Note on the complexity of deciding the rainbow (vertex-)connectedness for bipartite graphs

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Abstract

A path in an edge-colored graph is said to be a rainbow path if no two edges on the path share the same color. An edge-colored graph is (strongly) rainbow connected if there exists a rainbow (geodesic) path between every pair of vertices. The (strong) rainbow connection number of G, denoted by ($scr(G)$, respectively) $rc(G)$, is the smallest number of colors that are needed in order to make G (strongly) rainbow connected. A vertex-colored graph G is rainbow vertex-connected if any pair of vertices in G are connected by a path whose internal vertices have distinct colors. The rainbow vertex-connection number of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. Though for a general graph G it is NP-Complete to decide whether $rc(G) = 2$ (or $rvc(G) = 2$), in this paper, we show that the problem becomes easy when G is a bipartite graph. Whereas deciding whether $rc(G) = 3$ (or $rvc(G) = 3$) is still NP-Complete, even when G is a bipartite graph. Moreover, it is known that deciding whether a given edge(vertex)-colored (with an unbound number of colors) graph is rainbow (vertex-)connected is NP-Complete. We will prove that it is still NP-Complete even when the edge(vertex)-colored graph is bipartite. We also show that a few NP-hard problems on rainbow connection are indeed NP-Complete.

Keywords: (strong) rainbow connection; rainbow vertex-connection; bipartite graph; NP-Complete; polynomial-time

1. Introduction

We follow the terminology and notations of [2] and all graphs considered here are finite and simple.

As a means of strengthening the connectivity, Chartrand et al. in [6] first introduced the concepts of rainbow connection and strong rainbow connection. Let G be a nontrivial connected graph with an edge-coloring c: $E(G) \rightarrow \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. A path P in G is a *rainbow path* if no two edges of P are colored the same. The graph G is *rainbow connected* (*with respect to* c) if G contains a rainbow $u-v$ path for any pair of vertices u and v of G. In this case, the coloring c is called a *rainbow coloring* of G. If k colors are used, then c is a *rainbow k-coloring*. The *rainbow connection number* of G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. A *rainbow $u-v$ geodesic* in G is a *rainbow $u-v$ path* of length $d(u,v)$, where $d(u,v)$ is the distance between u and v. The graph G is *strongly rainbow connected* if there exists a rainbow $u-v$ geodesic for any two vertices u and v in G. In this case, the coloring c is called a *strong rainbow coloring* of G. Similarly, we define the *strong rainbow connection number* of a connected graph G, denoted by $src(G)$, as the smallest number of colors that are needed in order to make G strong rainbow connected. Clearly, we have $diam(G) \leq rc(G) \leq scr(G) \leq m$, where $diam(G)$ denotes the diameter of G and m is the number of edges of G. Moreover, it is easy to verify that $src(G) = rc(G) = 1$ if and only if G is a complete graph, that $rc(G) = 2$ if and only if $src(G) = 2$, and that $rc(G) = n-1$ if and only if G is a tree.

Similar to the concept of rainbow connection number, Krivelevich and Yuster [14] proposed the concept of rainbow vertex-connection number. Let G be a nontrivial connected graph with a vertex-coloring c: $V(G) \rightarrow \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$. A path P in G is *rainbow vertex-connected* if its internal vertices have distinct colors. The graph G is *rainbow vertex-connected* (*with respect to* c) if any pair of vertices are connected by a rainbow vertex-connected path. In this case, the coloring c is called a *rainbow vertex-coloring* of G. If k colors are used, then c is a *rainbow k-vertex-coloring* and G is *rainbow k-vertex-connected*. The *rainbow vertex-connection number* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. It is easy to observe that if G is of order n then $rvc(G) \leq n-2$, $rvc(G) = 0$ if and only if G is a complete graph, and $rvc(G) = 1$ if and only if $diam(G) = 2$. Notice that $rvc(G) \geq diam(G) - 1$ with equality if the diameter is 1 or 2. For more results on rainbow connection and rainbow vertex-connection, we refer to the survey [12] and the book [13].

