Sharp Upper Bounds for the Balaban Index of Bicyclic Graphs

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Abstract

A number of topological indices have recently been incorporated as distance-based descriptors. The Balaban index, a useful distance-based descriptor in Chemometrics, introduced by A.T. Balaban, is known to depend both on the number of nodes and topology of graphs. In this paper, we study the sharp upper bounds of Balaban index and Sum-Balaban index among all bicyclic graphs, by using some graph transformations. The graphs attaining these bounds are also characterized.

1 Introduction

Thousands of topological indices are introduced to characterize the physical-chemical properties of molecules [43]. These topological indices are mainly divided into three types: degree-based indices, distance-based indices and spectrum-based indices. Degree-based indices contain (general) Randić index [38], (general) zeroth order Randić index [38], Zagreb index [27], connective eccentricity index [48, 50] and so on. Distance-based indices [46] include the Balaban index [3, 4], the Wiener index [21], Wiener polarity index [16, 40], the Szeged index [26], the Kirchhoff index [20, 23, 24], the Harary index [1] and so forth [17]. Eigenvalues of graphs [34, 35, 49, 51], various of graph energies [7, 30, 31, 33, 39], Estrada index [28] and HOMO-LUMO index [32, 37] belong to spectrum-based indices. Actually, there are also some topological indices defined based on both degrees and distances [2,8,45], such as Gutman index [25], degree distance [19], graph entropies [6, 9, 10]. The main contribution of this paper is to prove bounds for the Balaban index and the Sum-Balaban index.

We first start by providing graph-theoretical preliminaries [29]. Let $G = (V, E)$ be a simple graph. The distance between vertices u and v is denoted by $d_G(u, v)$. Let $D_G(u) = \sum_{v \in V(G)} d_G(u, v)$, which is the distance sum of vertex u in G. Suppose $|V| := n,\ |E| := m$. The *cyclomatic number* μ of G is the minimum number of edges that must be removed from G in order to transform it to an acyclic graph. It is known that $\mu = m - n + 1$, see [43]. A bicyclic graph is a graph with $\mu = 2$. Denote by $deg(v)$ the degree of vertex v.

The *Balaban index* of a connected graph $G = (V, E)$ is defined as

$$J(G) = \frac{m}{\mu + 1} \sum_{uv \in E} \frac{1}{\sqrt{D_G(u) D_G(v)}}.$$

It has been proposed by Balaban [3,4] in 1982, which is also called the *average distance-sum connectivity index* or *Balaban J index*. Furthermore, Balaban et al. [5] proposed the concept of the *Sum-Balaban index* of a connected graph G. It has been defined by

$$SJ(G) = \frac{m}{\mu + 1} \sum_{uv \in E} \frac{1}{\sqrt{D_G(u) + D_G(v)}}.$$

We emphasize that many mathematical properties and results on the Balaban index and the Sum-Balaban index of trees and unicyclic graphs have been achieved, see [11–14, 18, 36, 41, 44, 47, 52]. he following theorem is an example.

Theorem 1.1 ([11–14, 36, 41, 44]) *If T is a tree with $n \geq 2$ vertices, then*

$$J(P_n) \leq J(T) \leq J(S_n), \;\; SJ(P_n) \leq SJ(T) \leq SJ(S_n),$$

and the left (right) equality holds if and only if $T \cong P_n$ ($T \cong S_n$), where P_n and S_n are the path graph and the star graph on n vertices, respectively.

In this paper, we study sharp upper bounds of the Balaban index and Sum-Balaban index by using all bicyclic graphs. We characterize the bicyclic graphs with the maximum Balaban index (Sum-Balaban index) among all bicyclic graphs, two cycles of those are with n_1, n_2 vertices, respectively. A plausible reason for using these graphs relates to the definition of the Balaban index and Sum-Balaban. As these indices are based on distances in a graph, special cyclic graphs are easy to calculate by using these quantities. Another reason is that mathematical properties of the Balaban index have been explored for trees extensively [11–14, 36, 41, 44]. The bounds we have proved help to better understand the mathematical framework that has already been proven useful, see, e.g., [3, 12, 15, 42]. Therefore, we now pursue studying mathematical properties of these quantities by using graphs containing cycles.

2 Lemmas

First of all, we list some useful lemmas.

Lemma 2.1 ([14]) *Let $a, a', b, b', w, x, y, z \in R^+$ such that $\frac{b}{x} \geq \frac{a}{w}, \frac{b'}{y} \geq \frac{a'}{z}, w \geq x, z \geq y$. Then $\frac{1}{\sqrt{(w+a)(z+a')}} + \frac{1}{\sqrt{xy}} \geq \frac{1}{\sqrt{wz}} + \frac{1}{\sqrt{(x+b)(y+b')}}$, the equality holds if and only if $b = a, b' = a', w = x, z = y$.*

Lemma 2.2 ([14]) *Let $a, x, y \in R^+$ such that $x \geq y + a$. Then $\frac{1}{\sqrt{xy}} \geq \frac{1}{\sqrt{(x-a)(y+a)}}$, and the equality holds if and only if $x = y + a$.*

Lemma 2.3 *Let $x_1, x_2, y_1, y_2 \in R^+$ such that $x_1 > y_1,\ x_2 - x_1 = y_2 - y_1 > 0$. Then* $\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{y_2}} < \frac{1}{\sqrt{x_2}} + \frac{1}{\sqrt{y_1}}$.

Proof. Let $s = x_2 - x_1 = y_2 - y_1 > 0$. Define a function $f(z) = \frac{1}{\sqrt{z}} - \frac{1}{\sqrt{z+s}}$. It is easy to verify that $f(z)$ is a decreasing function of z. Since $x_1 > y_1$, we have $\frac{1}{\sqrt{x_1}} - \frac{1}{\sqrt{x_1+s}} < \frac{1}{\sqrt{y_1}} - \frac{1}{\sqrt{y_1+s}}$. ∎

Now we recall a useful graph transformation introduced in [11] which we use extensively in our paper.

Definition 2.1 (The edge-lifting transformation) *Let G_1, G_2 be two graphs with $n_1 \geq 2,\ n_2 \geq 2$ vertices, respectively. Suppose $u_0 \in G_1$ and $v_0 \in G_2$. If G is the graph obtained from $G_1,\ G_2$ by adding an edge between u_0 and v_0, and G' is the graph obtained by identifying u_0 and v_0 and adding a pendent edge to $u_0(v_0)$, then G' is called the edge-lifting transformation of G (see Figure 2.1).*

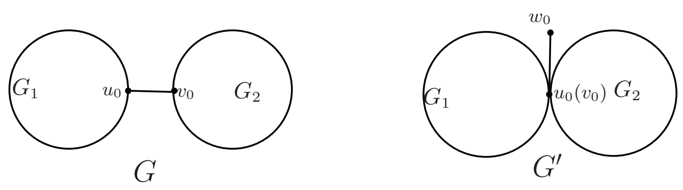

Figure 2.1: The edge-lifting transformation.

Lemma 2.4 ([11, 12]) *Let G' be the edge-lifting transformation of G. Then $J(G) < J(G')$ and $SJ(G) < JS(G')$.*

A rooted graph has one vertex called the root distinguished from the others [29]. Let $T_1, T_2, \ldots, T_k$ be k rooted trees with $|V(T_i)| \geq 2$ $(1 \leq i \leq k)$ and roots $u_1, u_2, \ldots, u_k$, respectively. Let C_r be a cycle with length $r(r \geq 3)$.

For $1 \leq k_1 \leq r_1 \leq n_1, 1 \leq k_2 \leq r_2 \leq n_2$, let $(G_1(n_1, r_1, k_1), G_2(n_2, r_2, k_2))$ be the bicyclic graph obtained from $C_{r_1}, C_{r_2}, T_1, T_2, \ldots, T_{k_1}, T'_1, T'_2, \ldots, T'_{k_2}$, by attaching k_1 rooted trees $T_1, T_2, \ldots, T_{k_1}$ to k_1 distinct vertices of C_{r_1} and attaching k_2 rooted trees $T'_1, T'_2, \ldots, T'_{k_1}$ to k_2 distinct vertices of C_{r_2}, where C_{r_1}, C_{r_2} are jointed.

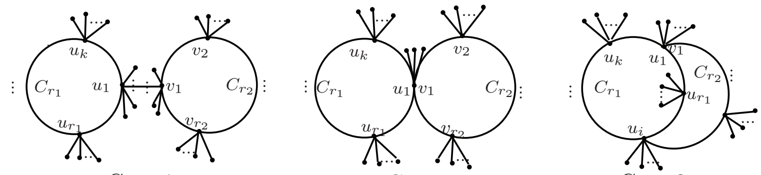

Figure 2.2: A graph $(G_1^*(n_1, r_1, k_1), G_2^*(n_2, r_2, k_2)) \in (\mathbb{G}_1^*(n_1, r_1, k_1), \mathbb{G}_2^*(n_2, r_2, k_2))$.

