Hypergraph Turán numbers of vertex disjoint cycles*

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Abstract

The Turán number of a k-uniform hypergraph H, denoted by $ex_k(n; H)$, is the maximum number of edges in any k-uniform hypergraph F on n vertices which does not contain H as a subgraph. Let $\mathcal{C}_\ell^{(k)}$ denote the family of all k-uniform minimal cycles of length ℓ, $\mathcal{S}(\ell_1, \ldots, \ell_r)$ denote the family of hypergraphs consisting of unions of r vertex disjoint minimal cycles of length $\ell_1, \ldots, \ell_r$, respectively, and $\mathbb{C}_\ell^{(k)}$ denote a k-uniform linear cycle of length ℓ. We determine precisely $ex_k(n; \mathcal{S}(\ell_1, \ldots, \ell_r))$ and $ex_k\left(n; \mathbb{C}_{\ell_1}^{(k)}, \ldots, \mathbb{C}_{\ell_r}^{(k)}\right)$ for sufficiently large n. The results extend recent results of Füredi and Jiang [Füredi, Z., Jiang, T. Hypergraph Turán numbers of linear cycles. J. Combin. Theory Ser. A, 123(1): 252–270 (2014)], in which the Turán numbers for single k-uniform minimal cycles and linear cycles are determined.

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1 Introduction

In this paper, we employ standard terminology and notation from hypergraph theory (see e.g.,[1]). A *hypergraph* is a pair $H = (V, E)$ consisting of a set V of vertices and a set $E \subseteq \mathcal{P}(V)$ of edges. If every edge contains exactly k vertices, then

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H is a k-*uniform hypergraph*. A *graph homomorphism* f from a graph $G = (V, E)$ to a graph $G' = (V', E')$ is a mapping $f\colon V \to V'$ from the vertex set of G to the vertex set of G' such that $uv \in E$ implies $f(u)f(v) \in E'$. For two hypergraphs G and H, we write $G \subseteq H$ if there is an injective homomorphism from G into H. We use $G \cup H$ to denote the disjoint union of (hyper)graphs G and H. By disjoint, we will always mean vertex disjoint. A *Berge path* of length ℓ is a family of distinct sets $\{F_1, \ldots, F_\ell\}$ and $\ell + 1$ distinct vertices $v_1, \ldots, v_{\ell+1}$ such that for each $i = 1, 2, \ldots, \ell$, F_i contains v_i and v_{i+1}. Let $\mathcal{B}_\ell^{(k)}$ denote the family of k-uniform Berge paths of length ℓ. A *linear path* of length ℓ is a family of sets $\{F_1, \ldots, F_\ell\}$ such that $|F_i \cap F_{i+1}| = 1$ for each i and $F_i \cap F_j = \emptyset$ whenever $|i - j| > 1$. Let $\mathbb{P}_\ell^{(k)}$ denote the k-uniform linear path of length ℓ. It is unique up to isomorphisms. A k-uniform *Berge cycle* of length ℓ is a cyclic list of distinct k-sets $A_1, \ldots, A_\ell$ and ℓ distinct vertices $v_1, \ldots, v_\ell$ such that for each $i = 1, 2, \ldots, \ell$, A_i contains v_i and v_{i+1} (where $v_{\ell+1} = v_1$). A k-uniform *minimal cycle* of length ℓ is a cyclic list of k-sets $A_1, \ldots, A_\ell$ such that consecutive sets intersect in at least one element and nonconsecutive sets are disjoint. Denote the family of all k-uniform minimal cycles of length ℓ by $\mathcal{C}_\ell^{(k)}$. A k-uniform *linear cycle* of length ℓ, denoted by $\mathbb{C}_\ell^{(k)}$, is a cyclic list of k-sets $A_1, \ldots, A_\ell$ such that consecutive sets intersect in exactly one element and nonconsecutive sets are disjoint.

The *Turán number*, or extremal number, of a k-uniform hypergraph H, denoted by $ex_k(n; H)$, is the maximum number of edges in any k-uniform hypergraph F on n vertices which does not contain H as a subgraph. This is a natural generalization of the classic Turán number for 2-uniform graphs; we restrict ourselves to the case of k-uniform hypergraphs. Let $ex_k(n; F_1, F_2, \ldots, F_r)$ denote the k-*uniform hypergraph Turán Number* of a list of k-uniform hypergraphs $F_1, F_2, \ldots, F_r$, i.e., $ex_k(n; F_1, F_2, \ldots, F_r) = ex_k(n; F_1 \cup F_2 \cup \ldots \cup F_r)$.