The computational complexity of rainbow (vertex-)connection number has been studied extensively. In [3], Caro et al. conjectured that computing $rc(G)$ is an NP-Hard problem, as well as that even deciding whether a graph has $rc(G) = 2$ is NP-Complete. In [4], Chakraborty et al. confirmed this conjecture. In [1], the complexity of computing $rc(G)$ and $src(G)$ was studied further. It was shown that given any natural number $k \geq 3$ and a graph G, it is NP-hard to determine whether $rc(G) \leq k$. Moreover, for $src(G)$, it was shown that given any natural number $k \geq 3$ and a graph G, determining whether $src(G) \leq k$ is NP-hard even when G is bipartite. In this paper, we will point out that the problems in [1] are, in fact, NP-Complete. Though for a general graph G it is NP-Complete to decide whether $rc(G) = 2$ [4], we show that the problem becomes easy when G is a bipartite graph. Whereas deciding whether $rc(G) = 3$ is still NP-Complete, even when G is a bipartite graph.

For the rainbow vertex-connection number, Chen et al. [8] showed that for a graph G, deciding whether $rvc(G) = 2$ is NP-Complete. Recently, Chen et al. [7] obtained a more general result: for any fixed integer $k \geq 2$, to decide whether $rvc(G) \leq k$ is NP-Complete. For more complexity results, we refer to [11, 9, 10, 5, 15].

In this paper, we continue focusing on the bipartite graph. Similarly, we obtain that deciding whether $rvc(G) = 2$ can be solved in polynomial time, whereas deciding whether $rvc(G) = 3$ is still NP-Complete when G is a bipartite graph. Moreover, it is NP-Complete to decide whether a given edge-colored (with an unbound number of colors) graph is rainbow connected [4] and it is also NP-Complete to decide whether a given vertex-colored graph is rainbow vertex-connected [8]. We will prove that the two problems are still NP-Complete even when the graph is bipartite.

2. Main results

At first, we restate several results in [4] and [1].

Lemma 1. *([4]) Given a graph G, deciding if $rc(G) = 2$ is NP-Complete. In particular, computing $rc(G)$ is NP-Hard.*

Lemma 2. *([1]) For every $k \geq 3$, deciding whether $rc(G) \leq k$ is NP-Hard.*

Lemma 3. *([1]) Deciding whether the rainbow connection number of a graph is at most 3 is NP-Hard even when the graph G is bipartite.*

Lemma 4. *([1]) For every $k \geq 3$, deciding whether $src(G) \leq k$ is NP-Hard even when G is bipartite.*

We will show that "NP-hard" in the above results can be replaced by "NP-Complete" if k is any fixed integer. It suffices to show that these problems belong to the class NP for any fixed k. In fact, from the proofs in [1], for the problems in Lemmas 2 and 4, "For every $k \geq 3$" can be replaced by "For any fixed $k \geq 3$".

Theorem 1. *For any fixed $k \geq 2$, given a graph G, deciding whether $rc(G) \leq k$ is NP-Complete.*

Proof. By Lemmas 1 and 2, it will suffice to show that the problem in Lemma 2 belongs to the class NP. Therefore, if given any instance of the problem whose answer is 'yes', namely a graph G with $rc(G) \leq k$, we want to show that there is a certificate validating this fact which can be checked in polynomial time.

Obviously, a rainbow k-coloring of G means that $rc(G) \leq k$. For checking a rainbow k-coloring, we only need to check whether k colors are used and for any two vertices u and v of G, whether there exists a rainbow $u-v$ path. Notice that for two vertices u, v, there are at most n^{l-1} $u-v$ paths of length l, since if let $P = ut_1t_2\cdots t_{l-1}v$, there are less than n choices for each t_i ($i \in \{1, 2, \ldots, l-1\}$). Therefore, G contains at most $\Sigma_{l=1}^{k} n^{l-1} \leq kn^{k-1} \leq n^k$ $u-v$ paths of length no more than k. Then check these paths in turn until we find one path whose edges have distinct colors or no such paths at all. It follows that the time used for checking is at most $O(n^k \cdot n \cdot n^2) = O(n^{k+3})$. Since k is a fixed integer, we conclude that the certificate, namely a rainbow k-coloring of G, can be checked in polynomial time. The proof is complete.∎

The next theorem can be obtained similarly.

Theorem 2. *For any fixed $k \geq 2$, given a graph G, deciding whether $src(G) \leq k$ is NP-Complete.*

Proof. Since $rc(G) = 2$ if and only if $src(G) = 2$, by Lemmas 1 and 4, it suffices to show that the problem in Lemma 4 belongs to the class NP.

