ON THE STRUCTURE OF LONG ZERO-SUM FREE SEQUENCES AND n-ZERO-SUM FREE SEQUENCES OVER FINITE CYCLIC GROUPS

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Abstract. In an additively written abelian group, a sequence is called zero-sum free if each of its nonempty subsequences has sum different from the zero element of the group. In this paper, we consider the structure of long zero-sum free sequences and n-zero-sum free sequences over finite cyclic groups $\mathbb{Z}_n$. Among which, we determines the structure of the long zero-sum free sequences of length between $n/3+1$ and $n/2$, where $n \geq 50$ is an odd integer, and we provide a general description on the structure of n-zero-sum free sequences of length $n+l$, where $\ell \geq n/p+p-2$ and p is the smallest prime dividing n.

1. Introduction and notation

Throughout this paper, let G be an additive finite abelian group. The cyclic group of order n is identified with the additive group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ of integers modulo n. Denote by $\mathcal{F}(G)$, the free abelian monoid with basis G and elements of $\mathcal{F}(G)$ are called *sequences* over G. A sequence of length l over G can be written in the form $S = g_1 \cdot \ldots \cdot g_l$ for some $g_1, \ldots, g_l \in G$. S is called a *zero-sum sequence* if the sum of all elements of S is zero, and a *zero-sum free* sequence if S does not contain a nonempty zero-sum subsequence. If S is a zero-sum sequence, but no proper nontrivial subsequence of S has sum zero, then S is called a *minimal zero-sum sequence*.

Recall that the index of a sequence S over G is defined as follows.

Definition 1. *Let $G = \mathbb{Z}_n$, for a sequence $S = (n_1 g) \cdot \ldots \cdot (n_l g)$, $1 \leq n_1, \ldots, n_l \leq n$, the index of S is defined by $ind(S) = \min\{\|S\|_g | g \in \mathbb{Z}_n,\ with\ \mathbb{Z}_n = \langle g \rangle\}$, where $\|S\|_g = \frac{n_1+\cdots+n_l}{ord(g)}$.*

Let S_1 and S_2 be two sequences over $\mathbb{Z}_n$. We say that S_1 is *equivalent* to S_2 and write $S_1 \sim S_2$ if S_2 can be obtained from S_1 through multiplication by an integer coprime to n and rearrangement of terms. Such multiplication is an affine map preserving all zero sums in $\mathbb{Z}_n$. Certainly, $\mathrm{ind}(S_1) = \mathrm{ind}(S_2)$.

The study of long zero-sum free sequences in $\mathbb{Z}_n$ has attracted considerable attention recently. There are serval related results on the structure of zero-sum free sequences, and among which W. Gao [6] characterized the zero-sum free sequences of lengths roughly greater than $\frac{2n}{3}$. S. Savchev and F. Chen [18], and P. Yuan [20] independently proved that each zero-sum free sequence S in $\mathbb{Z}_n$ with $|S| > \frac{n}{2}$ has index less than 1, where $|S|$ is the length of the sequence S. In this paper, we consider the general structure of the zero-sum free sequences S in $\mathbb{Z}_n$ of length $\frac{n}{3}+1 < |S| \leq \frac{n}{2}$,

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and our first main result (Theorem 3) shows that by removing at most two elements from S, the index of the remaining sequence is not more than $1-\frac{4}{n}$.

Similarly, the n-zero-sum free sequences in $\mathbb{Z}_n$ play an important role in the investigation of the structure of zero-sum free sequences. S. Savchev and F. Chen [19] characterized the n-zero-sum free sequences over $\mathbb{Z}_n$ of length greater than $\frac{3n}{2}-1$. In this paper, we investigate n-zero-sum free sequences of length $n+l$, where $\ell \geq \frac{n}{p}+p-2$, p is the smallest prime dividing n, and our second main result (Theorem 8) provides a general description of such sequences.