For the family of k-uniform Berge paths of length ℓ, Györi, Katona and Lemons [5] determined $ex_k(n; \mathcal{B}_\ell^{(k)})$ exactly for infinitely many n. In [2], Füredi, Jiang and Seiver established the following results.

Theorem 1 ([2]) *Let k, t be positive integers, where $k \geq 3$. For sufficiently large n, we have*

$$ex_k\left(n; \mathbb{P}_{2t+1}^{(k)}\right) = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \ldots + \binom{n-t}{k-1}.$$

The only extremal family consists of all the k-sets in $[n]$ that meet some fixed set S of t vertices. Also,

$$ex_k\left(n; \mathbb{P}_{2t+2}^{(k)}\right) = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \ldots + \binom{n-t}{k-1} + \binom{n-t-2}{k-2}.$$

The only extremal family consists of all the k-sets in $[n]$ that meet some fixed set S of t vertices plus all the k-sets in $[n] \setminus S$ that contain some two fixed elements.

For more results we refer to [2, 6].

For the minimal and linear cycles, Füredi and Jiang [3] determined the extremal numbers when the forbidden hypergraph is a single minimal cycle or a single linear cycle. This confirms, in a stronger form, a conjecture of Mubayi and Verstraëte [6] for $k \geq 5$ and adds to the limited list of hypergraphs whose Turán numbers have been known either exactly or asymptotically. Their main results are as follows.

Theorem 2 **([3])** *Let t be a positive integer, $k \geq 4$. For sufficiently large n, we have $ex_k\left(n; \mathcal{C}_{2t+1}^{(k)}\right) = \binom{n}{k} - \binom{n-t}{k}$, and $ex_k\left(n; \mathcal{C}_{2t+2}^{(k)}\right) = \binom{n}{k} - \binom{n-t}{k} + 1$. For $\mathcal{C}_{2t+1}^{(k)}$, the only extremal family consists of all the k-sets in $[n]$ that meet some fixed t-set S. For $\mathcal{C}_{2t+2}^{(k)}$, the only extremal family consists of all the k-sets in $[n]$ that intersect some fixed t-set S plus one additional k-set outside S.*

Theorem 3 **([3])** *Let t be a positive integer, $k \geq 5$. For sufficiently large n, we have $ex_k\left(n; \mathbb{C}_{2t+1}^{(k)}\right) = \binom{n}{k} - \binom{n-t}{k}$, and $ex_k\left(n, \mathbb{C}_{2t+2}^{(k)}\right) = \binom{n}{k} - \binom{n-t}{k} + \binom{n-t-2}{k-2}$. For $\mathbb{C}_{2t+1}^{(k)}$, the only extremal family consists of all the k-sets in $[n]$ that meet some fixed t-set S. For $\mathbb{C}_{2t+2}^{(k)}$, the only extremal family consists of all the k-sets in $[n]$ that intersect some fixed t-set S plus all the k-sets in $[n] \setminus S$ that contain some two fixed elements.*

From definition, two k-uniform minimal cycles of the same length may not be isomorphic. Hence, we define the following family of hypergraphs, where every member consists of r vertex disjoint cycles:

$$\mathcal{S}(\ell_1, \ldots, \ell_r) = \{C_1 \cup \ldots \cup C_r : C_i \in \mathcal{C}_{\ell_i}^{(k)} \ for \ i \in [r]\}.$$

Apart from the results above, we will need the following result, due to Keevash, Mubayi and Wilson [4].

Theorem 4 **([4])** *Let H be a k-uniform hypergraph on n vertices with no two edges intersecting in exactly one vertex, where $k \geq 3$. Then $|E(H)| \leq \binom{n}{k-2}$.*

Based on earlier work of Füredi and Jiang [3], in this paper we will determine precisely the exact Turán numbers when the forbidden hypergraphs are r vertex disjoint minimal cycles or r vertex disjoint linear cycles. Our main results are as follows.

