_{i=1}^{\ell} \binom{t_i - 1}{2}$, subject to $t_i - 1 \geq 2$, is maximized when $\ell - 1$ of the $t_i's$ are equal to 3, and one of the $t_i's$, say t_ℓ, is as large as it can be, namely, $t_\ell - 1$ is the largest integer smaller than $(t - 1 + \ell) - 2(\ell - 1) = t - \ell + 1$. Hence $t_\ell - 1 = t - \ell$. So

$$\begin{aligned}
\sum_{i=1}^{\ell} \binom{t_i - 1}{2} &\leq (\ell - 1) + \binom{t - \ell}{2} \\
&= \frac{1}{2}\left[\ell^2 + (3 - 2t)\ell + t^2 - t - 2\right] \\
&\leq \binom{t-1}{2} \quad \text{(take } \ell = 1\text{)} \\
&< \binom{t-1}{2} + 1.
\end{aligned}$$

Here we use the fact that the function $g(\ell) = \frac{1}{2}\left[\ell^2 + (3-2t)\ell + t^2 - t - 2\right]$ is decreasing when $1 \le \ell \le t-2$, and so is maximized at the point $\ell = 1$. For a contradiction, we just need to show that $\binom{t-1}{2} + 1 \le \binom{n}{2} - m$. In fact,

$$\begin{aligned}\binom{t-1}{2} + 1 + m &\le \binom{t-1}{2} + 1 + \binom{n-t}{2} + t(n-t) + (t-2)\\ &= \binom{n}{2}.\end{aligned}$$

Next we will show that the bound is sharp. Let G^* be the graph defined as follows: first take a complete $(n-t+1)$-partite graph with vertex classes $V_1, \ldots, V_{n-t+1}$ such that $|V_j| = 1$ for $1 \le j \le n-t$ and $|V_{n-t+1}| = t$; then add the (at most $t-2$) remaining edges to V_{n-t+1} randomly. Color all the edges between V_1 and V_{n-t+1} with one color, and every other edge with a distinct fresh color. It is easily checked that this is an MC-coloring of G^* using $m-t+1$ colors, which implies $mc(G^*) \ge m-t+1$. Hence $mc(G^*) = m-t+1$. □

With the aid of Lemma 5, we give the proof Theorem 4.

Proof of Theorem **4.** If $k = \binom{n}{2}$, then clearly $g(n,k) = \binom{n}{2}$.

If $\binom{n-t}{2} + t(n-t-1) + 1 \le k \le \binom{n-t}{2} + t(n-t) - 1$ for some $t \in \{2, \ldots, n-1\}$, it follows from Lemma 5 that if a connected graph G on n vertices satisfies $m(G) \le k+t-1 \left(\le \binom{n-t}{2} + t(n-t) + t - 2\right)$, then $mc(G) \le m(G) - t + 1 \le k$. Hence, $g(n,k) \ge k+t-1$. To prove $g(n,k) \le k+t-1$, it suffices to find a connected graph G on n vertices such that $m(G) = k+t$ and $mc(G) > k$. We can take the graph G^* described in Lemma 5 with $m(G^*) = k+t$. By Lemma 5, we have $mc(G^*) = m(G^*) - t + 1 = k+1 > k$ for $\binom{n-t}{2} + t(n-t-1) + 1 \le k \le \binom{n-t}{2} + t(n-t) - 2$, and $mc(G^*) = m(G^*) - (t-1) + 1 = k+2 > k$ for $k = \binom{n-t}{2} + t(n-t) - 1$. So $g(n,k) \le k+t-1$, and thus $g(n,k) = k+t-1$.

If $k = \binom{n-t}{2} + t(n-t)$ for some $t \in \{2, \ldots, n-1\}$, it follows from Lemma 5 that if a connected graph G on n vertices satisfies $m(G) \le k+t-2 \left(= \binom{n-t}{2} + t(n-t) + t - 2\right)$, then $mc(G) \le m(G) - t + 1 \le k - 1 < k$. Hence, $g(n,k) \ge k+t-2$. To prove $g(n,k) \le k+t-2$, it suffices to find a connected graph G on n vertices such that $m(G) = k+t-1$ and $mc(G) > k$. We can take the graph G^* described in Lemma 5 with $m(G^*) = k+t-1$. By Lemma 5, we have $mc(G^*) = m(G^*) - (t-1) + 1 = k+1 > k$. So $g(n,k) \le k+t-2$, and thus $g(n,k) = k+t-2$. □

References

[1] J.A. Bondy, U.S.R. Murty, *Graph Theory*, GTM 244, Springer, 2008.

[2] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, R. Yuster, *On rainbow connection*, Electron. J. Combin. 15(1)(2008), R57.

[3] Y. Caro, R. Yuster, *Colorful monochromatic connectivity*, Discrete Math. 311(2011), 1786-1792.

[4] G. Chartrand, G. Johns, K. McKeon, P. Zhang, *Rainbow connection in graphs*, Math. Bohem. 133(2008), 85-98.

[5] A. Kemnitz, I. Schiermeyer, *Graphs with rainbow connection number two*, Discuss. Math. Graph Theory, 31(2011), 313C320.

[6] M. Krivelevich, R. Yuster, *The rainbow connection of a graph is (at most) reciprocal to its minimum degree*, J. Graph Theory 63(3)(2010), 185-191.

[7] H. Li, X. Li, Y. Sun, Y. Zhao, *Note on minimally d-rainbow connected graphs*, Graphs & Combin. 30(4)(2014), 949-955.

[8] X. Li, M. Liu, I. Schiermeyer, *Rainbow connection number of dense graphs*, Discuss. Math. Graph Theory, 33(2013), 603C611.

[9] X. Li, Y. Shi, Y. Sun, *Rainbow connections of graphs: A survey*, Graphs & Combin. 29(2013), 1-38.

[10] X. Li, Y. Sun, *Rainbow Connections of Graphs*, SpringerBriefs in Math., Springer, New York, 2012.

[11] A. Lo, *A note on the minimum size of k-rainbow-connected graphs*, Discrete Math. 331(2014), 20C21.

[12] I. Schiermeyer, *On minimally rainbow k-connected graphs*, Discrete Appl. Math. 161(2013), 702C705.