Let $\mathbb{S} = \{S | S$ is a rooted star with the center as its root$\}$. Let $(\mathbb{G}_1^*(n_1, r_1, k_1), \mathbb{G}_2^*(n_2, r_2, k_2))$ be the set of all bicyclic graphs obtained from C_{r_1}, C_{r_2} by attaching k_1, k_2 rooted stars in $\mathbb{S}$ to k_1, k_2 distinct vertices of C_{r_1}, C_{r_2}, respectively (see Figure 2.2). By Lemma 2.4, we repeat the edge-lifting transformation to the rooted trees of

$$(G_1(n_1, r_1, k_1), G_2(n_2, r_2, k_2)),$$

and then obtain the following result.

Lemma 2.5 *Let* $n_1, k_1, r_1, n_2, k_2, r_2$ *be positive integers with* $1 \leq k_1 \leq r_1, 3 \leq r_1 \leq n_1 - k_1, 1 \leq k_2 \leq r_2, 3 \leq r_2 \leq n_2 - k_2$, *and* $(G_1(n_1, r_1, k_1), G_2(n_2, r_2, k_2))$ *defined as above. Let* $(G_1^*(n_1, r_1, k_1), G_2^*(n_2, r_2, k_2)) \in (\mathbb{G}_1^*(n_1, r_1, k_1), \mathbb{G}_2^*(n_2, r_2, k_2))$ *be obtained from* $(G_1(n_1, r_1, k_1),\ G_2(n_2, r_2, k_2))$ *by repeating the edge-lifting transformations. Then*

$$J(G_1(n_1, r_1, k_1), G_2(n_2, r_2, k_2)) \leq J(G_1^*(n_1, r_1, k_1), G_2^*(n_2, r_2, k_2)),$$

$$SJ(G_1(n_1, r_1, k_1), G_2(n_2, r_2, k_2)) \leq SJ(G_1^*(n_1, r_1, k_1), G_2^*(n_2, r_2, k_2)),$$

and the equality holds if and only if

$$(G_1(n_1, r_1, k_1), G_2(n_2, r_2, k_2)) \cong (G_1^*(n_1, r_1, k_1), G_2^*(n_2, r_2, k_2)).$$

Actually, Figure 2.2 depicts the three cases of bicyclic graphs obtained by repeating the edge-lifting transformations. Obviously, Case 1 can be exchanged to Case 2 by the edge-lifting transformations, so there are only two cases in the edge-lifting transformations of bicyclic graphs.

In the following, we define a new transformation, which is called *branch transformation*.

Definition 2.2 (Branch transformation) *Let* $G = (G_1^*(n_1, r_1, k_1), G_2^*(n_2, r_2, k_2)) \in (\mathbb{G}_1^*(n_1, r_1, k_1), \mathbb{G}_2^*(n_2, r_2, k_2))$ *be defined as above. For convenience, let* $m = \lfloor \frac{r_1}{2} \rfloor$ *and* $n = \lfloor \frac{r_2}{2} \rfloor$.

If r_1, r_2 *are even, define* $C_{r_1} = u_1, \ldots, u_m v_m, \ldots, v_1, C_{r_2} = x_1, \ldots, x_n y_n, \ldots, y_1$*;*
if r_1, r_2 *are odd, define* $C_{r_1} = u_1, \ldots, u_{m+1} v_m, \ldots, v_1, C_{r_2} = x_1, \ldots, x_{n+1} y_n, \ldots, y_1$*;*
if r_1 *is even,* r_2 *is odd, define* $C_{r_1} = u_1, \ldots, u_m v_m, \ldots, v_1, C_{r_2} = x_1, \ldots, x_{n+1} y_n, \ldots, y_1$*;*
if r_1 *is odd,* r_2 *is even, define* $C_{r_1} = u_1, \ldots, u_{m+1} v_m, \ldots, v_1, C_{r_2} = x_1, \ldots, x_n y_n, \ldots, y_1$*.*
The graph G' *is obtained from* G *by deleting the pendent edge* $v_i w$ *and adding a pendent edge* $u_i w$ *for any* $i \in \{1, 2, \ldots, m\}$ *(if such* $v_i w$ *exists), where* $w \in V(G_1^*(n_1, r_1, k_1)) \setminus V(C_{r_1})$*. We say that* G' *is obtained from* G *by branch transformation.*

Obviously, if G'' is obtained from G' by deleting the pendent edge $y_i w$ and adding the pendent edge $x_i w$ for any $i \in \{1, 2, \cdots, n\}$ (if such $y_i w$ exists), where $w \in V(G_2^*(n_2, r_2, k_2)) \setminus V(C_{r_2})$. We also say that G'' is obtained from G' by branch transformation. We refer to Figure 2.3.

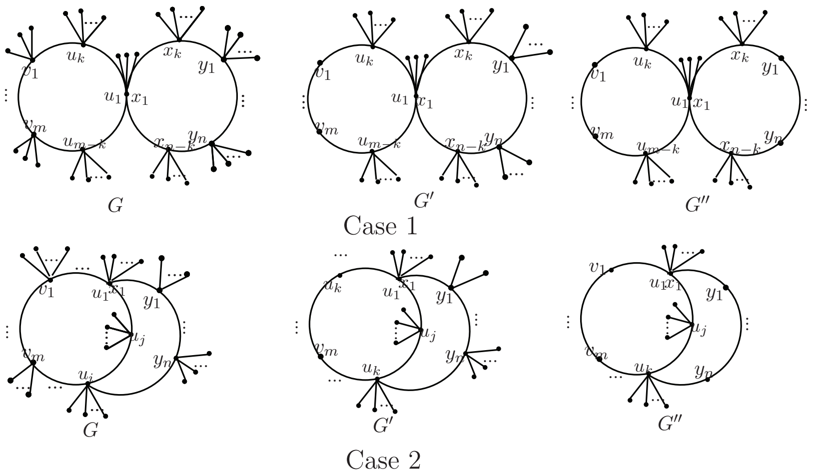

Figure 2.3: The branch transformation when r_1, r_2 are even.

Lemma 2.6 *Let* $n_1, k_1, r_1, n_2, k_2, r_2$ *be positive integers with* $1 \le k_1 \le r_1, 3 \le r_1 \le n_1 - k_1, 1 \le k_2 \le r_2, 3 \le r_2 \le n_2 - k_2$*. Suppose* $G = (G_1^*(n_1, r_1, k_1), G_2^*(n_2, r_2, k_2)) \in (\mathbb{G}_1^*(n_1, r_1, k_1), \mathbb{G}_2^*(n_2, r_2, k_2))$*. Let* G' *be the graph obtained from* G *by the branch transformation. Then* $J(G) \le J(G')$*.*

Proof. We suppose that the two cycles of the bicyclic graph have only one common vertex. Let $G_1 = G_1(n_1, r_1, k_1)$, $G_2 = G_2(n_2, r_2, k_2)$, $V_0 = \{v_1, v_2, \ldots, v_m\}$, $V_1 = \{w|v_i w \in E(G_1), deg(w) = 1, 1 \leq i \leq m\}$, $U_0 = \{u_1, u_2, \ldots, u_m\}$, $U_1 = \{w|u_i w \in E(G_1), deg(w) = 1, 1 \leq i \leq m\}$ when $r_1 = 2m$ is even, $U_1 = \{w|u_i w \in E(G_1), deg(w) = 1, 1 \leq i \leq m\} \bigcup \{u_{m+1}\}$ when $r_1 = 2m+1$ is odd, $Y_0 = \{y_1, y_2, \ldots, y_n\}$, $Y_1 = \{w|y_i w \in E(G_2), deg(w) = 1, 1 \leq i \leq n\}$, $X_0 = \{x_1, x_2, \ldots, x_n\}$, $X_1 = \{w|x_i w \in E(G_2), deg(w) = 1, 1 \leq i \leq n\}$ when $r_2 = 2n$ is even, $X_1 = \{w|x_i w \in E(G_2), deg(w) = 1, 1 \leq i \leq n\} \bigcup \{x_{n+1}\}$ when $r_2 = 2n+1$ is odd.

For any s with $1 \leq s \leq m$, it is clear that

$$D_G(v_s) = D_G(v_s, V_0) + D_G(v_s, U_0) + D_G(v_s, V_1) + D_G(v_s, U_1) + D_G(v_s, G_2) \quad (2.1)$$

and

$$D_{G'}(u_s) = D_{G'}(u_s, V_0) + D_{G'}(u_s, U_0) + D_{G'}(u_s, V_1) + D_{G'}(u_s, U_1) + D_{G'}(u_s, G_2). \quad (2.2)$$

Note that $D_G(v_s, U_0) = D_{G'}(u_s, V_0)$, $D_G(v_s, V_0) = D_{G'}(u_s, U_0)$, $D_G(v_s, V_1) = D_{G'}(u_s, V_1)$. Observe that $D_G(v_s, U_1) > D_{G'}(u_s, U_1)$, $D_G(v_s, G_2) > D_{G'}(u_s, G_2)$. Thus, we infer

$$D_G(v_s) - D_{G'}(u_s) = D_G(v_s, U_1) - D_{G'}(u_s, U_1) + D_G(v_s, G_2) - D_{G'}(u_s, G_2) > 0. \quad (2.3)$$

Similarly, we obtain

$$D_G(u_s) = D_G(u_s, V_0) + D_G(u_s, U_0) + D_G(u_s, V_1) + D_G(u_s, U_1) + D_G(u_s, G_2), \quad (2.4)$$

and

$$D_{G'}(v_s) = D_{G'}(v_s, V_0) + D_{G'}(v_s, U_0) + D_{G'}(v_s, V_1) + D_{G'}(v_s, U_1) + D_{G'}(v_s, G_2). \quad (2.5)$$