Theorem 5 *Let integers $k \geq 4$, $r \geq 1$, $\ell_1, \ldots, \ell_r \geq 3$, $t = \sum_{i=1}^{r} \left\lfloor \frac{\ell_i+1}{2} \right\rfloor - 1$, and $I = 1$ if all the $\ell_1, \ldots, \ell_r$ are even, and $I = 0$ otherwise. For sufficiently large n,*

$$ex_k\left(n; \mathcal{S}(\ell_1, \ldots, \ell_r)\right) = \binom{n}{k} - \binom{n-t}{k} + I.$$

Theorem 6 *Let integers $k \geq 5$, $r \geq 1$, $\ell_1, \ldots, \ell_r \geq 3$, $t = \sum_{i=1}^{r} \left\lfloor \frac{\ell_i+1}{2} \right\rfloor - 1$, and $J = \binom{n-t-2}{k-2}$ if all the $\ell_1, \ldots, \ell_r$ are even, and $J = 0$ otherwise. For sufficiently large n,*

$$ex_k\left(n; \mathbb{C}_{\ell_1}^{(k)}, \ldots, \mathbb{C}_{\ell_r}^{(k)}\right) = \binom{n}{k} - \binom{n-t}{k} + J.$$

Sometimes, we allow the hypergraph to contain less than r minimal or linear cycles, consider the Turán number in such cases, we have the following two corollaries. We use notation $r \cdot F$ to denote r vertex disjoint copies of a hypergraph F. Let $\ell_1 = \ldots = \ell_r = \ell$, we can immediately get the following two corollaries from Theorems 5 and 6.

Corollary 1 *Let integers $k \geq 4$, $r \geq 1$, $\ell \geq 3$, $t = r \left\lfloor \frac{\ell+1}{2} \right\rfloor - 1$, and $I = 1$ if ℓ is even, and $I = 0$ if ℓ is odd. For sufficiently large n,*

$$ex_k\left(n; r \cdot \mathcal{C}_{\ell}^{(k)}\right) = \binom{n}{k} - \binom{n-t}{k} + I.$$

Corollary 2 *Let integers $k \geq 5$, $r \geq 1$, $\ell \geq 3$, $t = r \left\lfloor \frac{\ell+1}{2} \right\rfloor - 1$, and $J = \binom{n-t-2}{k-2}$ if ℓ is even, and $J = 0$ if ℓ is odd. For sufficiently large n,*

$$ex_k\left(n; r \cdot \mathbb{C}_{\ell}^{(k)}\right) = \binom{n}{k} - \binom{n-t}{k} + J.$$

We can see that Theorem 2 and Theorem 3 are special cases of Corollary 1 and Corollary 2 (when $r = 1$), respectively. However, the methods we used in the proofs of Theorem 5 and Theorem 6 are quite different from Theorem 2 and Theorem 3.

2 Proof of Theorem 5

For convenience, we define $f(n,k,\{\ell_1,\ldots,\ell_r\}) = \binom{n}{k} - \binom{n-t}{k} + I$. Note that the hypergraph on n vertices that has every edge incident to some fixed t-set S, along with one additional edge disjoint from S when all of $\ell_1,\ldots,\ell_r$ are even, has exactly $f(n,k,\{\ell_1,\ldots,\ell_r\})$ edges and does not contain a copy of any member of $\mathcal{S}(\ell_1,\ldots,\ell_r)$.

Thus, to prove Theorem 5, it suffices to show that $ex_k\left(n;\mathcal{S}(\ell_1,\ldots,\ell_r)\right) \le \binom{n}{k} - \binom{n-t}{k} + I$, i.e., any hypergraph on n vertices with more than $f(n,k,\{\ell_1,\ldots,\ell_r\})$ edges must contain a member of $\mathcal{S}(\ell_1,\ldots,\ell_r)$. We use induction on r. From Theorem 2, the case $r=1$ has been proved. Assume that $r\ge 2$, and Theorem 5 holds for smaller r.