We next recall a few more standard notations and terminologies (which follow from that in [10] and [16]).

We denote by $\sigma(S)$ the sum of all terms of S. The maximum multiplicity of a term in S is denoted as $h(S)$. We denote by $\sum(S)$ the set of all subsums of S, and $\sum_k(S)$ the set of k-term subsums of S, where $k \in \mathbb{N}$. The union of two sequences S_1 and S_2, denoted S_1S_2, is the sequence formed by appending the terms of S_2 to S_1.

To study the index problem of minimal zero-sum sequences, we usually first investigate the structure of unsplittable minimal zero-sum sequences first. Let S be a minimal zero-sum (resp. zero-sum free) sequence of elements over abelian group G. An element g_0 in S is called *splittable* if there exist two elements $x, y \in G$ such that $x+y=g_0$ and $Sg_0^{-1}xy$ is a minimal zero-sum (resp. zero-sum free) sequence as well; otherwise, g_0 is called *unsplittable*. S is called *splittable* if at least one of the elements of S is splittable; otherwise, it is called *unsplittable*.

We state this easy observation: if S is a minimal zero-sum sequence and S' is obtained from S by splitting some elements of S, then $|S| \leq |S'|$ and $\mathrm{ind}(S) \leq \mathrm{ind}(S')$.

The paper is organized as follows. In the next section, based on a recent result of Y. Li and P. Yuan [21](see also [**?**]), we give a proof for Theorem 3, and obtain a corresponding proposition on the maximum multiplicities of a term in such zero-sum free sequences. In section 3, we provide a general description on the structure of n-zero-sum free sequences of length $n+l$ over a finite abelian group G of order n, where $\ell \geq n/p+p-2$ and p is the smallest prime dividing n. In the last section, we discuss when subsums of a long zero-sum free sequence may form an interval.

2. On long zero-sum free sequences over $\mathbb{Z}_n$

For a long zero-sum free sequence S over $\mathbb{Z}_n$, if $|S|>n/2$, then $\mathrm{ind}(S)<1$ [18, 20]. In this section, we consider long zero-sum free sequences of length between $n/3+1$ and $n/2$ when $n \geq 50$ is an odd integer.

We recall the following lemma, which is essential to approach our main results.

Lemma 2. [**?**] *Let $n \geq 50$ be an odd integer, and S be an unsplittable minimal zero-sum sequence of length $|S| > \lfloor \frac{n}{3} \rfloor + 2$ over $\mathbb{Z}_n$. If ind$(S) \geq 2$, then*

$$S \sim g^{\alpha}(\frac{n+s}{2}g)^{2t}((\frac{n-s}{2}+1)g),$$

where g is a generator of $\mathbb{Z}_n$, $t \geq 1$, s is odd with $s \geq 3$ and $2\alpha+2ts+2-s=n$. Moreover, ind$(S)=2$.

Theorem 3. *Let $n \geq 50$ be an odd integer, and S be a zero-sum free sequence of length $\lfloor \frac{n}{3} \rfloor + 1 < |S| \leq \lfloor \frac{n}{2} \rfloor$ over $\mathbb{Z}_n$. Then, by removing at most two elements from S, the remaining sequence is equivalent to a sequence whose index is not more than $1 - 4/n$.*

Proof. Let S be a zero-sum free sequence over $\mathbb{Z}_n$ with $l = |S| > \lfloor \frac{n}{3} \rfloor + 1$. Then $S_1 = S(-\sigma(S))$ is a minimal zero-sum sequence of length $|S_1| > \lfloor \frac{n}{3} \rfloor + 2$. If S_1 is splittable, then there exist $g_0 \in S_1$ and two elements $x, y \in G$ such that $x + y = g_0$ and $S_2 = S_1 g_0^{-1} xy$ is a minimal zero-sum sequence as well. Continuing the above process, we eventually obtain an unsplittable minimal zero-sum sequence S' from S_1. Then, we have $|S_1| \leq |S'|$ and

$$\mathrm{ind}(S) < \mathrm{ind}(S_1) \leq \mathrm{ind}(S'). \tag{1}$$

We divide the proof into two cases.