Hence,

$$D_{G'}(v_s) - D_G(u_s) = D_{G'}(v_s, U_1) - D_G(u_s, U_1) + D_{G'}(v_s, G_2) - D_G(u_s, G_2) > 0. \quad (2.6)$$

Therefore, we get

$$D_G(v_s) - D_{G'}(u_s) = D_{G'}(v_s) - D_G(u_s) > 0. \quad (2.7)$$

From Eqs (2.1)–(2.5), we get

$$D_{G'}(v_s) - D_G(v_s) = D_G(u_s) - D_{G'}(u_s) > 0. \tag{2.8}$$

For any edge $u_s u_t \in E(G_1[U_0])$, $v_s v_t \in E(G_1[V_0])$, take $x = D_{G'}(u_s)$, $y = D_{G'}(u_t)$, $w = D_G(v_s)$, $z = D_G(v_t)$, $a = D_{G'}(v_s) - D_G(v_s)$, $a' = D_{G'}(v_t) - D_G(v_t)$, $b = D_G(u_s) - D_{G'}(u_s)$, $b' = D_G(u_t) - D_{G'}(u_t)$. Then $b = a > 0$, $b' = a' > 0$ by (2.8). It is obvious that $a, a', b, b', w, x, y, z \in R^+$, $w > x$, $z > y$, which implies that $\frac{b}{x} \geq \frac{a}{w}, \frac{b'}{y} \geq \frac{a'}{z}$. Therefore, by Lemma 2.1, we obtain

$$\frac{1}{\sqrt{D_{G'}(u_s) D_{G'}(u_t)}} + \frac{1}{\sqrt{D_{G'}(v_s) D_{G'}(v_t)}} > \frac{1}{\sqrt{D_G(u_s) D_G(u_t)}} + \frac{1}{\sqrt{D_G(v_s) D_G(v_t)}}. \tag{2.9}$$

Similarly, for any vertex $w \in U_1 \cup V_1$ and any edge $u_s w \in E(G)$, we get $D_G(w) > D_{G'}(w)$ and $D_G(u_s) > D_{G'}(u_s)$ by (2.8). Thus,

$$\frac{1}{\sqrt{D_{G'}(u_s) D_{G'}(w)}} > \frac{1}{\sqrt{D_G(u_s) D_G(w)}}. \tag{2.10}$$

For any edge $v_s w \in E(G)$ and $u_s w \in E(G)$, we obtain

$$D_G(w) = D_G(w, V_0) + D_G(w, U_0) + D_G(w, V_1) + D_G(w, U_1) + D_G(w, G_2) \tag{2.11}$$

and

$$D_{G'}(w) = D_{G'}(w, V_0) + D_{G'}(w, U_0) + D_{G'}(w, V_1) + D_{G'}(w, U_1) + D_{G'}(w, G_2). \tag{2.12}$$

So,

$$D_G(w) - D_{G'}(w) = D_G(w, U_1) - D_{G'}(w, U_1) + D_G(w, G_2) - D_{G'}(w, G_2) > 0. \tag{2.13}$$

By Eq. (2.7), we have $D_G(v_s) > D_{G'}(u_s)$, and then

$$\frac{1}{\sqrt{D_{G'}(u_s) D_{G'}(w)}} > \frac{1}{\sqrt{D_G(v_s) D_G(w)}}. \tag{2.14}$$

For any edge $u_{m+1} w \in E(G)$ (if such an edge exists), it is obvious that

$$\frac{1}{\sqrt{D_{G'}(u_{m+1}) D_{G'}(w)}} = \frac{1}{\sqrt{D_G(u_{m+1}) D_G(w)}}. \tag{2.15}$$

For edge u_1v_1, by Lemma 2.3 and Eq. (2.8), we obtain

$$\frac{1}{\sqrt{D_{G'}(u_1)D_{G'}(v_1)}} > \frac{1}{\sqrt{D_G(u_1)D_G(v_1)}}. \tag{2.16}$$

For any $x_s, y_s \in E(G)$, it is clear that $D_G(x_s) > D_{G'}(x_s), D_G(y_s) > D_{G'}(y_s)$, and so

$$\frac{1}{\sqrt{D_{G'}(x_s)D_{G'}(x_t)}} > \frac{1}{\sqrt{D_G(x_s)D_G(x_t)}}. \tag{2.17}$$

$$\frac{1}{\sqrt{D_{G'}(y_s)D_{G'}(y_t)}} > \frac{1}{\sqrt{D_G(y_s)D_G(y_t)}}. \tag{2.18}$$

From (2.10)–(2.18), we obtain $J(G) < J(G')$ by the definition of Balaban index.

By applying a similar method, we can also prove the case that the two cycles of a bicyclic graph have k common vertices. ∎

Similarly, we infer $J(G') < J(G'')$.

Lemma 2.7 *Let $n_1, k_1, r_1, n_2, k_2, r_2$ be positive integers with $1 \leq k_1 \leq r_1, 3 \leq r_1 \leq n_1 - k_1, 1 \leq k_2 \leq r_2, 3 \leq r_2 \leq n_2 - k_2$. Suppose $G = (G_1^*(n_1, r_1, k_1), G_2^*(n_2, r_2, k_2)) \in (\mathbb{G}_1^*(n_1, r_1, k_1), \mathbb{G}_2^*(n_2, r_2, k_2))$. Let G' be the graph obtained from G by the branch transformation. Then $SJ(G) \leq SJ(G')$.*

Proof. We suppose that the two cycles of the bicyclic graph have only one common vertex. Let $U_0, U_1, V_0, V_1, a, a', b, b'$ be defined as in Lemma 2.6. Let $f(x) = \frac{1}{\sqrt{x}} - \frac{1}{x+b+b'}$. Observe that $f(x)$ is a decreasing function of x. Note that $D_G(v_s)+D_G(v_t) > D_{G'}(u_s)+D_{G'}(u_t) = D_G(u_s) + D_G(u_t) - b - b'$, we have

$$\begin{aligned}
&\frac{1}{\sqrt{D_G(v_s) + D_G(v_t)}} - \frac{1}{\sqrt{D_G(v_s) + D_G(v_t) + b + b'}} \\
<&\frac{1}{\sqrt{D_G(u_s) + D_G(u_t) - b - b'}} - \frac{1}{\sqrt{D_G(u_s) + D_G(u_t)}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{\sqrt{D_{G'}(v_s) + D_{G'}(v_t)}} + \frac{1}{\sqrt{D_{G'}(u_s) + D_{G'}(u_t)}} \\
>&\frac{1}{\sqrt{D_G(v_s) + D_G(v_t)}} + \frac{1}{\sqrt{D_G(u_s) + D_G(u_t)}}.
\end{aligned}$$

Similarly, for any vertex $w \in U_1 \bigcup V_1$ and any edge $u_s w \in E(G)$, $D_G(w) > D_{G'}(w)$, $D_G(u_s) > D_{G'}(u_s)$. Then, we obtain

$$\frac{1}{\sqrt{D_{G'}(u_s) + D_{G'}(w)}} > \frac{1}{\sqrt{D_G(u_s) + D_G(w)}}. \tag{2.19}$$

For any edge $v_s w \in E(G)$, then $u_s w \in E(G)$, we have $D_G(v_s) > D_{G'}(u_s)$, and thus

$$\frac{1}{\sqrt{D_{G'}(u_s) + D_{G'}(w)}} > \frac{1}{\sqrt{D_G(v_s) + D_G(w)}}. \tag{2.20}$$

For any edge $u_{m+1} w \in E(G)$ (if such an edge exists), it is obvious that

$$\frac{1}{\sqrt{D_{G'}(u_{m+1}) + D_{G'}(w)}} = \frac{1}{\sqrt{D_G(u_{m+1}) + D_G(w)}}. \tag{2.21}$$

For edge $u_1 v_1$, by (2.8), we obtain

$$\frac{1}{\sqrt{D_{G'}(u_1) + D_{G'}(v_1)}} = \frac{1}{\sqrt{D_G(u_1) + D_G(v_1)}}. \tag{2.22}$$

For any $x_s, y_s \in E(G)$, it is clear that $D_G(x_s) > D_{G'}(x_s)$ and $D_G(y_s) > D_{G'}(y_s)$, which implies that

$$\frac{1}{\sqrt{D_{G'}(x_s) + D_{G'}(x_t)}} > \frac{1}{\sqrt{D_G(x_s) + D_G(x_t)}}. \tag{2.23}$$

$$\frac{1}{\sqrt{D_{G'}(y_s) + D_{G'}(y_t)}} > \frac{1}{\sqrt{D_G(y_s) + D_G(y_t)}}. \tag{2.24}$$

From (2.19)–(2.24), we obtain $SJ(G) < SJ(G')$ by the definition of Sum-Balaban index.

Similarly, we can prove the case that the two cycles of the bicyclic graph have k common vertices. ∎

Similarly, we infer $SJ(G') < SJ(G'')$.