From the proof of Theorem 1, it is clear that for any two vertices u and v of G, the existence of a $u-v$ path of length l ($\leq k$) can be decided in time $O(n^{l-1})$. Therefore, if we check each integer $l \leq k$ in turn, we can either find an integer l such that there is a $u-v$ path of length l but no $u-v$ path of length less than l, or conclude that there is no $u-v$ path of length at most k. In the former case, the integer l is exactly the distance $d(u, v)$ and then check the colors of edges of each $u-v$ path of length $d(u, v)$ in turn. Similar to the proof of Theorem 1, we can obtain that the certificate, namely, a strong rainbow k-coloring of G, can be checked in polynomial time. The proof is complete. ∎

We know that for a given graph G, deciding if $rc(G) = 2$ is NP-Complete. Surprisingly, if G is a bipartite graph, the problem turns out to be easy. Before giving the proof, we first introduce the following result stated in [6].

Lemma 5. *([6]) For integers s and t with $2 \leq s \leq t$,*

$$rc(K_{s,t}) = min\{\lceil \sqrt[s]{t}\ \rceil,\ 4\}.$$

Theorem 3. *For a bipartite graph G, deciding whether $rc(G) = 2$ can be solved in polynomial time.*

Proof. Obviously if G is not a complete bipartite graph, there must exist two nonadjacent vertices x and y in the different parts of G. But then the distance $d(x, y)$ must be at least 3. We know that $d(x, y) \leq diam(G) \leq rc(G)$. It follows that $rc(G) \neq 2$. Therefore, only when G is a complete bipartite graph $K_{s,t}$ ($s \leq t$), it is possible that $rc(G) = 2$. If $s = 1$, then G is a star and $rc(G) = t$. Otherwise by Lemma 5, $rc(G) = min\{\lceil \sqrt[s]{t}\ \rceil,\ 4\}$. One only needs to check if $1 < t \leq 2^s$, which can be done by simple computations and comparisons. Moreover, it is clear that checking whether G is a complete bipartite graph can be done in polynomial time. The proof is complete. ∎

Then by Lemma 3 and Theorem 1, the following result is immediate.

Corollary 1. *Given a bipartite graph G, deciding if $rc(G) = 3$ is NP-Complete.*

For a general graph, the computational complexity of rainbow vertex-connection number has been completely solved.

Lemma 6. *([8]) Given a graph G, deciding whether $rvc(G) = 2$ is NP-Complete. Thus, computing $rvc(G)$ is NP-Hard.*

Later, Chen et al. obtained a more general result:

Lemma 7. *([7]) For every integer $k \geq 2$, to decide whether $rvc(G) \leq k$ is NP-Hard. Moreover, for any fixed integer $k \geq 2$, the problem belongs to NP-class, and therefore it is NP-Complete.*

Now, we restrict our attention to the bipartite graph and obtain similar results to those for rainbow connection number.

Theorem 4. *For a bipartite graph G, deciding whether $rvc(G) = 2$ can be solved in polynomial time.*

Proof. We know that computing the diameter $diam(G)$ of G can be done in polynomial time.

If $diam(G) \geq 4$, then $rvc(G) \geq diam(G) - 1 \geq 3$.

If $diam(G) = 2$, namely G is a complete bipartite graph, then $rvc(G) = 1$.

If $diam(G) = 3$, then $rvc(G) \geq diam(G) - 1 \geq 2$. Let $G = G[X, Y]$, where (X, Y) is a bipartition of G. We give a vertex-coloring c of G as follows: $c(u) = 1$ for $u \in X$ and $c(u) = 2$ for $u \in Y$. For any two vertices u and v in G, $d(u, v) \leq diam(G) = 3$. Furthermore, if u and v belong to the same part, then $d(u, v) = 2$ and if u and v belong to the different parts, then $d(u, v) = 1$ or 3. Obviously, in this coloring c, G is rainbow vertex-connected. So $rvc(G) \leq 2$ and it follows that $rvc(G) = 2$.

To summarize, when G is a bipartite graph, for deciding whether $rvc(G) = 2$, we only need to check whether $diam(G) = 3$. The proof is complete. ∎

Though for a bipartite graph G it is easy to check whether $rvc(G) = 2$, we will show that the problem of deciding whether $rvc(G) = 3$ is still hard. Before proceeding, let us recall the 3-vertex-coloring problem: given a graph G, is G 3-colorable, i.e., does there exist an assignment of at most 3 colors to the vertices of G such that no pair of adjacent vertices are colored the same. It is known that this 3-vertex-coloring problem is NP-Complete.