Case 1: If $\mathrm{ind}(S') \leq 1$, then $\mathrm{ind}(S) < 1$ by (1). Let h be an element of G with $\mathrm{ord}(h) = n$, such that $S \sim (n_1 h) \cdot \ldots \cdot (n_l h)$ and $\mathrm{ind}(S) = \|S\|_h = \frac{n_1 + \cdots + n_l}{n}$. If there are n_i and n_j such that $n_i + n_j \geq 4$ for $1 \leq i \neq j \leq l$, then $\frac{\sum_{k=1}^{l} n_k - n_i - n_j}{n} \leq \frac{\sum_{k=1}^{l} n_k - 4}{n} < 1 - \frac{4}{n}$. If for any n_i, n_j, we have $n_i + n_j \leq 3$. Then there exists at most one i such that $n_i = 2$, and for all other $j \neq i$, $n_j = 1$. So $\sum_{k=1}^{l} n_k \leq \lfloor \frac{n}{2} \rfloor + 1$ (as $l \leq \lfloor \frac{n}{2} \rfloor$), and thus $\mathrm{ind}(S) \leq \frac{\lfloor \frac{n}{2} \rfloor + 1}{n} \leq \frac{1}{2} + \frac{1}{n} < 1 - \frac{4}{n}$. Therefore, by removing at most two elements from S, the index of the remaining sequence is not more than $1 - \frac{4}{n}$.

Case 2: If $\mathrm{ind}(S') > 1$, then by Lemma 2,

$$S' \sim g^{\alpha} (\frac{n+s}{2} g)^{2t} ((\frac{n-s}{2} + 1) g),$$

where $\mathbb{Z}_n = \langle g \rangle$, $t \geq 1$, $s \geq 3$ and $2\alpha + 2ts + 2 - s = n$. Thus

$$S' \sim (2g)^{\alpha} (sg)^{2t} ((n - s + 2) g).$$

Let $T = (2g)^{\alpha} (sg)^{2t} ((n - s + 2) g)$. It is easy to see that if we remove three elements from T, including sg and $(n - s + 2)g$, then the remaining sequence has index not more than $1 - 4/n$. If we remove two elements from S such that, including the elements sg and $(n - s + 2)g$, at least three terms of T are left out, then the remaining sequence of S has index not more than $1 - 4/n$, as desired. □

We next estimate a lower bound for $h(S)$. Let S be a zero-sum free sequence over $\mathbb{Z}_n$. An extensively used result of Bovey et al [1] on the constant $h(S)$ states that if $|S| = l > n/2$, then $h(S) \geq 2l - n + 1$. S. Savchev and F. Chen [18], and P. Yuan [20] gave a more precisely lower bound on $h(S)$ for zero-sum free sequences of length $l > n/2$ independently. The following theorem provides a lower bound for $h(S)$ when the length of a zero-sum free sequence S is between $\lfloor \frac{n}{3} \rfloor + 1$ and $\lfloor \frac{n}{2} \rfloor$.

Theorem 4. *Let $n \geq 50$ be an odd integer and l be an integer satisfying $\lfloor \frac{n}{3} \rfloor + 1 < l \leq \lfloor \frac{n}{2} \rfloor$. Let S be a zero-sum free sequence of length l over $\mathbb{Z}_n$, and i be an positive integer such that $1 \leq i \leq 3$ and $4l - 4 - n \equiv i (\mod 3)$. If ind$(S) > 1$, then*

(a) $h(S) \geq 3l - 2 - n$, if $\frac{2n+5-i}{5} \leq l \leq \lfloor \frac{n}{2} \rfloor$;

(b) $h(S) \geq \lfloor \frac{4l-4-n}{3} \rfloor + 1$, if $\lfloor \frac{n}{3} \rfloor + 1 < l < \frac{2n+5-i}{5}$.