Lemma 2.8 *Let $n_1, k_1, r_1, n_2, k_2, r_2$ be positive integers with $1 \leq k_1 \leq r_1, 3 \leq r_1 \leq n_1 - k_1, 1 \leq k_2 \leq r_2, 3 \leq r_2 \leq n_2 - k_2$. Suppose $G = (G_1^*(n_1, r_1, k_1), G_2^*(n_2, r_2, k_2)) \in (\mathbb{G}_1^*(n_1, r_1, k_1), \mathbb{G}_2^*(n_2, r_2, k_2))$. Let G' be the unique graph obtained from G by repeating the branch transformations. Then*

(1) *$G' \in (\mathbb{G}_1^*(n_1, r_1, 1), \mathbb{G}_2^*(n_2, r_2, 1))$ (see Figure 2.4).*

(2) *$J(G) \leq J(G')$, and the equality holds if and only if $G \cong G'$.*

(3) *$SJ(G) \leq SJ(G')$, and the equality holds if and only if $G \cong G'$.* ∎

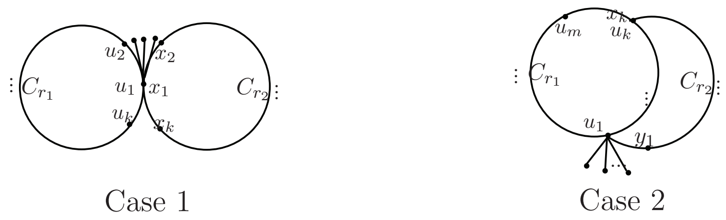

Figure 2.4: A graph $(G_1^*(n_1, r_1, 1), G_2^*(n_2, r_2, 2)) \in (\mathbb{G}_1^*(n_1, r_1, 1), \mathbb{G}_2^*(n_2, r_2, 2))$.

In order to pursue, we introduce two new transformations.

Figure 2.5: The crossing-edge-lifting transformation.

Definition 2.3 (The crossing-edge-lifting transformation) *Let $G \in (\mathbb{G}_1^*(n_1, r_1, 1), \mathbb{G}_2^*(n_2, r_2, 1))$ be a bicyclic graph such that the two cycles have k common vertices. As shown in Figure 2.5, let u_0 the last common vertex of the two cycles, which is adjacent to v_0, u_1, u_2, where $u_1 \in C_{r_1}, u_2 \in C_{r_2}$ and v_0 is also a common vertex.w is a vertex adjacent to u_0 and $deg(w) = 1$. Denote by G' the graph obtained from G by deleting edges u_0u_1, u_0u_2, u_0w and adding edges v_0u_1, v_0u_2, v_0w. We say that G' is the crossing-edge-lifting transformation of G (see Figure 2.5).* ∎

Definition 2.4 (The cycle-edge-lifting transformation) *Let $G = C_{n_1,n_2}$ be a bicyclic graph such that all the vertices lie on the cycles C_{n_1,n_2} with $n_1, n_2 \geq 3$,and the cycles have only one crossing point u_0. If G' is the graph obtained from G by deleting a vertex in G apart from u_0,and adding a edge between u_0 and a new vertex v_0,then connecting the vertices adjacent to the deleted vertex. Then G' is called the cycle-edge-lifting transformation of G. (see Figure 2.6).* ∎

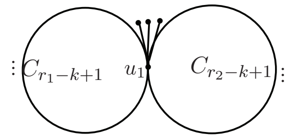

Figure 2.6: The cycle-edge-lifting transformation.

Lemma 2.9 *Let G' be the crossing-edge-lifting transformation of G. Then*

$$J(G) \leq J(G'),\ \ SJ(G) \leq SJ(G')$$

Proof. Denote by K the set of k common vertices. Let $U_0 = C_{r_1} \backslash K,\ V_0 = C_{r_2} \backslash K,\ W = \{w | wu_0 \in G, deg(w) = 1\}$.

Case 1: For any vertex $u \in U_0$, we have

$$D_G(u) = D_G(u, U_0) + D_G(u, V_0) + D_G(u, W) + D_G(u, K),$$

$$D_{G'}(u) = D_{G'}(u, U_0) + D_{G'}(u, V_0) + D_{G'}(u, W) + D_{G'}(u, K).$$

Since $D_G(u, U_0) = D_{G'}(u, U_0)$, $D_G(u, V_0) = D_{G'}(u, V_0)$, $D_G(u) - D_{G'}(u) = D_G(u, W) + D_G(u, K) - (D_{G'}(u, W) + D_{G'}(u, K)) \geq (n_1 + n_2 - r_1 - r_2)(\lfloor \frac{r_1}{2} \rfloor + 1) - \lfloor \frac{r_1}{2} \rfloor \geq 0$, we have $D_G(u) \geq D_{G'}(u)$.

Case 2: For any vertex $v \in V_0$, it is similar as that in **Case 1**, and so we have $D_G(v) \geq D_{G'}(v)$.

Case 3: For any vertex $p \in K$, we get

$$D_G(p) = D_G(p, U_0) + D_G(p, V_0) + D_G(p, W) + D_G(p, K),$$

$$D_{G'}(p) = D_{G'}(p, U_0) + D_{G'}(p, V_0) + D_{G'}(p, W) + D_{G'}(p, K).$$

Since $D_G(p, U_0) > D_{G'}(p, U_0)$, $D_G(p, V_0) > D_{G'}(p, V_0)$, $D_G(p, W) = D_{G'}(p, W)$ and $D_G(p, K) = D_{G'}(p, K)$, we have $D_G(p) > D_{G'}(p)$.

Case 4: For any vertex $w \in W$, obviously, $D_G(w) > D_{G'}(w)$.

Combining the above arguments and by using the definitions of the Balaban index and Sum-Balaban index, we obtain $J(G) \leq J(G'),\ SJ(G) \leq SJ(G')$. ∎

Lemma 2.10 *Let $n_1, k_1, r_1, n_2, k_2, r_2$ be positive integers with $1 \leq k_1 \leq r_1, 3 \leq r_1 \leq n_1 - k_1, 1 \leq k_2 \leq r_2, 3 \leq r_2 \leq n_2 - k_2$. Suppose $G = (G_1^*(n_1, r_1, 1), G_2^*(n_2, r_2, 1)) \in (\mathbb{G}_1^*(n_1, r_1, 1), \mathbb{G}_2^*(n_2, r_2, 1))$ is the bicyclic graph whose two cycles have k common vertices. Let G' be the unique graph obtained from G by repeating the crossing-edge-lifting transformations until the two cycles of G' have only one crossing point. Then we have*

(1) *$G' \in (\mathbb{G}_1^*(n_1, r_1 - k + 1, 1), \mathbb{G}_2^*(n_2, r_2 - k + 1, 1))$, with one common vertex (see Figure 2.7).*

(2) *$J(G) \leq J(G')$, and the equality holds if and only if $G \cong G''$.*

(3) *$SJ(G) \leq SJ(G')$, and the equality holds if and only if $G \cong G'$.* ∎

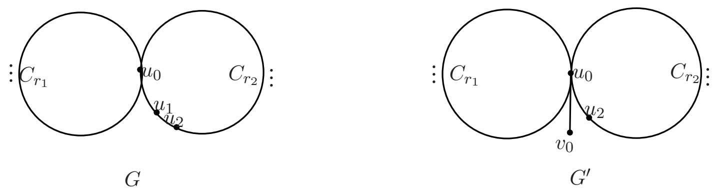

Figure 2.7: A graph $(G_1^*(n_1, r_1 - k + 1, 1), G_2^*(n_2, r_2 - k + 1, 2)) \in (\mathbb{G}_1^*(n_1, r_1 - k + 1, 1), \mathbb{G}_2^*(n_2, r_2 - k + 1, 2))$.

Lemma 2.11 *Let G' be the cycle-edge-lifting transformation of G. Then*

$$J(G) \leq J(G') \text{ and } SJ(G) \leq SJ(G')$$

Proof. Let $U_0 = V(C_{n_1}), V_0 = V(C_{n_2})$. For any $u \in U_0,\ v \in V_0$, it is clearly that

$$D_G(u) = D_G(u, U_0) + D_G(u, V_0),\ D_G(v) = D_G(v, U_0) + D_G(v, V_0),$$

$$D_{G'}(u) = D_{G'}(u, U_0) + D_{G'}(u, V_0),\ D_{G'}(v) = D_{G'}(v, U_0) + D_{G'}(v, V_0),$$

Obviously, $D_G(u, U_0) = D_{G'}(u, U_0)$, so $D_G(u) - D_{G'}(u) = D_G(u, V_0) - D_{G'}(u, V_0) = \lfloor \frac{n_2}{2} \rfloor - 1 \geq 0$, and $D_G(v) - D_{G'}(v) \geq \lfloor \frac{n_2}{2} \rfloor - \lfloor \frac{n_2}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor - 1 - \lfloor \frac{n_2}{2} \rfloor + 1 = 0$ for $v \in V_0, v \neq v_0$. Thus, we have

$$\frac{1}{\sqrt{D_G(u_i) D_G(u_j)}} < \frac{1}{\sqrt{D_{G'}(u_i) D_{G'}(u_j)}}, \tag{2.25}$$

$$\frac{1}{\sqrt{D_G(v_i)D_G(v_j)}} < \frac{1}{\sqrt{D_{G'}(v_i)D_{G'}(v_j)}}, \tag{2.26}$$

for $u_i, u_j \in U_0, v_i, v_j \in V_0$ and $u_i \sim u_j, v_i \sim v_j$

$$\frac{1}{\sqrt{D_G(u_{\lfloor \frac{n_2}{2} \rfloor})D_G(u_{\lfloor \frac{n_2}{2} \rfloor}+1)}} < \frac{1}{\sqrt{D_{G'}(u_0)D_{G'}(v_0)}}. \tag{2.27}$$

From (2.25)–(2.27), we obtain $J(G) \leq J(G')$ by using the definition of the Balaban index.

