Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. x No. x (201x), pp. xx-xx.

www.combinatorics.ir

www.ui.ac.ir

05C20, 05C50

SKEW EQUIENERGETIC DIGRAPHS

HARISHCHANDRA S. RAMANE*, K. CHANNEGOWDA NANDEESH,
IVAN GUTMAN AND XUELIANG LI

Communicated by

Abstract. Let D be a digraph with skew-adjacency matrix $S(D)$. The skew energy of D is defined as the sum of the norms of all eigenvalues of $S(D)$. Two digraphs are said to be skew equienergetic if their skew energies are equal. We establish an expression for the characteristic polynomial of the skew adjacency matrix of the join of two digraphs, and for the respective skew energy, and thereby construct non-cospectral, skew equienergetic digraphs on n vertices, for all $n \geq 6$. Thus we arrive at the solution of some open problems proposed in [X. Li, H. Lian, A suvey on the skew energy of oriented graphs, arXiv:1304.5707].

1. Introduction

Let D be a digraph with vertex set $V(D) = \{v_1, v_2, ..., v_n\}$ and arc set $\Gamma(D)$. The skew-adjacency matrix of D is the $n \times n$ matrix $S(D) = [s_{ij}]$ in which $s_{ij} = 1$ if (v_i, v_j) is an arc of D, $s_{ij} = -1$, if (v_j, v_i) is an arc of D, and $s_{ij} = 0$, otherwise. The characteristic polynomial of $S(D)$ is defined as $\phi_s(D : \lambda) = det(\lambda I - S(D))$, where I is an identity matrix of order n. The eigenvalues of $S(D)$ are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$. These are all pure imaginary or zeros, since $S(D)$ is skew symmetric. Two digraphs are said to be cospectral if they have same eigenvalues. The skew energy of the digraph D, denoted by $E_s(D)$, is defined as the sum of the norms of all eigenvalues of $S(D)$, that is,

$$E_s(D) = \sum_{i=1}^{n} |\lambda_i| \,.$$

MSC(2010): Primary: 05C20; Secondary: 05C50.

Keywords: energy of graph, skew energy, skew equienergetic digraphs.

Received: 19 December 2014, Accepted: dd mmmm yyyy.

*Corresponding author.

Two digraphs D_1 and D_2 are said to be skew equienergetic if $E_s(D_1) = E_s(D_2)$. If two digraphs are cospectral, then in a trivial manner, they are skew equienergetic. Therefore, in what follows, we are interested in finding skew equienergetic non-cospectral digraphs.

The energy of a simple undirected graph was introduced by one of the present authors [8]; for details of the theory of graph energy see [9, 20] and the references cited therein. In theoretical chemistry, the energy of a given molecular graph is related to the total π-electron energy of the molecule represented by that graph [10]. Recently, other graph energies were considered, such as the Laplacian energy [11], signless Laplacian energy [2], incidence energy [15], distance energy [14], and skew energy [1].

Adiga et al. [1] introduced the concept of skew energy of a simple digraph and obtained bounds for it. They have also calculated the skew energy of directed trees and directed cycles. For other results on the skew energy, see [3, 4, 5, 6, 7, 12, 13, 16, 21, 19, 22, 23, 24, 25, 26] and the survey [18].

Adiga et al. [1] showed that the skew energy of an oriented tree is independent of its orientation. With this, Li and Lian [18] proposed the following problem:

Problem 1. [18] How to construct families of oriented graphs such that they have equal skew energy, but they do not have the same spectra?

In this paper we establish expressions for the characteristic polynomial of the skew adjacency matrix of a join of digraphs and for the skew energy of join of digraphs. Further, we construct non-cospectral skew equienergetic digraphs on n vertices for all $n \geq 6$. By this we get the complete solution of the Problem 1 from [18].

2. Spectra and skew energy of join of digraphs

The indegree of a vertex u, denoted by $id_D(u)$ in a digraph D is the number of arcs coming to u. The outdegree of u, denoted by $od_D(u)$ is the number of arcs going out from u.