Moreover, these estimates are best possible.

Proof. By the assumption and Theorem 3, we may assume that S is of the form

$$S=(2g)^u(sg)^vWg_1g_2,\ ||S(g_1g_2)^{-1}||_g\le 1-\frac{4}{n},\ \text{and}\ \mathrm{ind}(S)=\|S\|_g,$$

where $s\ge 3$, $W|S$, $1\ge |W|\ge 0$ and $2g, sg\notin W$. We claim that $g\notin S$. In fact, let S' be given as in the proof of Theorem 3. Then $\mathrm{ind}(S')=||S'||_g=2$ and $S'=(2g)^{\alpha}(sg)^{2t}((n-s+2)g)$. If $g\in S$, then since $g\notin S'$ g is splittable in S. Thus there exists a subsequence S_1' of S' such that $\sigma(S_1')=g$. Since $||S'||_g=2$, we must have $n||S_1'||_g=n+1$, so $n||S'(S_1')^{-1}||_g=(n-1)$. Therefore, $n||S||_g=n||g||_g+n||Sg^{-1}||_g\le 1+n||S'(S_1')^{-1}||_g=n$, contradicting $||S||_g=\mathrm{ind}(S)>1$, and the claim is proved. If $kg\in W$, by the same argument as above we can show that $k>s$. Then

$$n-4\ge n||S(g_1g_2)^{-1}||_g\ge 2u+3(l-2-u)=3l-6-u. \tag{2}$$

$$n-4\ge n||S(g_1g_2)^{-1}||_g\ge 2u+3v+4(l-2-u-v)=4l-8-2u-v. \tag{3}$$

The above inequalities yield that $u\ge 3l-2-n$ and $2u+v\ge 4l-4-n$, respectively. Since $h(S)\ge\max\{u,v\}$, it follows that $h(S)\ge\max\{3l-2-n,\lceil\frac{4l-4-n}{3}\rceil\}$.

We now consider whether we can find some extremal sequences S with $h(S)=3l-2-n$ and $h(S)=\lceil\frac{4l-4-n}{3}\rceil$, respectively.

(a) If $3l-2-n\ge\lceil\frac{4l-4-n}{3}\rceil$, that is, $l\ge\frac{2n+5-i}{5}$, then we have $h(S)\ge 3l-2-n$. Suppose there is an extremal sequence S with $h(S)=3l-2-n$. By (2), $S=(2g)^{3l-2-n}(3g)^{n-2l}g_1g_2$. It is easy to check that $S=(2g)^{3l-2-n}(3g)^{n-2l+1}((n-1)g)$ since S is zero-sum free and $\mathrm{ind}(S)>1$. So S is the only extremal sequence with $h(S)=3l-2-n$ when $l\ge\frac{2n+5-i}{5}$.

(b) If $3l-2-n<\lceil\frac{4l-4-n}{3}\rceil$, then $\lfloor\frac{n}{3}\rfloor+1<l<\frac{2n+5-i}{5}$. Suppose S is an extremal sequence with $h(S)=\lceil\frac{4l-4-n}{3}\rceil$. We consider the following cases:

Case 1. If $i=1$, that is, $\lceil(\frac{4l-4-n}{3}\rceil=\frac{4l-2-n}{3}$, then by (3) and the assumption $u=v$, we have $S=(2g)^{\frac{4l-2-n}{3}}(3g)^{\frac{4l-2-n}{3}}(4g)^{l-2-2\frac{4l-2-n}{3}}(g_1)(g_2)$. Since S is zero-sum free, $\mathrm{ind}(S)>1$ and $h(S)=\frac{4l-2-n}{3}$, we infer that either $S=(2g)^{\frac{4l-2-n}{3}}(3g)^{\frac{4l-2-n}{3}}\ (4g)^{l-1-2\frac{4l-2-n}{3}}((n-1)g)$, or $S=(2g)^{\frac{4l-2-n}{3}}(3g)^{\frac{4l-2-n}{3}}(4g)^{l-2-2\frac{4l-2-n}{3}}(5g)((n-1)g)$. These are two extremal sequences with $h(S)=\frac{4l-2-n}{3}=\lfloor\frac{4l-4-n}{3}\rfloor+1$.