We have $\frac{1}{\sqrt{D_G(u_i)+D_G(u_j)}} < \frac{1}{\sqrt{D_{G'}(u_i)+D_{G'}(u_j)}}$, $\frac{1}{\sqrt{D_G(v_i)+D_G(v_j)}} < \frac{1}{\sqrt{D_{G'}(v_i)+D_{G'}(v_j)}}$ and $\frac{1}{\sqrt{D_G(u_{\lfloor \frac{n_2}{2} \rfloor})+D_G(u_{\lfloor \frac{n_2}{2} \rfloor}+1)}} < \frac{1}{\sqrt{D_{G'}(u_0)+D_{G'}(v_0)}}$, similarly. Then we obtain that $SJ(G) \leq SJ(G')$ by the definition of the Sum-Balaban index. ∎

3 Bicyclic Graphs

There are three types of bicyclic graphs according to the number of common vertices of two cycles. In this section, we will determine the graph which has the maximum Balaban index among all bicyclic graphs with n vertices.

The preceding discussion shows that the Balaban index of a bicyclic graph G is lower than the Balaban index of G' obtained from G by repeating edge-lifting transformations, branch transformations and crossing-edge-lifting transformations. Thus, the bicyclic graph which has the maximum Balaban index among all bicyclic graphs on $n_1 + n_2 - k$ vertices is the bicyclic graph such that the two cycles have only one common vertex. Now we only need to prove $J(G) \leq J(G_1^*(n_1, 3, 1), G_2^*(n_2, 3, 1))$, where G is the bicyclic graph such that the two cycles have only one common vertex.

Let G be a bicyclic graph on $n_1 + n_2 - 1$ vertices. Then $|E(G)| = n_1 + n_2$, $\mu = 2$, and then $J(G) = \frac{n_1+n_2}{3} \sum_{uv \in E(G)} \frac{1}{\sqrt{D_G(u)D_G(v)}}$.

Lemma 3.1 *Let the two cycles of the bicyclic graph have only one common vertex. Let* n_1, r_1, n_2, r_2 *be positive integers with* $1 \leq r_1 \leq n_1, 1 \leq r_2 \leq n_2 - k_2$ *and* $G = (G_1^*(n_1, r_1, 1), G_2^*(n_2, r_2, 1)) \in (\mathbb{G}_1^*(n_1, r_1, 1), \mathbb{G}_2^*(n_2, r_2, 1))$ *(see Case 1 of Figure 2.4). We get that*

(1). *if* r_1, r_2 *are even, then*

$$\frac{3J(G)}{n_1+n_2} = \frac{n_1+n_2-r_1-r_2}{\sqrt{\left(\frac{r_1^2}{4}+\frac{r_2^2}{4}-r_1-r_2+2(n_1+n_2)-3\right)\left(\frac{r_1^2}{4}+\frac{r_2^2}{4}-r_1-r_2+n_1+n_2\right)}} + \sum_{1\le i\le \frac{r_1}{2}} \frac{2}{\sqrt{D_G^1(u_i)D_G^1(u_{i+1})}} + \sum_{1\le j\le \frac{r_2}{2}} \frac{2}{\sqrt{D_G^1(v_j)D_G^1(v_{j+1})}};$$

(2). *if* r_1, r_2 *are odd, then*

$$\frac{3J(G)}{n_1+n_2} = \frac{n_1+n_2-r_1-r_2}{\sqrt{\left(\frac{r_1^2}{4}+\frac{r_2^2}{4}-r_1-r_2+2(n_1+n_2)-\frac{7}{2}\right)\left(\frac{r_1^2}{4}+\frac{r_2^2}{4}-r_1-r_2+n_1+n_2-\frac{1}{2}\right)}} + \sum_{1\le i\le \frac{r_1-1}{2}} \frac{2}{\sqrt{D_G^2(u_i)D_G^2(u_{i+1})}} + \frac{1}{\frac{r_1^2}{4}+\frac{r_2^2}{4}+\frac{r_2+1}{2}(n_1+n_2-r_2-1)-r_1+\frac{1}{2}} + \sum_{1\le j\le \frac{r_2-1}{2}} \frac{2}{\sqrt{D_G^2(v_j)D_G^2(v_{j+1})}} + \frac{1}{\frac{r_1^2}{4}+\frac{r_2^2}{4}+\frac{r_1+1}{2}(n_1+n_2-r_1-1)-r_2+\frac{1}{2}};$$

(3). *if* r_1 *is odd and* r_2 *is even, then*

$$\frac{3J(G)}{n_1+n_2} = \frac{n_1+n_2-r_1-r_2}{\sqrt{\left(\frac{r_1^2}{4}+\frac{r_2^2-1}{4}-r_1-r_2+2(n_1+n_2)-3\right)\left(\frac{r_1^2}{4}+\frac{r_2^2-1}{4}-r_1-r_2+n_1+n_2\right)}} + \sum_{1\le i\le \frac{r_1-1}{2}} \frac{2}{\sqrt{D_G^3(u_i)D_G^3(u_{i+1})}} + \sum_{1\le j\le \frac{r_2}{2}} \frac{2}{\sqrt{D_G^3(v_j)D_G^3(v_{j+1})}} + \frac{1}{\frac{r_1^2}{4}+\frac{r_2^2}{4}+\frac{r_1+1}{2}(n_1+n_2-r_1-1)-r_2+\frac{1}{2}};$$

(4). *if* r_1 *is even and* r_2 *is odd, then*

$$\frac{3J(G)}{n_1+n_2} = \frac{n_1+n_2-r_1-r_2}{\sqrt{\left(\frac{r_1^2-1}{4}+\frac{r_2^2}{4}-r_1-r_2+2(n_1+n_2)-3\right)\left(\frac{r_1^2}{4}+\frac{r_2^2-1}{4}-r_1-r_2+n_1+n_2\right)}} + \sum_{1\le i\le \frac{r_1}{2}} \frac{2}{\sqrt{D_G^4(u_i)D_G^4(u_{i+1})}} + \sum_{1\le j\le \frac{r_2-1}{2}} \frac{2}{\sqrt{D_G^4(v_j)D_G^4(v_{j+1})}} + \frac{1}{\frac{r_1^2}{4}+\frac{r_2^2}{4}+\frac{r_2+1}{2}(n_1+n_2-r_2-1)-r_1+\frac{1}{2}},$$

where $D_G^1(u_i) = \frac{r_1^2}{4}+\frac{r_2^2}{4}+i(n_1+n_2-r_1-1)-r_2+1$, $D_G^1(v_j) = \frac{r_1^2}{4}+\frac{r_2^2}{4}+j(n_1+n_2-r_2-1)-r_1+1$, $D_G^2(u_i) = \frac{r_1^2}{4}+\frac{r_2^2}{4}+i(n_1+n_2-r_1-1)-r_2+\frac{1}{2}$, $D_G^2(v_j) = \frac{r_1^2}{4}+\frac{r_2^2}{4}+j(n_1+n_2-r_2-1)-r_1+\frac{1}{2}$, $D_G^3(u_i) = \frac{r_1^2}{4}+\frac{r_2^2-1}{4}+i(n_1+n_2-r_1-1)-r_2+1$, $D_G^3(v_j) = \frac{r_1^2-1}{4}+\frac{r_2^2}{4}+j(n_1+n_2-r_2-1)-r_1+1$, $D_G^4(u_i) = \frac{r_1^2}{4}+\frac{r_2^2-1}{4}+i(n_1+n_2-r_1-1)-r_2+1$, $D_G^4(v_j) = \frac{r_1^2-1}{4}+\frac{r_2^2}{4}+j(n_1+n_2-r_2-1)-r_1+1$.

Proof. We calculate $D_G(u)$ for any vertex $u \in V(G)$.

Case 1. r_1, r_2 is even.

Subcase 1.1. $u \in V(G) \setminus V(C_{r_1}) \bigcup V(C_{r_2})$.

In this subcase, we have $D_G(u) = \frac{r_1^2}{4} + \frac{r_2^2}{4} - r_1 - r_2 + 2(n_1 + n_2) - 3$.

Subcase 1.2. $u = u_i \in V(C_{r_1})$.

Note that $D_G(u_i) = D_G(u_{r_1-i+2})$, we only need to calculate $D_G(u_i)$ for $1 \le i \le \frac{r_1+2}{2}$. Clearly, when $1 \le i \le \frac{r_1+2}{2}$, we have $D_G(u_i) = \frac{r_1^2}{4} + \frac{r_2^2}{4} + i(n_1 + n_2 - r_1 - 1) - r_2 + 1$.

Subcase 1.3. $u = v_j \in V(C_{r_2})$.