Theorem 5. *Given a bipartite graph G, deciding whether $rvc(G) = 3$ is NP-Complete.*

Proof. By Lemma 7, it is clear that the problem belongs to class NP. So it suffices to show that the 3-vertex-coloring problem is polynomially reducible to this problem.

Let $G = (V, E)$ be any instance of the 3-vertex-coloring problem. We will construct a corresponding bipartite graph G' such that $rvc(G') \leq 3$ if and only if G is 3-colorable.

Define the bipartite graph $G' = G'[X, Y]$, where the two parts X and Y are defined as follows:

$X = X_1 \cup X_2 \cup X_3$,
$X_1 = V(G) = \{v_i : i \in \{1, 2, \ldots, n\}\}$
$X_2 = \{x_{ij} : v_iv_j \notin E(G)\} \cup \{x_i : d(v_i) = n - 1\}$
$X_3 = \{t, u_1, u_2, z_1, z_2\}$;
$Y = Y_1 \cup Y_2 \cup Y_3$,
$Y_1 = \{v'_i : i \in \{1, 2, \ldots, n\}\}$
$Y_2 = \{y_{ij} : x_{ij} \in X_2\} \cup \{y_i : x_i \in X_2\}$
$Y_3 = \{a, b, c, w_1, w_2\}$.

Moreover, the edge set $E(G')$ is defined as follows:

$E(G') = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7$,
$E_1 = \{v_iv'_i : i \in \{1, 2, \ldots, n\}\}$
$E_2 = \{v'_ix_{ij}, v'_jx_{ij} : v_iv_j \notin E(G)\} \cup \{v'_ix_i : d(v_i) = n - 1\}$
$E_3 = \{x_{ij}y_{ij} : x_{ij} \in X_2\} \cup \{x_iy_i : x_i \in X_2\}$
$E_4 = \{av_i, bv_i, cv_i : i \in \{1, 2, \ldots, n\}\}$
$E_5 = \{w_1u, w_2u : u \in X_2\}$
$E_6 = \{z_1u, z_2u : u \in Y_2\}$
$E_7 = \{ta, tb, tc, tw_1, tw_2, u_1w_1, u_2w_2, z_1a, z_1b, z_1c, z_2a, z_2b, z_2c\}$.

Suppose that $rvc(G') \leq 3$ and c' is a rainbow 3-vertex-coloring of G'. Since $d(u_1) = d(u_2) = 1$, any rainbow vertex-connected $u_1 - u_2$ path must go through w_1 and w_2. So $c'(w_1) \neq c'(w_2)$ and without loss of generality, let $c'(w_1) = 1$ and $c'(w_2) = 2$. Note that any vertex y of Y_2 corresponds to the only vertex x of X_2 such that $xy \in E(G')$, and vice versa. For any vertex $y \in Y_2$, it can be seen that there is only one $u_1 - y$ path of length ≤ 4, namely u_1w_1xy, where $x \in X_2$ is the corresponding vertex of y. It follows that $c'(w_1) \neq c'(x)$. Similarly, there is only one $u_2 - y$ path of length ≤ 4, namely u_2w_2xy and so $c'(w_2) \neq c'(x)$. So $c'(x) \neq 1$ and 2, and we can obtain that $c'(x) = 3$, for each vertex x of X_2. Now we give the 3-vertex-coloring

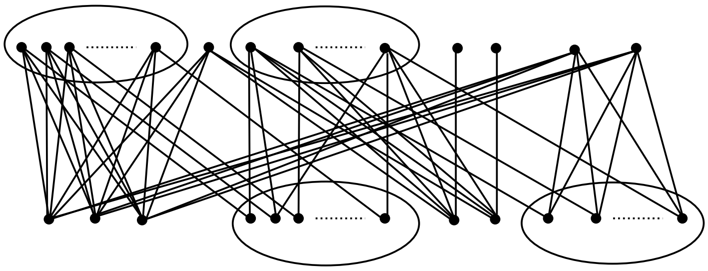

Figure 1: The bipartite graph G'.

c of G by $c(v_i) = c'(v_i)$, for $i \in \{1, 2, \dots, n\}$. For every $v_iv_j \in E(G)$, notice that there is no vertex adjacent to both v'_i and v'_j and so the rainbow vertex-connected path between v'_i and v'_j has length four. We can let the $v'_i - v'_j$ path be $v'_im_1m_2m_3v'_j$. We know that $N(Y_1) = X_1 \cup X_2$ and it is impossible that both m_1 and m_3 belong to X_2, since each vertex of X_2 has the same color 3. Moreover, the case that $m_1 \in X_1$ and $m_3 \in X_2$ is also impossible. It follows that m_1 must be $v_i \in X_1$ and m_3 must be $v_j \in X_1$. Now we can conclude that $c'(v_i) \neq c'(v_j)$, namely $c(v_i) \neq c(v_j)$ and so the 3-vertex-coloring c of G is proper.