Case 2. If $i=2$, then $\lceil\frac{4l-4-n}{3}\rceil=\frac{4l-3-n}{3}$. Using the same method as in Case 1, we derive that $S=(2g)^{\frac{4l-3-n}{3}}(3g)^{\frac{4l-3-n}{3}}\ (4g)^{l-1-2\frac{4l-3-n}{3}}((n-1)g)$, which is an extremal sequence with $h(S)=\frac{4l-3-n}{3}=\lfloor\frac{4l-4-n}{3}\rfloor+1$.

Case 3. If $i=3$, then $\lceil\frac{4l-4-n}{3}\rceil=\frac{4l-4-n}{3}$, and $S=(2g)^{\frac{4l-4-n}{3}}(3g)^{\frac{4l-4-n}{3}}(4g)^{l-2-2\frac{4l-4-n}{3}}g_1g_2$, where $g_1,g_2\in\{2g,3g,(n-1)g\}$ and at least one of g_i is in $\{2g,3g\}$, for $i=1,2$. So, there is no extremal sequence S with $h(S)=\frac{4l-4-n}{3}$, and thus $h(S)\ge\lfloor\frac{4l-4-n}{3}\rfloor+1$. In addition, $S=(2g)^{\frac{4l-4-n}{3}+1}(3g)^{\frac{4l-4-n}{3}}\ (4g)^{l-2-2\frac{4l-4-n}{3}}(n-1)g$ and $S=(2g)^{\frac{4l-4-n}{3}+1}(3g)^{\frac{4l-4-n}{3}+1}(4g)^{l-2-2\frac{4l-4-n}{3}}$ are two extremal sequences with $h(S)=\lfloor\frac{4l-4-n}{3}\rfloor+1$.

Therefore, if $\lfloor\frac{n}{3}\rfloor+1<l<\frac{2n+5-i}{5}$, then we have $h(S)\ge\lfloor\frac{4l-4-n}{3}\rfloor+1$, so the lower bound is best possible.

□

3. On long n-zero-sum free sequences over $\mathbb{Z}_n$

In this section, we consider the structure of a long n-zero-sum free sequence over $\mathbb{Z}_n$. Prior to this, we give some properties on a sequence which contains (or does not contain) a zero-sum subsequence of length $|G|$ over a finite abelian group G.

Let G be a finite abelian group, $S = g_1 \cdot \ldots \cdot g_m$ be a sequence over G and $A \subset G$. Let $v_A(S) = |\{1 \le i \le m : g_i \in A\}|$. If $A = \{g\}$, we simply let $v_g(S) = v_{\{g\}}(S)$. For every subgroup H of G, let S_H denote the subsequence of S consisting of all terms of S in H. We need the following beautiful result due to DeVos, Goddyn and Mohar [3]:

Lemma 5. *Let ℓ be a positive integer, S be a sequence over a finite abelian group G, and let $H = stab(\sum_\ell(S))$. Then,*

$$|\sum_\ell(S)| \ge |H| \left(1 - \ell + \sum_{Q\in G/H} \min\{\ell, v_Q(S)\}\right).$$

Lemma 6. *Let G be a finite abelian group, and p be the smallest prime dividing $|G|$. Let $\ell \ge |G|/p + p - 2$ be an integer and S be a sequence over G of length $|S| = |G| + \ell$. If $h(S) \le \ell$, then S contains a zero-sum subsequence of length $|G|$.*