Definition: The join of a digraph D_1 to D_2, denoted by $D_1 \to D_2$, is a graph obtained from D_1 and D_2 by adding an arcs from each vertex of D_1 to all vertices of D_2 . An example is depicted in Fig. 1.

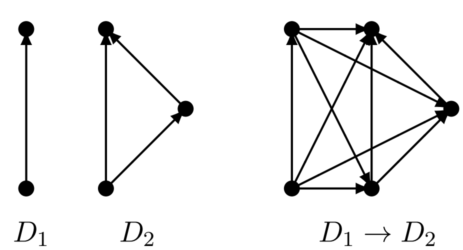

Fig.1

Theorem 2.1. *Let for $i = 1, 2$, D_i be a digraph on n_i vertices and $id_{D_i}(u) = od_{D_i}(u)$ for all $u \in V(D_i)$. Then the characteristic polynomial of the skew adjacency matrix of $D_1 \to D_2$ is*

$$\phi_s(D_1 \to D_2 : \lambda) = \frac{\lambda^2 + n_1 n_2}{\lambda^2} \phi_s(D_1 : \lambda) \phi_s(D_2 : \lambda)\,. \tag{2.1}$$

Proof.

$$\phi_s(D_1 \to D_2 : \lambda) = \det(\lambda I - S(D_1 \to D_2)) = \begin{vmatrix} \lambda I_{n_1} - S(D_1) & -J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & \lambda I_{n_2} - S(D_2) \end{vmatrix} \tag{2.2}$$

where J is a matrix whose all entries are equal to one.

The determinant (2.2) can be written as

$$\begin{vmatrix} \lambda & -s_{12} & \dots & -s_{1n_1} & -1 & -1 & \dots & -1 \\ -s_{21} & \lambda & \dots & -s_{2n_1} & -1 & -1 & \dots & -1 \\ \vdots & & \vdots & & & & \vdots & \\ -s_{n_11} & -s_{n_12} & \dots & \lambda & -1 & -1 & \dots & -1 \\ 1 & 1 & \dots & 1 & \lambda & -s'_{12} & \dots & -s'_{1n_2} \\ 1 & 1 & \dots & 1 & -s'_{21} & \lambda & \dots & -s'_{2n_2} \\ \vdots & & \vdots & & & & \vdots & \\ 1 & 1 & \dots & 1 & -s'_{n_21} & -s'_{n_22} & \dots & \lambda \end{vmatrix} \tag{2.3}$$

where s_{ij} is the (i, j)-th entry in D_1, $i, j = 1, 2, \dots, n_1$ and s'_{ij} is the (i, j)-th entry in D_2, $i, j = 1, 2, \dots, n_2$.

Since $id_{D_i}(u) = od_{D_i}(u)$ for all $u \in V(D_i)$, $i = 1, 2$, it easily follows that

$$\sum_{j=1}^{n_1} s_{ij} = 0 \qquad \text{for} \quad i = 1, 2, \dots, n_1 \tag{2.4}$$

$$\sum_{j=1}^{n_2} s'_{ij} = 0 \qquad \text{for} \quad i = 1, 2, \dots, n_2\,. \tag{2.5}$$

We now perform the number of operations on the determinant (2.3).

Subtract the row $(n_1 + 1)$ from the rows $(n_1 + 2), (n_1 + 3), \dots, (n_1 + n_2)$ of (2.3) to obtain (2.6):

$$\begin{vmatrix} \lambda & -s_{12} & \dots & -s_{1n_1} & -1 & -1 & \dots & -1 \\ -s_{21} & \lambda & \dots & -s_{2n_1} & -1 & -1 & \dots & -1 \\ \vdots & & \vdots & & & & \vdots & \\ -s_{n_11} & -s_{n_12} & \dots & \lambda & -1 & -1 & \dots & -1 \\ 1 & 1 & \dots & 1 & \lambda & -s'_{12} & \dots & -s'_{1n_2} \\ 0 & 0 & \dots & 0 & -s'_{21} - \lambda & \lambda + s'_{12} & \dots & -s'_{2n_2} + s'_{1n_2} \\ \vdots & & \vdots & & & & \vdots & \\ 0 & 0 & \dots & 0 & -s'_{n_21} - \lambda & -s'_{n_22} + s'_{12} & \dots & \lambda + s'_{1n_2} \end{vmatrix}\,. \tag{2.6}$$