It is similar as **Subcase 1.2**. When $1 \le j \le \frac{r_2+2}{2}$, we have $D_G(v_j) = \frac{r_1^2}{4} + \frac{r_2^2}{4} + j(n_1 + n_2 - r_2 - 1) - r_1 + 1$.

Case 2. r_1, r_2 are odd.

Subcase 2.1. $u \in V(G) \setminus V(C_{r_1}) \bigcup V(C_{r_2})$.

In this subcase, we have $D_G(u) = \frac{r_1^2}{4} + \frac{r_2^2}{4} - r_1 - r_2 + 2(n_1 + n_2) - \frac{7}{2}$.

Subcase 2.2. $u = u_i \in V(C_{r_1})$.

Note that $D_G(u_i) = D_G(u_{r_1-i+2})$, we only need to calculate $D_G(u_i)$ for $1 \le i \le \frac{r_1+1}{2}$. Clearly, when $1 \le i \le \frac{r_1+1}{2}$, we have $D_G(u_i) = \frac{r_1^2}{4} + \frac{r_2^2}{4} + i(n_1 + n_2 - r_1 - 1) - r_2 + \frac{1}{2}$.

Subcase 2.3. $u = v_j \in V(C_{r_2})$.

It is similar as **Subcase 2.2**. When $1 \le j \le \frac{r_2+1}{2}$, we have $D_G(v_j) = \frac{r_1^2}{4} + \frac{r_2^2}{4} + j(n_1 + n_2 - r_2 - 1) - r_1 + \frac{1}{2}$.

Case 3. r_1 is odd and r_2 is even.

Subcase 3.1. $u \in V(G) \setminus V(C_{r_1}) \bigcup V(C_{r_2})$.

In this subcase, we have $D_G(u) = \frac{r_1^2}{4} + \frac{r_2^2}{4} - r_1 - r_2 + 2(n_1 + n_2) - \frac{13}{4}$.

Subcase 3.2. $u = u_i \in V(C_{r_1})$.

Note that $D_G(u_i) = D_G(u_{r_1-i+2})$, we only need to calculate $D_G(u_i)$ for $1 \le i \le \frac{r_1+1}{2}$. Clearly, when $1 \le i \le \frac{r_1+1}{2}$, we have $D_G(u_i) = \frac{r_1^2}{4} + \frac{r_2^2}{4} + i(n_1 + n_2 - r_1 - 1) - r_2 + \frac{3}{4}$.

Subcase 3.3. $u = v_j \in V(C_{r_2})$.

Note that $D_G(v_j) = D_G(u_{r_2-j+2})$, we only need to calculate $D_G(v_j)$ for $1 \leq j \leq \frac{r_2+2}{2}$. Clearly, when $1 \leq j \leq \frac{r_2+2}{2}$, we have $D_G(v_j) = \frac{r_1^2}{4} + \frac{r_2^2}{4} + j(n_1+n_2-r_2-1) - r_1 + \frac{3}{4}$.

Case 4. r_1 is even and r_2 is odd.

Obviously, this case is similar as **Case 3**.

By combining the above arguments the proof is thus completed. ∎

Theorem 3.2 *Let n_1, r_1, n_2, r_2 be positive integers with $1 \leq r_1 + k - 1 \leq n_1, 1 \leq r_2 + k - 1 \leq n_2 - k_2$. Let G be a connected bicyclic graph on $n_1 + n_2 - 1$ vertices such that the two cycles have k common vertices and $r_1 + k - 1,\ r_2 + k - 1$ vertices, respectively. Then $J(G) \leq \frac{n_1+n_2}{3}(A + B + C)$, and the equality holds if and only if $G \cong (G_1^*(n_1, 3, 1), G_2^*(n_2, 3, 1))$, where $A = \frac{2}{2(n_1+n_2)-6}$, $B = \frac{n_1+n_2-6}{\sqrt{(2(n_1+n_2)-5)(n_1+n_2-2)}}$ and $C = \frac{4}{\sqrt{(2(n_1+n_2)-6)(n_1+n_2-2)}}$.*

Proof. Let $G \ncong C_{n_1,n_2}$. There exist positive integers $n_1, k_1, r_1, n_2, k_2, r_2$ with $1 \leq k_1 \leq r_1 + k - 1, 3 \leq r_1 + k - 1 \leq n_1 - k_1, 1 \leq k_2 \leq r_2 + k - 1, 3 \leq r_2 + k - 1 \leq n_2 - k_2$ such that $G = (G_1(n_1, r_1 + k - 1, k_1), G_2(n_2, r_2 + k - 1, k_2))$.

By Lemma 2.5, there exists a graph $G_1 \in (\mathbb{G}_1^*(n_1, r_1+k-1, k_1), \mathbb{G}_2^*(n_2, r_2+k-1, k_2))$ such that G_1 is obtained from G by repeating edge-lifting transformations. Then $J(G) \leq J(G_1)$, and the equality holds if and only if $G \cong G_1$. By Lemma 2.8, we obtain a graph $G_2 = (G_1^*(n_1, r_1 + k - 1, 1), G_2^*(n_2, r_2 + k - 1, 1)) \in (\mathbb{G}_1^*(n_1, r_1 + k - 1, 1), \mathbb{G}_2^*(n_2, r_2 + k - 1, 1))$ from G_1 by repeating branch transformations such that $J(G_1) \leq J(G_2)$, with equality if and only if $G_1 \cong G_2$. By Lemma 2.10, we obtain a graph $G_3 \in (\mathbb{G}_1^*(n_1, r_1, 1), \mathbb{G}_2^*(n_2, r_2, 1))$ such that the two cycles have only one common vertex, from G_2 by repeating crossing-edge-lifting transformations such that $J(G_2) \leq J(G_3)$, with equality if and only if $G_2 \cong G_3$. By Lemma 3.1, we need to prove that

$$\begin{aligned} &J(G_1^*(n_1, r_1, 1), G_2^*(n_2, r_2, 1)) \\ \leq &\max\{J(G_1^*(n_1, 3, 1), G_2^*(n_2, 3, 1)), J(G_1^*(n_1, 3, 1), G_2^*(n_2, 4, 1)), J(G_1^*(n_1, 4, 1), \\ &\quad G_2^*(n_2, 3, 1)), J(G_1^*(n_1, 4, 1), G_2^*(n_2, 4, 1))\}. \end{aligned}$$

We consider the following cases.

Case 1. r_1, r_2 is even.

Let $f(r_1,r_2)=(\frac{r_1^2}{4}+\frac{r_2^2}{4}-r_1-r_2+2(n_1+n_2)-3)(\frac{r_1^2}{4}+\frac{r_2^2}{4}-r_1-r_2+n_1+n_2)$, $g_i(r_1,r_2)=(\frac{r_1^2}{4}+\frac{r_2^2}{4}+i(n_1+n_2-r_1-1)-r_2+1)(\frac{r_1^2}{4}+\frac{r_2^2}{4}+(i+1)(n_1+n_2-r_1-1)-r_2+1)$ for $1\le i\le \frac{r_1}{2}$, and $h_j(r_1,r_2)=(\frac{r_1^2}{4}+\frac{r_2^2}{4}+j(n_1+n_2-r_1-1)-r_1+1)(\frac{r_1^2}{4}+\frac{r_2^2}{4}+(j+1)(n_1+n_2-r_1-1)-r_1+1)$ for $1\le j\le \frac{r_2}{2}$.

It is obvious that $f'_{r_1}>0$, $f'_{r_2}>0$, $g'_{1r_1}>0$, $\cdots$, $g'_{\frac{r_1}{2}r_1}>0$, $g'_{1r_2}>0$, $\cdots$, $g'_{\frac{r_1}{2}r_2}>0$, $h'_{1r_1}>0$, $\cdots$, $h'_{\frac{r_2}{2}r_1}>0$, $h'_{1r_2}>0$, $\cdots$, $h'_{\frac{r_2}{2}r_2}>0$. So $J(G_1^*(n_1,r_1,1),G_2^*(n_2,r_2,1))$ is a decreasing function of r_1,r_2. Thus, we have

$$\begin{aligned}J(G_1^*(n_1,4,1),G_2^*(n_2,4,1))>&J(G_1^*(n_1,6,1),G_2^*(n_2,6,1))>\cdots\\&>J(G_1^*(n_1,2\lfloor\frac{n_1-1}{2}\rfloor,1),G_2^*(n_2,2\lfloor\frac{n_2-1}{2}\rfloor,1)).\end{aligned}$$

Case 2. r_1, r_2 is odd.

Let $f(r_1,r_2)=(\frac{r_1^2}{4}+\frac{r_2^2}{4}-r_1-r_2+2(n_1+n_2)-\frac{7}{2})(\frac{r_1^2}{4}+\frac{r_2^2}{4}-r_1-r_2+n_1+n_2-\frac{1}{2})$, $g_i(r_1,r_2)=(\frac{r_1^2}{4}+\frac{r_2^2}{4}+i(n_1+n_2-r_1-1)-r_2+\frac{1}{2})(\frac{r_1^2}{4}+\frac{r_2^2}{4}+(i+1)(n_1+n_2-r_1-1)-r_2+\frac{1}{2})$ for $1\le i\le\frac{r_1+1}{2}$, $h_j(r_1,r_2)=(\frac{r_1^2}{4}+\frac{r_2^2}{4}+j(n_1+n_2-r_1-1)-r_1+\frac{1}{2})(\frac{r_1^2}{4}+\frac{r_2^2}{4}+(j+1)(n_1+n_2-r_1-1)-r_1+\frac{1}{2})$ for $1\le j\le\frac{r_2+1}{2}$, $p(r_1,r_2)=\frac{r_1^2}{4}+\frac{r_2^2}{4}+\frac{r_1+1}{2}(n_1+n_2-r_1-1)-r_2+\frac{1}{2}$ and $q(r_1,r_2)=\frac{r_1^2}{4}+\frac{r_2^2}{4}+\frac{r_2+1}{2}(n_1+n_2-r_2-1)-r_1+\frac{1}{2}$.