Proof. Let $n = |G|$. Assume to the contrary that $0 \notin \sum_n(S)$. Then, $|\sum_n(S)| \le n - 1$. Since $\sum_\ell(S) = \sigma(S) - \sum_n(S)$, we have $|\sum_\ell(S)| = |\sum_n(S)| \le n - 1$. Let $H = \mathrm{stab}(\sum_\ell(S)$. Now by Lemma 5 we obtain that

$$|H| \left(1 - \ell + \sum_{Q\in G/H} \min\{\ell, v_Q(S)\}\right) \le n - 1. \tag{4}$$

From $|\sum_\ell(S)| \le n-1$ we have $H \ne G$. If $H = \{0\}$, then by the hypothesis $h(S) \le \ell$, we infer that

$$|H| \left(1 - \ell + \sum_{Q\in G/H} \min\{\ell, v_Q(S)\}\right) = 1 - \ell + \sum_{Q\in G/H} \min\{\ell, v_Q(S)\} = 1 - \ell + |S| = n + 1,$$

a contradiction to (4). This proves that $H \ne \{0\}$. Hence, H is a nontrivial proper subgroup of G.

Let $w = |\{Q \in G/H : v_Q(S) \ge \ell + 1\}|$. Now we distinguish three cases to derive a contradiction.

Case 1. If $w = 0$, then,

$$|H| \left(1 - \ell + \sum_{Q\in G/H} \min\{\ell, v_Q(S)\}\right) = |H|(1 - \ell + |S|) > n + 1,$$

a contradiction to (4).

Case 2. If $w \ge 2$, then $1-\ell+\sum_{Q\in G/H} \min\{\ell, v_Q(S)\} \ge 1-\ell+2\ell = \ell+1$. Since $\ell \ge n/p+p-2 \ge n/|H| + |H| - 2$, we infer that

$$|H| \left(1 - \ell + \sum_{Q\in G/H} \min\{\ell, v_Q(S)\}\right) \ge |H|(\ell + 1) > n,$$

a contradiction to (4).

Case 3. If $w = 1, then$ let $R \in G/H$ be the unique coset such that $v_R(S) \geq \ell + 1$. Thus

$$1 - \ell + \sum_{Q \in G/H} \min\{\ell, v_Q(S)\} = 1 - \ell + \ell + (|S| - v_R(S)) = n + \ell + 1 - v_R(S).$$

It follows from (4) that $|H|(n + \ell + 1 - v_R(S)) \leq n - 1$. Therefore,

$$\begin{aligned} v_R(S) &\geq n + \ell + 1 - \tfrac{n}{|H|} + \tfrac{1}{|H|} \geq n + (n/p + p - 2) + 1 - \tfrac{n}{|H|} + \tfrac{1}{|H|} \\ &\geq n + (\tfrac{n}{|H|} + |H| - 2) + 1 - \tfrac{n}{|H|} + \tfrac{1}{|H|} > n + |H| - 1. \end{aligned}$$

Hence, $v_R(S) \geq n + |H|$. Let $R = g + H$ with $g \in G$ and $T = -g + S$. Then,

$$v_H(T) = v_R(S) \geq n + |H|.$$

By using the Erdős-Ginzburg-Ziv theorem on the subsequence T_H of T we can find $\frac{n}{|H|}$ disjoint zero-sum subsequences $T_1, \cdots, T_{\frac{n}{|H|}}$ such that $|T_i| = |H|$ for every $i \in [1, \frac{n}{|H|}]$. Now $T_1 T_2 \cdots T_{\frac{n}{|H|}}$ is a zero-sum subsequence of T of length n. Therefore, $g + T_1 T_2 \cdots T_{\frac{n}{|H|}}$ is a zero-sum subsequence of S of length n, a contradiction. □