Adding the columns $(n_1+2), (n_1+3), \ldots, (n_1+n_2)$ to the column (n_1+1) of (2.6), using Eq. (2.5), and noting that $s'_{ij} = -s'_{ji}$ we arrive at the determinant (2.7):

$$\tag{2.7} \begin{vmatrix} \lambda & -s_{12} & \ldots & -s_{1n_1} & -n_2 & -1 & \ldots & -1 \\ -s_{21} & \lambda & \ldots & -s_{2n_1} & -n_2 & -1 & \ldots & -1 \\ \vdots & & \vdots & & & & \vdots & \\ -s_{n_11} & -s_{n_12} & \ldots & \lambda & -n_2 & -1 & \ldots & -1 \\ 1 & 1 & \ldots & 1 & \lambda & -s'_{12} & \ldots & -s'_{1n_2} \\ 0 & 0 & \ldots & 0 & 0 & \lambda + s'_{12} & \ldots & -s'_{2n_2} + s'_{1n_2} \\ \vdots & & \vdots & & & & \vdots & \\ 0 & 0 & \ldots & 0 & 0 & -s'_{n_22} + s'_{12} & \ldots & \lambda + s'_{1n_2} \end{vmatrix}$$

which is equal to (2.8):

$$\tag{2.8} \begin{vmatrix} \lambda & -s_{12} & \ldots & -s_{1n_1} & -n_2 \\ -s_{21} & \lambda & \ldots & -s_{2n_1} & -n_2 \\ \vdots & & \vdots & & \\ -s_{n_11} & -s_{n_12} & \ldots & \lambda & -n_2 \\ 1 & 1 & \ldots & 1 & \lambda \end{vmatrix} |B|$$

where

$$\tag{2.9} |B| = \begin{vmatrix} \lambda + s'_{12} & -s'_{23} + s'_{13} & \ldots & -s'_{2n_2} + s'_{1n_2} \\ -s'_{32} + s'_{12} & \lambda + s'_{13} & \ldots & -s'_{3n_2} + s'_{1n_2} \\ \vdots & & \vdots & \\ -s'_{n_22} + s'_{12} & -s'_{n_23} + s'_{13} & \ldots & \lambda + s'_{1n_2} \end{vmatrix} .$$

The first determinant in (2.8) is of order (n_1+1). Subtract the first row from the rows $2, 3, \ldots, n_1$, in (2.8) to obtain (2.10):

$$\tag{2.10} \begin{vmatrix} \lambda & -s_{12} & \ldots & -s_{1n_1} & -n_2 \\ -s_{21} - \lambda & \lambda + s_{12} & \ldots & -s_{2n_1} + s_{1n_1} & 0 \\ \vdots & & \vdots & & \\ -s_{n_11} - \lambda & -s_{n_12} + s_{12} & \ldots & \lambda + s_{1n_1} & 0 \\ 1 & 1 & \ldots & 1 & \lambda \end{vmatrix} |B| .$$

Adding columns $2, 3, \ldots, n_1$ to the first column of (2.10) and using Eq. (2.4) we get (2.11):

$$\left|\begin{array}{ccccc} \lambda & -s_{12} & \ldots & -s_{1n_1} & -n_2 \\ 0 & \lambda + s_{12} & \ldots & -s_{2n_1} + s_{1n_1} & 0 \\ \vdots & & \vdots & & \\ 0 & -s_{n_1 2} + s_{12} & \ldots & \lambda + s_{1n_1} & 0 \\ n_1 & 1 & \ldots & 1 & \lambda \end{array}\right| |B| \,. \tag{2.11}$$