In the other direction, assume that G is 3-colorable and let c be a proper 3-vertex-coloring of G. We define the vertex-coloring c' of G' as follows: $c'(v_i) = c(v_i)$, for $i \in \{1, 2, \dots, n\}$; $c'(x) = 3$, for each vertex $x \in X_2$; $c'(t) = 3$, $c'(z_1) = 1$, $c'(z_2) = 2$, $c'(a) = 1$, $c'(b) = 2$, $c'(c) = 3$, $c'(w_1) = 1$, $c'(w_2) = 2$; $c'(v) = 1$, for each remaining vertex v. We now show that the 3-vertex-coloring c' makes G' rainbow vertex-connected. Let u, v be any two vertices of G'.

Case 1: $u, v \in X_i$ or Y_i, where $i \in \{1, 2, 3\}$.

It can be seen that when $U = X_1, X_2, Y_2$ or Y_3, there always exists a vertex adjacent to every vertex in U. In this situation, any two vertices in U are clearly rainbow vertex-connected.

For $u, v \in X_3$, if $u = t$ and $v = u_i(z_i)$, then $tw_iu_i(taz_i)$ is a rainbow vertex-connected $u - v$ path, for $i \in \{1, 2\}$; if $u = u_1(u_2)$ and $v = z_i$, then $u_1w_1tbz_i(u_2w_2taz_i)$ is a rainbow vertex-connected $u - v$ path, for $i \in \{1, 2\}$; if $u = u_1$ and $v = u_2$, then $u_1w_1xw_2u_2$ is a rainbow vertex-connected $u - v$ path, where x is an arbitrary vertex in X_2; if $u = z_1$ and $v = z_2$, then z_1az_2

is a rainbow vertex-connected $u-v$ path.

For $u, v \in Y_1$, let $u = v'_i$ and $v = v'_j$. If $v_iv_j \notin E(G)$, then $v'_ix_{ij}v'_j$ is a rainbow vertex-connected $u-v$ path. If $v_iv_j \in E(G)$, by the definition of c', $c'(v_i) \neq c'(v_j)$. Choose a vertex $w \in \{a, b, c\}$ such that $c'(w) \neq c'(v_i) \neq c'(v_j)$ and then $v'_iv_iwv_jv'_j$ is a rainbow vertex-connected $u-v$ path.

Case 2: $u \in X_1$ and $v \in V(G')\backslash X_1$.

If $v \in X_2$, then $uatw_2v$ is a rainbow vertex-connected $u-v$ path.

For $v \in X_3$, if $v = t, z_1$ or z_2, then uav is a rainbow vertex-connected $u-v$ path; if $v = u_1(u_2)$, then $ubtw_1u_1(uatw_2u_2)$ is a rainbow vertex-connected $u-v$ path.

If $v \in Y_1$, let $v = v'_i$. u and v are obviously rainbow vertex-connected if $u = v_i$; otherwise, choose a vertex $w \in \{a, b, c\}$ such that $c'(w) \neq c'(v_i)$ and then uwv_iv is a rainbow vertex-connected $u-v$ path.

If $v \in Y_2$, then uaz_2v is a rainbow vertex-connected $u-v$ path.

For $v \in Y_3$, if $v = a, b$ or c, then u and v are adjacent and so rainbow vertex-connected; if $v = w_1$ or w_2, then $uatv$ is a rainbow vertex-connected $u-v$ path.

Case 3: $u \in X_2$ and $v \in V(G')\backslash X_1 \cup X_2$.

For $v \in X_3$, if $v = u_i(t)$, then $uw_iu_i(uw_1t)$ is a rainbow vertex-connected $u-v$ path, for $i \in \{1, 2\}$; if $v = z_1$ or z_2, then uyv is a rainbow vertex-connected $u-v$ path, where $y \in Y_2$ is the corresponding vertex of $u \in X_2$.