Clearly, the partial derivative of $f(r_1,r_2), g_i(r_1,r_2), h_j(r_1,r_2), p(r_1,r_2), q(r_1,r_2)$ for r_1,r_2 is positive, so $J(G_1^*(n_1,r_1,1),G_2^*(n_2,r_2,1))$ is decreasing. Thus we have

$$\begin{aligned}J(G_1^*(n_1,3,1),G_2^*(n_2,3,1))>&J(G_1^*(n_1,5,1),G_2^*(n_2,5,1))>\cdots\\&>J(G_1^*(n_1,2\lfloor\frac{n_1-2}{2}\rfloor+1,1),G_2^*(n_2,2\lfloor\frac{n_2-2}{2}\rfloor+1,1)).\end{aligned}$$

Case 3. r_1 is odd and r_2 is even.

Let $f(r_1,r_2)=(\frac{r_1^2}{4}+\frac{r_2^2}{4}-r_1-r_2+2(n_1+n_2)-\frac{13}{4})(\frac{r_1^2}{4}+\frac{r_2^2}{4}-r_1-r_2+n_1+n_2-\frac{1}{4})$, $g_i(r_1,r_2)=(\frac{r_1^2}{4}+\frac{r_2^2}{4}+i(n_1+n_2-r_1-1)-r_2+\frac{3}{4})(\frac{r_1^2}{4}+\frac{r_2^2}{4}+(i+1)(n_1+n_2-r_1-1)-r_2+\frac{3}{4})$ for $1\le i\le\frac{r_1+1}{2}$, $h_j(r_1,r_2)=(\frac{r_1^2}{4}+\frac{r_2^2}{4}+j(n_1+n_2-r_1-1)-r_1+\frac{3}{4})(\frac{r_1^2}{4}+\frac{r_2^2}{4}+(j+1)(n_1+n_2-r_1-1)-r_1+\frac{3}{4})$ for $1\le j\le\frac{r_2}{2}$ and $p(r_1,r_2)=\frac{r_1^2}{4}+\frac{r_2^2}{4}+\frac{r_1+1}{2}(n_1+n_2-r_1-1)-r_2+\frac{1}{2}$.

Clearly, the partial derivative of $f(r_1, r_2), g_i(r_1, r_2), h_j(r_1, r_2), p(r_1, r_2)$ for r_1, r_2 is larger than 0, so $J(G_1^*(n_1, r_1, 1), G_2^*(n_2, r_2, 1))$ is a decreasing function. Thus, we have

$$\begin{aligned} J(G_1^*(n_1,3,1), G_2^*(n_2,4,1)) >& J(G_1^*(n_1,5,1), G_2^*(n_2,6,1)) > \cdots \\ & > J(G_1^*(n_1, 2\lfloor\frac{n_1-2}{2}\rfloor+1, 1), G_2^*(n_2, 2\lfloor\frac{n_2-1}{2}\rfloor, 1)). \end{aligned}$$

Case 4. r_1 is even and r_2 is odd.

Similarly as that in **Case 3**, we have

$$\begin{aligned} J(G_1^*(n_1,4,1), G_2^*(n_2,3,1)) >& J(G_1^*(n_1,6,1), G_2^*(n_2,5,1)) > \cdots \\ & > J(G_1^*(n_1, 2\lfloor\frac{n_1-1}{2}\rfloor, 1), G_2^*(n_2, 2\lfloor\frac{n_2-2}{2}\rfloor+1, 1)). \end{aligned}$$

On the other hand, by performing some elementary calculations, we get

$$\begin{aligned} & \frac{3}{n_1+n_2}(J(G_1^*(n_1,3,1), G_2^*(n_2,3,1)) - J(G_1^*(n_1,4,1), G_2^*(n_2,3,1))) \\ =& \frac{3}{n_1+n_2}\left(J(G_1^*(n_1,3,1), G_2^*(n_2,3,1)) - J(G_1^*(n_1,3,1), G_2^*(n_2,4,1))\right) > 0 \end{aligned}$$

and

$$\begin{aligned} & \frac{3}{n_1+n_2}\left(J(G_1^*(n_1,3,1), G_2^*(n_2,3,1)) - J(G_1^*(n_1,3,1), G_2^*(n_2,4,1))\right) \\ =& \frac{n_1+n_2-6}{\sqrt{(2(n_1+n_2)-5)(n_1+n_2-2)}} + \frac{4}{\sqrt{(2(n_1+n_2)-6)(n_1+n_2-2)}} \\ & + \frac{2}{2(n_1+n_2)-6} - \frac{n_1+n_2-7}{\sqrt{(2(n_1+n_2)-4)(n_1+n_2-1)}} \\ & - \frac{2}{\sqrt{(2(n_1+n_2)-6)(n_1+n_2-1)}} - \frac{2}{\sqrt{(2(n_1+n_2)-6)(3(n_1+n_2)-1)}} \\ & - \frac{2}{\sqrt{(2(n_1+n_2)-5)(n_1+n_2-1)}} - \frac{1}{2(n_1+n_2)-5} > 0. \end{aligned}$$

Moreover, we have

$$\begin{aligned} & \frac{3}{n_1+n_2}(J(G_1^*(n_1,3,1), G_2^*(n_2,3,1)) - J(G_1^*(n_1,4,1), G_2^*(n_2,4,1))) \\ =& \frac{n_1+n_2-6}{\sqrt{(2(n_1+n_2)-5)(n_1+n_2-2)}} + \frac{4}{\sqrt{(2(n_1+n_2)-6)(n_1+n_2-2)}} \\ & - \frac{n_1+n_2-8}{\sqrt{(2(n_1+n_2)-3)(n_1+n_2)}} - \frac{4}{\sqrt{(2(n_1+n_2)-5)(n_1+n_2)}} \\ & - \frac{4}{\sqrt{(2(n_1+n_2)-5)(3(n_1+n_2)-10)}} + \frac{2}{2(n_1+n_2)-6} > 0. \end{aligned}$$

From the above arguments, we have

$$\begin{aligned}&J(G_1^*(n_1,r_1,1),G_2^*(n_2,r_2,1))\\ \le&\max\{J(G_1^*(n_1,3,1),G_2^*(n_2,3,1)),J(G_1^*(n_1,3,1),G_2^*(n_2,4,1)),\\&\quad J(G_1^*(n_1,4,1),G_2^*(n_2,3,1)),J(G_1^*(n_1,4,1),G_2^*(n_2,4,1))\}\\ =&J(G_1^*(n_1,3,1),G_2^*(n_2,3,1)).\end{aligned}$$

If $G \cong C_{n_1,n_2}$, by Lemma 2.11, we get

$$J(G) \le J(G_1^*(n_1,3,1),G_2^*(n_2,3,1)).$$

Therefore, if G is a bicyclic graph such that the two cycles have k common vertices, then $J(G) \le J(G_1^*(n_1,3,1),G_2^*(n_2,3,1))$. ∎

4 Maximum Sum-Balaban Index of Bicyclic Graphs

In this section, we will determine the graph which has the maximum Sum-Balaban index among all bicyclic graphs with n vertices.

Similar to the arguments of the maximum Balaban index of bicyclic graphs, we see that the bicyclic graph which has the maximum Sum-Balaban index among all bicyclic graphs on n_1+n_2-k vertices is the bicyclic graph such that the two cycles have only one common vertex. Now we only need to prove $SJ(G) \le SJ(G_1^*(n_1,3,1),G_2^*(n_2,3,1))$, where G is the bicyclic graph such that the two cycles have only one common vertex.

Let $G=(V,E)$ be a bicyclic graph on n_1+n_2-1 vertices. Suppose $|E|=n_1+n_2$, $\mu=2$ and then $J(G)=\frac{n_1+n_2}{3}\sum_{uv\in E(G)}\frac{1}{\sqrt{D_G(u)+D_G(v)}}$. Similarly, we obtain the following results straightforwardly.