Expand it along the first column to obtain (2.12):

$$\{\lambda\, \Delta_1 + (-1)^{n_1}\, n_1\, \Delta_2\}\, |B| \tag{2.12}$$

where

$$\Delta_1 = \left|\begin{array}{ccccc} \lambda + s_{12} & -s_{23} + s_{13} & \ldots & -s_{2n_1} + s_{1n_1} & 0 \\ -s_{32} + s_{12} & \lambda + s_{13} & \ldots & -s_{3n_1} + s_{1n_1} & 0 \\ \vdots & & \vdots & & \\ -s_{n_1 2} + s_{12} & -s_{n_1 3} + s_{13} & \ldots & \lambda + s_{1n_1} & 0 \\ 1 & 1 & \ldots & 1 & \lambda \end{array}\right|$$

and

$$\Delta_2 = \left|\begin{array}{ccccc} -s_{12} & -s_{13} & \ldots & -s_{1n_1} & -n_2 \\ \lambda + s_{12} & -s_{23} + s_{13} & \ldots & -s_{2n_1} + s_{1n_1} & 0 \\ -s_{32} + s_{12} & \lambda + s_{13} & \ldots & -s_{3n_1} + s_{1n_1} & 0 \\ \vdots & & \vdots & & \\ -s_{n_1 2} + s_{12} & -s_{n_1 3} + s_{13} & \ldots & \lambda + s_{1n_1} & 0 \end{array}\right| .$$

The expression (2.12) can be rewritten as

$$\{\lambda^2 |A| + (-1)^{n_1} n_1\, (-1)^{n_1} n_2 |A|\} |B| = (\lambda^2 - n_1\, n_2) |A| |B| \tag{2.13}$$

where

$$|A| = \left|\begin{array}{cccc} \lambda + s_{12} & -s_{23} + s_{13} & \ldots & -s_{2n_1} + s_{1n_1} \\ -s_{32} + s_{12} & \lambda + s_{13} & \ldots & -s_{3n_1} + s_{1n_1} \\ \vdots & & \vdots & \\ -s_{n_1 2} + s_{12} & -s_{n_1 3} + s_{13} & \ldots & \lambda + s_{1n_1} \end{array}\right| . \tag{2.14}$$

The determinant (2.14) can be written as

$$|A| = \frac{1}{\lambda} \left|\begin{array}{ccccc} \lambda & -s_{12} & -s_{13} & \ldots & -s_{1n_1} \\ 0 & \lambda + s_{12} & -s_{23} + s_{13} & \ldots & -s_{2n_1} + s_{1n_1} \\ 0 & -s_{32} + s_{12} & \lambda + s_{13} & \ldots & -s_{3n_1} + s_{1n_1} \\ \vdots & & & \vdots & \\ 0 & -s_{n_1 2} + s_{12} & -s_{n_1 3} + s_{13} & \ldots & \lambda + s_{1n_1} \end{array}\right| . \tag{2.15}$$

From Eq. (2.4), the sum of the i-th row in (2.15) is $\lambda + s_{i1}$ for $i = 2,3,\ldots,n_1$. Therefore, by subtracting the columns $2,3,\ldots,n_1$ of (2.15) from the first column, we obtain (2.16):

$$|A| = \frac{1}{\lambda}\begin{vmatrix} \lambda & -s_{12} & -s_{13} & \ldots & -s_{1n_1} \\ -\lambda - s_{21} & \lambda + s_{12} & -s_{23} + s_{13} & \ldots & -s_{2n_1} + s_{1n_1} \\ -\lambda - s_{31} & -s_{32} + s_{12} & \lambda + s_{13} & \ldots & -s_{3n_1} + s_{1n_1} \\ \vdots & & & \vdots & \\ -\lambda - s_{n_1 1} & -s_{n_1 2} + s_{12} & -s_{n_1 3} + s_{13} & \ldots & \lambda + s_{1n_1} \end{vmatrix}. \tag{2.16}$$

Add the first row of (2.16) to the rows $2,3,\ldots,n_1$ to obtain (2.17):

$$|A| = \frac{1}{\lambda}\begin{vmatrix} \lambda & -s_{12} & -s_{13} & \ldots & -s_{1n_1} \\ -s_{21} & \lambda & -s_{23} & \ldots & -s_{2n_1} \\ -s_{31} & -s_{32} & \lambda & \ldots & -s_{3n_1} \\ \vdots & & & \vdots & \\ -s_{n_1 1} & -s_{n_1 2} & -s_{n_1 3} & \ldots & \lambda \end{vmatrix} = \frac{1}{\lambda}\,\phi_s(D_1 : \lambda)\,. \tag{2.17}$$