For $v \in Y_1$, notice that by the construction of G', each vertex v of Y_1 has at least one neighbor in X_2. If u and v are adjacent, then they are obviously rainbow vertex-connected; otherwise, uw_1n_vv is a rainbow vertex-connected $u-v$ path, where $n_v \in N(v) \cap X_2$.

For $v \in Y_2$, If u and v are adjacent, there is nothing to prove; otherwise, uw_1xv is a rainbow vertex-connected $u-v$ path, where $x \in X_2$ is the corresponding vertex of $v \in Y_2$.

For $v \in Y_3$, if $v = w_1$ or w_2, u and v are clearly rainbow vertex-connected; if $v = a, b$ or c, then uw_1tv is a rainbow vertex-connected $u-v$ path.

Case 4: $u \in X_3$ and $v \in Y_i$, for $i \in \{1, 2, 3\}$.

If $v \in Y_1$, let $v = v'_i$. If $u = t, z_1$ or z_2, choose a vertex $w \in \{a, b, c\}$ such that $c'(w) \neq c'(v_i)$ and then uwv_iv is a rainbow vertex-connected $u-v$ path; if $u = u_i$, then $u_iw_in_vv$ is a rainbow vertex-connected $u-v$ path, where $n_v \in N(v) \cap X_2$ and $i \in \{1, 2\}$.

For $v \in Y_2$, if $u = z_1$ or z_2, then u and v are adjacent; if $u = u_i(t)$, then $u_iw_ixv(taz_2v)$ is a rainbow vertex-connected $u-v$ path, where $x \in X_2$ is the corresponding vertex of $v \in Y_2$ and $i \in \{1, 2\}$.

For $v \in Y_3$, if $u = t$, then u and v are adjacent; if $u = u_i$ and $v = w_i$, then u and v are also adjacent, for $i \in \{1, 2\}$; if $u = u_1(u_2)$ and $v = a, b, c$ or $w_2(a, b, c$ or $w_1)$, then $u_1w_1tv(u_2w_2tv)$ a rainbow vertex-connected $u - v$ path; if $u = z_1$ or z_2 and $v = a, b$ or c, then u and v are adjacent; if $u = z_1$ or z_2 and $v = w_1$ or w_2, then $uatv$ is a rainbow vertex-connected $u - v$ path.

Case 5: $u \in Y_1$ and $v \in Y_2 \cup Y_3$.

Let $u \in Y_1$ be v_i'. If $v \in Y_2$, we first choose a vertex $z \in \{z_1, z_2\}$ such that $c'(z) \neq c'(v_i)$ and then choose a vertex $w \in \{a, b, c\}$ such that $c'(w) \neq c'(v_i)$ and $c'(w) \neq c'(z)$. Now $v_i'v_iwzv$ is a rainbow vertex-connected $u - v$ path;

For $v \in Y_3$, if $v = a, b$ or c, then $v_i'v_iv$ is a rainbow vertex-connected $u - v$ path; if $v = w_1$ or w_2, then un_uv is a rainbow vertex-connected $u - v$ path, where $n_u \in N(u) \cap X_2$.

Case 6: $u \in Y_2$ and $v \in Y_3$.

If $v = a, b$ or c, then uz_1v is a rainbow vertex-connected $u - v$ path; if $v = w_1$ or w_2, then uz_1btv is a rainbow vertex-connected $u - v$ path.

We have considered all the cases and so c' is a rainbow 3-vertex-coloring of G'. The proof is complete. ∎

As shown in the proof of Theorem 1, given an edge-coloring of a graph, if the number of colors is constant, then we can verify whether the colored graph is rainbow connected in polynomial time. However, in [4], Chakraborty et al. showed that if the coloring is arbitrary, the problem becomes NP-Complete.

Lemma 8. *([4]) The following problem is NP-Complete: given an edge-colored graph G, check whether the given coloring makes G rainbow connected.*

Now we prove that even when G is bipartite, the problem is still NP-Complete.

Theorem 6. *Given an edge-colored bipartite graph G, checking whether the given coloring makes G rainbow connected is NP-Complete.*

Proof. By Lemma 8, it will suffice by showing a polynomial reduction from the problem in Lemma 8.

Given a graph $G = (V, E)$ and an edge-coloring c of G, we will construct an edge-colored bipartite graph G' such that G is rainbow connected if and only if G' is rainbow connected.