Lemma 4.1 *Suppose the two cycles of the bicyclic graph have only one common vertex. Let* n_1, r_1, n_2, r_2 *be positive integers with* $1 \le r_1 \le n_1, 1 \le r_2 \le n_2 - k_2$, $G = (G_1^*(n_1,r_1,1),G_2^*(n_2,r_2,1)) \in (\mathbb{G}_1^*(n_1,r_1,1),\mathbb{G}_2^*(n_2,r_2,1))$ *(see Case 1 of Figure 2.4). We have*

(1). *if* r_1, r_2 *are even, then*

$$\frac{3SJ(G)}{n_1+n_2} = \frac{n_1+n_2-r_1-r_2}{\sqrt{\frac{r_1^2}{2}+\frac{r_2^2}{2}-2(r_1+r_2)+3(n_1+n_2)-3}} + \sum_{1\le i\le \frac{r_1}{2}} \frac{2}{\sqrt{D_G^1(u_i)+D_G^1(u_{i+1})}} + \sum_{1\le j\le \frac{r_2}{2}} \frac{2}{\sqrt{D_G^1(v_j)+D_G^1(v_{j+1})}};$$

(2). *if* r_1, r_2 *are odd, then*

$$\begin{aligned}\frac{3SJ(G)}{n_1+n_2} =& \frac{n_1+n_2-r_1-r_2}{\sqrt{\frac{r_1^2}{2}+\frac{r_2^2}{2}-2(r_1+r_2)+3(n_1+n_2)-4}} \\ &+ \frac{1}{\sqrt{\frac{r_1^2}{2}+\frac{r_2^2}{2}+(r_1+1)(n_1+n_2-r_1-1)-2r_2+1}} \\ &+ \frac{1}{\sqrt{\frac{r_1^2}{2}+\frac{r_2^2}{2}+(r_2+1)(n_1+n_2-r_2-1)-2r_1+1}} \\ &+ \sum_{1\le i\le \frac{r_1-1}{2}} \frac{2}{\sqrt{D_G^2(u_i)+D_G^2(u_{i+1})}} + \sum_{1\le j\le \frac{r_2-1}{2}} \frac{2}{\sqrt{D_G^2(v_j)+D_G^2(v_{j+1})}};\end{aligned}$$

(3). *if* r_1 *is odd and* r_2 *is even, then*

$$\begin{aligned}\frac{3SJ(G)}{n_1+n_2} =& \frac{n_1+n_2-r_1-r_2}{\sqrt{\frac{r_1^2}{2}+\frac{r_2^2}{2}-2(r_1+r_2)+3(n_1+n_2)-\frac{7}{2}}} \\ &+ \frac{1}{\sqrt{\frac{r_1^2}{2}+\frac{r_2^2}{2}+(r_1+1)(n_1+n_2-r_1-1)-2r_2+1}} \\ &+ \sum_{1\le i\le \frac{r_1-1}{2}} \frac{2}{\sqrt{D_G^3(u_i)+D_G^3(u_{i+1})}} + \sum_{1\le j\le \frac{r_2}{2}} \frac{2}{\sqrt{D_G^3(v_j)+D_G^3(v_{j+1})}};\end{aligned}$$

(4). *if* r_1 *is even and* r_2 *is odd, then*

$$\begin{aligned}\frac{3SJ(G)}{n_1+n_2} =& \frac{n_1+n_2-r_1-r_2}{\sqrt{\frac{r_1^2}{2}+\frac{r_2^2}{2}-2(r_1+r_2)+3(n_1+n_2)-\frac{7}{2}}} \\ &+ \frac{1}{\sqrt{\frac{r_1^2}{2}+\frac{r_2^2}{2}+(r_2+1)(n_1+n_2-r_2-1)-2r_1+1}} \\ &+ \sum_{1\le i\le \frac{r_1}{2}} \frac{2}{\sqrt{D_G^4(u_i)+D_G^4(u_{i+1})}} + \sum_{1\le j\le \frac{r_2-1}{2}} \frac{2}{\sqrt{D_G^4(v_j)+D_G^4(v_{j+1})}},\end{aligned}$$

where $D_G^1(u_i) = \frac{r_1^2}{4}+\frac{r_2^2}{4}+i(n_1+n_2-r_1-1)-r_2+1$, $D_G^1(v_j) = \frac{r_1^2}{4}+\frac{r_2^2}{4}+j(n_1+n_2-r_2-1)-r_1+1$, $D_G^2(u_i) = \frac{r_1^2}{4}+\frac{r_2^2}{4}+i(n_1+n_2-r_1-1)-r_2+\frac{1}{2}$, $D_G^2(v_j) = \frac{r_1^2}{4}+\frac{r_2^2}{4}+j(n_1+n_2-r_2-1)-r_1+\frac{1}{2}$, $D_G^3(u_i) = \frac{r_1^2}{4}+\frac{r_2^2-1}{4}+i(n_1+n_2-r_1-1)-r_2+1$, $D_G^3(v_j) = \frac{r_1^2-1}{4}+\frac{r_2^2}{4}+j(n_1+n_2-r_2-1)-r_1+1$, $D_G^4(u_i) = \frac{r_1^2}{4}+\frac{r_2^2-1}{4}+i(n_1+n_2-r_1-1)-r_2+1$, $D_G^4(v_j) = \frac{r_1^2-1}{4}+\frac{r_2^2}{4}+j(n_1+n_2-r_2-1)-r_1+1$. ∎

Note that

$$
\begin{aligned}
&\frac{3}{n_1+n_2}\left(SJ(G_1^*(n_1,3,1),G_2^*(n_2,3,1))-SJ(G_1^*(n_1,3,1),G_2^*(n_2,4,1))\right)\\
=&\frac{n_1+n_2-6}{\sqrt{3(n_1+n_2)-7}}+\frac{2}{\sqrt{4(n_1+n_2)-12}}+\frac{4}{\sqrt{3(n_1+n_2)-8}}-\frac{n_1+n_2-7}{\sqrt{3(n_1+n_2)-5}}\\
&-\frac{2}{\sqrt{3(n_1+n_2)-6}}-\frac{1}{\sqrt{4(n_1+n_2)-10}}-\frac{2}{\sqrt{3(n_1+n_2)-7}}-\frac{2}{\sqrt{5(n_1+n_2)-17}}\\
>&\,0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{3}{n_1+n_2}(SJ(G_1^*(n_1,3,1),G_2^*(n_2,3,1))-SJ(G_1^*(n_1,4,1),G_2^*(n_2,3,1)))\\
=&\frac{3}{n_1+n_2}(SJ(G_1^*(n_1,3,1),G_2^*(n_2,3,1))-SJ(G_1^*(n_1,3,1),G_2^*(n_2,4,1)))>0.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\frac{3}{n_1+n_2}(SJ(G_1^*(n_1,3,1),G_2^*(n_2,3,1))-SJ(G_1^*(n_1,4,1),G_2^*(n_2,4,1)))\\
=&\frac{n_1+n_2-6}{\sqrt{3(n_1+n_2)-7}}+\frac{2}{\sqrt{4(n_1+n_2)-12}}+\frac{4}{\sqrt{3(n_1+n_2)-8}}\\
&-\left(\frac{n_1+n_2-8}{\sqrt{3(n_1+n_2)-3}}+\frac{4}{\sqrt{3(n_1+n_2)-5}}+\frac{4}{\sqrt{5(n_1+n_2)-15}}\right)>0.
\end{aligned}
$$

Therefore, we infer the following theorem.

Theorem 4.2 *Let n_1, r_1, n_2, r_2 be positive integers with $1 \le r_1+k-1 \le n_1, 1 \le r_2+k-1 \le n_2-k_2$. Suppose G is a connected bicyclic graph on n_1+n_2-1 vertices such that the two cycles have k common vertices and r_1+k-1, r_2+k-1 vertices, respectively. Then*

$$
SJ(G) \le \frac{n_1+n_2}{3}\left(\frac{n_1+n_2-6}{\sqrt{3(n_1+n_2)-7}}+\frac{2}{\sqrt{4(n_1+n_2)-12}}+\frac{4}{\sqrt{3(n_1+n_2)-8}}\right),
$$

and the equality holds if and only if $G \cong (G_1^*(n_1,r_1,1),G_2^*(n_2,r_2,1))(k=1)$.

Proof. Similar to the proof of Theorem 3.2, we have

$$\begin{aligned}
&J(G_1^*(n_1, r_1 + k - 1, 1), G_2^*(n_2, r_2 + k - 1, 1)) \\
\leq &\max\{SJ(G_1^*(n_1, 3, 1), G_2^*(n_2, 3, 1)), SJ(G_1^*(n_1, 3, 1), G_2^*(n_2, 4, 1)), \\
&\quad SJ(G_1^*(n_1, 4, 1), G_2^*(n_2, 3, 1)), SJ(G_1^*(n_1, 4, 1), G_2^*(n_2, 4, 1))\} \\
= &SJ(G_1^*(n_1, 3, 1), G_2^*(n_2, 3, 1))(k = 1).
\end{aligned}$$

If $G \cong C_{n_1,n_2}$, then by Lemma 2.11, we obtain

$$SJ(G) \leq SJ(G_1^*(n_1, 3, 1), G_2^*(n_2, 3, 1)).$$

Combining the above cases, we complete the proof. ∎

5 Conclusion

In this paper, we studied sharp upper bounds for the Balaban index and the Sum-Balaban index among all bicyclic graphs, by using some transformations. The graphs attaining these bounds were also characterized. An important question is how general the bounds are. Obviously, the proof techniques use structural properties of the graphs under consideration and it may be intricate to extend the techniques when using more general graphs.

Consequently we will consider the extremal problems of the Balaban index for general graphs, i.e., the graphs with the cyclomatic number $\mu = k$ for any integer $k \geq 3$, and also some special networks as future work. Further, we would like to explore advanced structural properties of the Balaban index, and relations between the Balaban index and some other topological indices.

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