In a similar manner we can show that from (2.9) it follows

$$|B| = \frac{1}{\lambda}\,\phi_s(D_2 : \lambda)\,. \tag{2.18}$$

Substituting (2.17) and (2.18) back into (2.13) gives Eq. (2.1). □

Let W be a subset of the vertex set $V(D)$ of a digraph D and $\overline{W} = V(D) \setminus W$. Let D' be the digraph obtained from D by reversing the directions of all arcs between W and $\overline{W}$. As usual [18], we say that D' has been obtained from D by switching with respect to W.

Two digraphs D and D' are said to be switching equivalent if D' can be obtained from D by a sequence of switching. Hou, Shen and Zhang [13] showed that if D and D' are switching equivalent, then D and D' have the same spectra and equal skew energy. Thus for any two digraphs D_1 and D_2, the joins $D_1 \to D_2$ and $D_2 \to D_1$ are switching equivalent and therefore have same characteristic polynomials and $E_s(D_1 \to D_2) = E_s(D_2 \to D_1)$.

Theorem 2.2. *Let for $i = 1,2$, D_i be a digraph on n_i vertices and $id_{D_i}(u) = od_{D_i}(u)$ for all $u \in V(D_i)$. Then*

$$E_s(D_1 \to D_2) = E_s(D_1) + E_s(D_2) + 2\sqrt{n_1\,n_2}\,. \tag{2.19}$$

Proof. From Theorem 2.1,

$$\phi_s(D_1 \to D_2 : \lambda) = \frac{(\lambda^2 + n_1\,n_2)}{\lambda^2}\,\phi_s(D_1 : \lambda)\,\phi_s(D_2 : \lambda)$$

which yields

$$\lambda^2\,\phi_s(D_1 \to D_2 : \lambda) = (\lambda^2 + n_1\,n_2)\,\phi_s(D_1 : \lambda)\,\phi_s(D_2 : \lambda)\,.$$

Let

$$P_1(\lambda) = \lambda^2 \, \phi_s(D_1 \to D_2 : \lambda)$$

and

$$P_2(\lambda) = (\lambda^2 + n_1\, n_2)\, \phi_s(D_1 : \lambda)\, \phi_s(D_2 : \lambda)\,.$$

The roots of the equation $P_1(\lambda) = 0$ are 0 (2 times) and the eigenvalues of $S(D_1 \to D_2)$. Therefore the sum of the absolute values of the roots of $P_1(\lambda) = 0$ is

$$E_s(D_1 \to D_2)\,. \tag{2.20}$$

The roots of $P_2(\lambda) = 0$ are $\sqrt{n_1\, n_2}$, $-\sqrt{n_1\, n_2}$, and the eigenvalues of $S(D_1)$ and $S(D_2)$. Therefore the sum of the absolute values of the roots of $P_2(\lambda) = 0$ is

$$E_s(D_1) + E_s(D_2) + 2\sqrt{n_1\, n_2}\,. \tag{2.21}$$

Since $P_1(\lambda) = P_2(\lambda)$, equating Eqs. (2.20) and (2.21) we arrive at Eq. (2.19). □

Let $\overline{K_n}$ be the totally disconncted graph on n vertices, that is a graph without edges.

Corollary 2.3. $E_s(\overline{K_p} \to \overline{K_q}) = 2\sqrt{pq}$.

Corollary 2.4. *If H_1 and H_2 are non cospectral, skew-equienergetic digraphs on n vertices such that $id_{H_i}(u) = od_{H_i}(u)$ for all $u \in V(H_i)$, $i = 1, 2$, then for any digraph G with $id_G(v) = od_G(v)$, $v \in V(G)$, $E_s(H_1 \to G) = E_s(H_2 \to G)$.*

3. Construction of skew equienergetic digraphs

Theorem 3.1. *There exist pairs of non cospectral, skew equienergetic digraphs on n vertices for all $n \geq 6$.*

Proof. Consider the digraphs D_a and D_b as depicted in Fig. 2.