Now for each edge $e \in E(G)$, subdivide e by a new vertex v_e. The obtained graph is exactly G' and (X, Y) is a bipartition of G', where $X =$

$V(G)$ and $Y = \{v_e : e \in E(G)\}$. Then the edge-coloring c' of G' is defined by for each edge $e = v_iv_j \in E(G)$ $(i \leq j)$, $c'(v_iv_e) = c(e)$ and $c'(v_jv_e) = l_e$, where l_e is a new color and different from the colors used in c and if $e \neq e'$, then $l_e \neq l_{e'}$.

If c' is a rainbow coloring of G', then any two vertices u and v are connected by a rainbow path $P'_{u,v}$, including every pair of vertices in $X = V(G)$. Clearly, by contracting edges which are assigned new colors, $P'_{u,v}$ can be converted to a rainbow path $P_{u,v}$ of G (with respect to c), where $u, v \in V(G)$. It follows that the coloring c makes G rainbow connected.

To prove the other direction, assume that for every two vertices v_t and $v_{t'}$ of G, there always exists a rainbow path $P_{v_tv_{t'}} = v_tv_{t_1}v_{t_2}\ldots v_{t'}$. Now for each pair $(v_t, v_{t'})$ of vertices in $V(G')$, if $v_t, v_{t'} \in X = V(G)$, then $P'_{v_tv_{t'}} = v_tv_{e_{m_1}}v_{t_1}v_{e_{m_2}}v_{t_2}\ldots v_{e_{m_j}}v_{t'}$ is a rainbow path in G', where the vertex $v_{e_{m_i}}$ subdivides the edge $e_{m_i} = v_{t_{i-1}}v_{t_i}$ of G, $i \in \{1, \ldots, j\}$ (when $i = 1$, the edge is $v_tv_{t_1}$ and when $i = j$, the edge is $v_{t_{j-1}}v_{t'}$). If $v_t, v_{t'} \in Y$, then there exist two edges $e_1 = v_{i_1}v_{j_1}$ and $e_2 = v_{i_2}v_{j_2}$ ($i_1 \leq j_1$ and $i_2 \leq j_2$) such that in G' v_t and $v_{t'}$ subdivide e_1 and e_2, respectively. Since $v_{j_1}, v_{j_2} \in X = V(G)$, we can find a rainbow path $P'_{v_{j_1}v_{j_2}}$ in G'. If $v_t \in P'_{v_{j_1}v_{j_2}}$, then delete the edge $v_{j_1}v_t$ from the path $P'_{v_{j_1}v_{j_2}}$. Otherwise, add the edge $v_tv_{j_1}$ to the path $P'_{v_{j_1}v_{j_2}}$. Similarly, if $v_{t'} \in P'_{v_{j_1}v_{j_2}}$, then delete the edge $v_{t'}v_{j_2}$ from the path $P'_{v_{j_1}v_{j_2}}$. Otherwise, add the edge $v_{j_2}v_{t'}$ to the path $P'_{v_{j_1}v_{j_2}}$. Then $P'_{v_{j_1}v_{j_2}}$ can always be converted to a rainbow $v_t - v_{t'}$ path. The proof of the case that $v_t \in X$ and $v_{t'} \in Y$ is similar. Therefore G' is rainbow connected with respect to c'. The proof is complete. ∎

For the rainbow vertex-connection, the conclusions are similar.

Lemma 9. *([8]) The following problem is NP-Complete: given a vertex-colored graph G, check whether the given coloring makes G rainbow vertex-connected.*

Similarly, we can also prove that even when G is bipartite, the problem is still NP-Complete.

Theorem 7. *Given a vertex-colored bipartite graph G, checking whether the given coloring makes G rainbow vertex-connected is NP-Complete.*

Proof. By Lemmas 8 and 9, it will suffice by showing a polynomial reduction from the problem in Lemma 8.

Given a graph $G = (V, E)$ and an edge-coloring c of G, we let G' be obtained by subdividing every edge of G. Then we define a vertex-coloring c' of G' as follows. Each vertex of G' that comes from subdividing an edge e of G gets the color $c(e)$. Each vertex of G' that comes from a vertex of G, gets a unique distinct color. It is easy to see that c' makes G' rainbow vertex-connected if and only of c makes G rainbow connected. The proof is similar to that of Theorem 6. ∎

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