Lemma 7. [5] *Let G be a finite abelian group, and let $S = 0^h T$ be a sequence over G with $h = h(S)$. Then,*

$$\sum_{|G|}(S) = \sum_{\geq |G| - h}(S).$$

Theorem 8. *Let G, p, ℓ be as in Lemma 6. Let S be a sequence over G of length $|S| = n + \ell$. Suppose that $0 \notin \sum_{|G|}(S)$. Then, there is an element $g \in G$ such that*

$$-g + S = 0^h T S'$$

with $h \geq \ell + 1$, T is a zero-sum sequence of length $|T| \leq |G| - h - 1$, and S' is zero-sum free of length $|S'| \geq \ell + 1$.

Proof. The result follows immediately from Lemma 6 and Lemma 7. □

Corollary 9. *Let p be the smallest prime dividing n, and S be an n-zero-sum free sequence over $\mathbb{Z}_n$ of length $|S| = n + \ell$, where $\ell \geq n/p + p - 2$ is an integer. Then, there exists $g \in \mathbb{Z}_n$ such that*

$$-g + S = 0^h T S'$$

with $h \geq \ell + 1$, T is a zero-sum sequence of length $|T| \leq n - h - 1$, and S' is zero-sum free with $|S'| \geq \ell + 1$.

Proof. The result follows from Theorem 8. □

4. Concluding Remarks

In this section, we discuss the question of whether for each long zero-sum free sequence S, there exists a sequence T, such that $T \sim S$ and subsums of T form an interval (such T is referred as to a smooth sequence). In [18], S. Savchev and F. Chen showed that, for each zero-sum free sequence S of length $|S| > \frac{n}{2}$ over $\mathbb{Z}_n$, there exists a sequence T, such that $T \sim S$ and $\sum(T) = \{1, 2, \cdots, \sigma(T)\} = [1, \sigma(T)]$ is an interval. Essentially, they proved that if T is a zero-sum free sequence of length $|T| > \frac{n}{2}$ over $\mathbb{Z}_n$ such that $\sigma(T) < n$ (as positive integers), then $1 \in T$ and $\sum(T) = [1, \sigma(T)]$ is an interval, (i.e., T is smooth). In fact, the same result holds under the weaker assumption that $|T| \geq (n+2)/3$. In what follows, we will show that if S is a zero-sum free sequence over $\mathbb{Z}_n$ of length $|S| \geq \frac{n+2}{3}$ such that $1 \in S$ and $\sigma(S) < n$ (as positive integers), then $\sum(S)$ is almost an interval except for some special cases.

Lemma 10. *Let S be a sequence with positive integer terms of length $|S| \geq 2$ such that $\sum(S)$ is an interval. Then, for any positive integer g, $\sum(gS)$ is an interval if and only if $g \leq \sigma(S)+1$. In particular, for any $g \in S$, $\sum(gS)$ is an interval.*

Proof. Since $|S| \geq 2$, we may assume $\sum(S) = [a, \sigma(S)]$ with $a < \sigma(S)$. Since $\sigma(S) - 1 \in [a, \sigma(S)]$, $1 \in [a, \sigma(S)]$, forcing $a = 1$. For any positive integer g, $\sum(gS) = [1, \sigma(S)] \cup \{g\} \cup [1+g, \sigma(S)+g]$. Thus $\sum(gS)$ is an interval $[1, \sigma(S)+g]$ if and only if $g \leq \sigma(S)+1$. In particular, if $g \in S$, we have $g < \sigma(S)+1$, so $\sum(gS)$ is an interval. □

Theorem 11. *Let S be a sequence with positive integer terms of length $|S| = t > \frac{n+2}{3}$, and $\sigma(S) < n$. If $1 \in S$, then $\sum(S)$ is an interval except for the case when $S = S_0 n_l$, where $\sum(S_0)$ is an interval and $n_l > \sigma(S_0)+1$.*