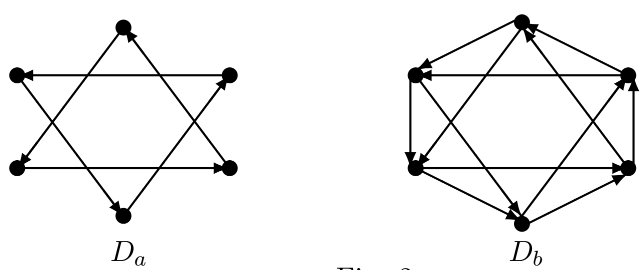

Fig. 2

By direct computation,

$$\phi_s(D_a : \lambda) = \lambda^4\, (\lambda^2 + 12) \tag{3.1}$$

$$\phi_s(D_b : \lambda) = \lambda^2\, (\lambda^4 + 6\lambda^2 + 9)\,. \tag{3.2}$$

Both D_a and D_b are digraphs on 6 vertices. Also $id_{D_i}(u) = od_{D_i}(u)$, for all $u \in D_i$, $i = a, b$, and $E_s(D_a) = E_s(D_b) = 4\sqrt{3}$.

Let D be any digraph on $p \geq 1$ vertices and $id_D(u) = od_D(u)$, $u \in V(D)$. Then by Theorem 2.2,

$$E_s(D_a \to D) = E_s(D_b \to D) = 4\sqrt{3} + E_s(D) + 2\sqrt{6p}.$$

Thus, $D_a \to D$ and $D_b \to D$ are skew equienergetic. By Eqs. (3.1) and (3.2), D_a and D_b are non-cospectral. Then by Theorem 2.1, $D_a \to D$ and $D_b \to D$ are also non cospectral. Further $D_a \to D$ and $D_b \to D$ possesses equal number of vertices $n = 6 + p$, $p = 0, 1, 2, \ldots$. □

Let $\overline{K_p}$ be the totally disconnected digraph on p vertices. In this $id_{\overline{K_p}}(u) = od_{\overline{K_p}}(u) = 0$ for all $u \in V(\overline{K_p})$ and $\phi_s(\overline{K_p} : \lambda) = \lambda^p$. Therefore $E_s(\overline{K_p}) = 0$. Using this in Theorem 2.2 we have following result.

Corollary 3.2. *If D_a and D_b are the digraphs shown in Fig. 2, then*

$$E_s(D_a \to \overline{K_p}) = E_s(D_b \to \overline{K_p}) = 4\sqrt{3} + 2\sqrt{6p} \quad , \quad p \geq 0.$$

4. Conclusions

Using Corollary 2.4, it is easy to construct a pair of non cospectral, skew equienergetic digraphs. In particular by means of Theorem 3.1 and Corollary 3.2 pairs of non-cospectral, skew equienergetic n-vertex digraphs can be constructed for all $n \geq 6$. Thus Problem 1. from [18] has been completely solved.

Acknowledgments

The authors HSR and KCN are thankful to the University Grants Commission (UGC), Govt. of India, for support through research grant under UPE FAR-II grant No. F 14-3/2012 (NS/PE).

References

[1] C. Adiga, R. Balakrishnan and W. So, The skew energy of a digraph, *Linear Algebra Appl.*, **432** (2010) 1825–1835.

[2] N. Abreu, D. M. Cardoso, I. Gutman, E. A. Martins and M. Robbiano, Bounds for the signless Laplacian energy, *Linear Algebra Appl.*, **435** (2011) 2365–2374.

[3] X. Chen, X. Li and H. Lian, The skew energy of random oriented graphs, *Linear Algebra Appl.*, **438** (2013) 4547–4556.

[4] X. Chen, X. Li, and H. Lian, 4-Regular oriented graphs with optimum skew energy, *Linear Algebra Appl.*, **439** (2013) 2948–2960.

[5] S. Gong, X. Li and G. Xu, On oriented graphs with minimal skew energy, arXiv:1304.2458.

[6] S. Gong and G. Xu, 3-Regular digraphs with optimum skew energy, *Linear Algebra Appl.*, **436** (2012) 465–471.

[7] S. Gong, W. Zhang and G. Xu, 4-Regular oriented graphs with optimum skew energies, *Eur. J. Combin.*, **36** (2014) 77–85.

[8] I. Gutman, The energy of a graph, *Ber. Math. Stat. Sekt. Forschungsz. Graz*, **103** (1978) 1–22.

[9] I. Gutman, X. Li and J. Zhang, Graph energy, in: M. Dehmer and F. Emmert-Streib (Eds.) *Analysis of Complex Networks: From Biology to Linguistics*, Wiley/VCH, Weinheim, 2009, pp. 145–174.

[10] I. Gutman and O. E. Polansky, *Mathematical concepts in organic chemistry*, Springer, Berlin, 1986.

[11] I. Gutman and B. Zhou, Laplacian energy of a graph, *Linear Algebra Appl.*, **414** (2006) 29–37.

[12] J. He and T. Z. Huang, Note on the skew energy of oriented graphs, *Trans. Comb.*, **4** No. 1 (2015) 13–17.

[13] Y. Hou, X. Shen and C. Zhang, Oriented unicyclic graphs with extremal skew energy, arXiv:1108.6229.

[14] G. Indulal, I. Gutman and A. Vijaykumar, On the distance energy of a graph, *MATCH Commun. Math. Comput. Chem.*, **60** (2008) 461–472.

[15] M. R. Jooyandeh, D. Kiani and M. Mirzakhah, Incidence energy of a graph, *MATCH Commun. Math. Comput. Chem.*, **62** (2009) 561–572.

[16] J. Li, X. Li and H. Lian, Extremal skew energy of digraphs with no even cycles, *Trans. Comb.*, **3** No 1 (2014) 37–49.

[17] H. Lian and X. Li, Skew–spectra and skew energy of various products of graphs, *Trans. Comb.*, **4** No. 2 (2015) 13–21.

[18] X. Li and H. Lian, A survey on the skew energy of oriented graphs, arXiv:1304.5707.

[19] X. Li and H. Lian, Skew energy of graph products and skew weighing matrix, arXiv:1305.7305.

[20] X. Li, Y. Shi and I. Gutman, *Graph Energy*, Springer, New York, 2012.

[21] H. Lian and X. Li, Skew–spectra and skew energy of various products of graphs, *Trans. Comb.*, **4** No. 2 (2015) 13–21.

[22] H. Mao and Y. Hou, Minimal skew energy of unicyclic graphs with prescribed girth and pendent vertices, *J. Hunan Normal Univ.*, to appear.

[23] X. Shen, Y. Hou and C. Zhang, Bicyclic digraphs with extremal skew energy, *El. J. Linear Algebra*, **23** (2012) 340–355.

[24] G. X. Tian, On the skew energy of orientation of hypercubes, *Linear Algebra Appl.*, **435** (2011) 2140–2149.

[25] X. Yang, S. Gong and G. Xu, Minimal skew energy of oriented unicyclic graphs with fixed diameter, *J. Inequal. Appl.*, **418** (2013) 1–11.

[26] J. Zhu, Oriented unicyclic graphs with the first $\lfloor \frac{n-9}{2} \rfloor$ largest skew energies, *Linear Algebra Appl.*, **437** (2012) 2630–2649.

Harishchandra S. Ramane

Department of Mathematics, Karnatak University, Dharwad - 580003, India

Email: hsramane@yahoo.com

K. Channegowda Nandeesh

Department of Mathematics, Karnatak University, Dharwad - 580003, India

Email: nandeeshkc@yahoo.com

Ivan Gutman

Faculty of Science, University of Kragujevac, 34000 Kragujevac, Serbia

Email: gutman@kg.ac.rs

Xueliang Li

Center for Combinatorics and LPMC-TJKLC, Nankai University, Tianjin 300071, P.R. China

Email: lxl@nankai.edu.cn