Proof. Suppose $S = (n_1)^{t_1} \cdot \ldots \cdot (n_l)^{t_l}$, where $1 = n_1 < \cdots < n_l$, $\sum_{i=1}^{l} t_i = t > \frac{n}{3}$ and $\sigma(S) = \sum_{i=1}^{l} t_i n_i < n$. Set $\sum(S) = \{m_1, \cdots, m_k\}$ with $1 = m_1 < \cdots < m_k$. Then $\sum(S)$ is an interval if and only if $m_{i+1} - m_i = 1$ for $i = 1, \cdots, k-1$. Assume to the contrary that $\sum(S)$ is not an interval. Let v be the smallest positive integer such that $m_{v+1} - m_v \geq 2$, and let S_0 be the subsequence of S with largest length such that $supp(S_0) \subseteq \{m_1, \cdots, m_v\} = [1, m_v] \subseteq \sum(S_0)$. Set $supp(S_0) = \{n_{i_1}, \ldots, n_{i_r}\}$, where $n_{i_1} < \ldots < n_{i_r}$, and then $S_0 = n_{i_1}^{t_{i_1}} \ldots n_{i_r}^{t_{i_r}}$. Evidently, $n_{i_1} = n_1 = 1$. We consider the following two cases.

Case 1. If $v = 1$, then $t_1 = 1$ and $n_2 \geq 3$; otherwise, $\{1, 2\}$ are consecutive integers, which is a contradiction to $v = 1$. Thus

$$n > \sigma(S) = \sum_{i=1}^{l} t_i n_i \geq 1 + 3(t-1) = 3t - 2 \geq 3(\frac{n+2}{3}) - 2 = n,$$

a contradiction.

Case 2. If $v \geq 2$, we first show the following:

Claim. $\sum(S_0) = \{m_1, \cdots, m_v\}$

We now find a subsequence S_1 of S_0, such that $n_1 \in S_1$ and $\sum(S_1)(\subseteq \{m_1, \cdots, m_v\})$ forms an interval. In fact, if $t_1 \geq 2$, then $\sum(1^{t_1})$ is a desired interval; if $t_1 = 1$, we have $n_{i_2} = 2$ since $v \geq 2$, and thus $\sum(\{1, 2\}) = [1, 3]$ is an interval.

By the definition of S_1 and Lemma 10, $\sum\{S_1 \cup \{n_{i_k}\}\} \subseteq \{m_1, \cdots, m_v\}$ is an interval, where $n_{i_k} = \min\{n_{i_j} | n_{i_j} \in S_0(S_1)^{-1}\}$, so by applying Lemma 10 repeatedly, we conclude that $\sum S_1 \cdot n_{i_k}^{t_{i_k}}$, and thus $\sum(n_{i_1}^{t_{i_1}} \cdot \ldots \cdot n_{i_r}^{t_{i_r}})$ forms an interval. Therefore, $\sum(S_0) = \{m_1, \cdots, m_v\}$ and the claim is confirmed.

Let $n_j \in S{S_0}^{-1}$, and let $|S_0| = u$. Again, by Lemma 10, we derive that $n_j \geq \sigma(S_0) + 2 \geq u + 2$, so $u + (t-u)(u+2) \leq \sigma(S) \leq n - 1$. Then $u \geq \frac{2t-3}{2}$ since $t > \frac{n}{3}$, that is, $u = t-1$ or $u = t$. If $u = t$, then $S = S_0$ and $\sum(S)$ is an interval. If $u = t-1$, then we have $S = S_0 n_l$, where $\sum(S_0)$ is an interval and $n_l > \sigma(S_0)+1$.

□

We remark that the sequence S in the above theorem can be regarded as a zero-sum free sequence S over $\mathbb{Z}_n$ of length $|S| \geq \frac{n+2}{3}$ such that $1 \in S$ and $\sigma(S) < n$ (as positive integers). Such S has $ind(S) < 1$ and it is almost smooth.

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