Let H be a hypergraph on n vertices with m edges and $m > f(n,k,\{\ell_1,\ldots,\ell_r\})$. Since $f(n,k,\{\ell_1,\ldots,\ell_r\}) > f(n,k,\ell_1)$ for sufficiently large n, there exists at least one k-uniform minimal ℓ_1-cycle in H. Take one of them, denote its vertex set by C, so $\ell_1 \le |C| \le (k-1)\ell_1$. We have $|E(H\backslash C)| \le f(n-|C|,k,\{\ell_2,\ldots,\ell_r\})$, since otherwise, by induction hypothesis, we can find vertex disjoint copies of $\mathcal{C}^{(k)}_{\ell_2}\cup\ldots\cup\mathcal{C}^{(k)}_{\ell_r}$ in H, plus the minimal ℓ_1-cycle on C, and then there is a copy of a member of $\mathcal{S}(\ell_1,\ldots,\ell_r)$ in H already.

Let m_C denote the number of edges in H incident to vertices in C. Then,

$$\begin{aligned} m_C &\ge m - f(n-|C|,k,\{\ell_2,\ldots,\ell_r\}) &(1)\\ &\ge f(n,k,\{\ell_1,\ldots,\ell_r\}) - f(n-\ell_1,k,\{\ell_2,\ldots,\ell_r\}) &(2)\\ &= \frac{\left\lfloor\frac{\ell_1+1}{2}\right\rfloor}{(k-1)!}n^{k-1} + O\left(n^{k-2}\right). &(3)\end{aligned}$$

We call an edge in H a *terminal edge* if it contains exactly one vertex in C. Let T denote the set of all terminal edges in H. For every $(k-1)$-set R in $V(H)\setminus C$, define

$$T_R = \{E\in T:\ R\subseteq E\}.$$

According to the size of each set T_R, we divide all the $(k-1)$-sets in $V(H)\setminus C$ into two sets, such that

$$X = \{R\subseteq V(H)\setminus C \textit{ and } |R| = k-1 : |T_R| \le \left\lfloor\frac{\ell_1+1}{2}\right\rfloor - 1\}$$

$$Y = \{R\subseteq V(H)\setminus C \textit{ and } |R| = k-1 : |T_R| \ge \left\lfloor\frac{\ell_1+1}{2}\right\rfloor\}.$$

It is not difficult to give an upper bound of m_C in terms of $|X|$ and $|Y|$ as follows:

$$\begin{aligned} m_C &\leq \binom{|C|}{2}\binom{n-2}{k-2} + |X|\left(\left\lfloor \frac{\ell_1+1}{2} \right\rfloor - 1\right) + |Y|\cdot|C| \\ &\leq \binom{|C|}{2}\binom{n-2}{k-2} + \binom{n}{k-1}\left(\left\lfloor \frac{\ell_1+1}{2} \right\rfloor - 1\right) + |Y|\cdot \ell_1\,(k-1)\,. \end{aligned}$$

Combine with (3), we have

$$|Y| \geq \frac{n^{k-1}}{(k-1)\,\ell_1\,(k-1)!} + O(n^{k-2}). \tag{4}$$

For any $(k-1)$-set $R \in Y$, there are at least $\lfloor \frac{\ell_1+1}{2} \rfloor$ vertices in C that can form terminal edges with R. We choose exactly $\lfloor \frac{\ell_1+1}{2} \rfloor$ of them, and call the vertex set of these $\lfloor \frac{\ell_1+1}{2} \rfloor$ vertices *terminal set* relative to R. Since the number of $\lfloor \frac{\ell_1+1}{2} \rfloor$-sets in C is at most $\binom{|C|}{\lfloor \frac{\ell_1+1}{2} \rfloor}$, we can get that some elements in Y may have the same terminal set. And it is easy to derive that the number of $(k-1)$-sets in Y with the same terminal set is at least

$$\frac{n^{k-1}}{(k-1)\,\ell_1\,(k-1)!}\binom{|C|}{\lfloor \frac{\ell_1+1}{2} \rfloor}^{-1} + O(n^{k-2}) \geq \frac{n^{k-1}}{(k-1)\,\ell_1\,(k-1)!}\binom{(k-1)\,\ell_1}{\lfloor \frac{\ell_1+1}{2} \rfloor}^{-1} + O(n^{k-2}).$$

Choose one terminal set U in C, such that there are at least $\frac{n^{k-1}}{(k-1)\ell_1(k-1)!}\binom{(k-1)\ell_1}{\lfloor \frac{\ell_1+1}{2} \rfloor}^{-1} + O(n^{k-2})$ $(k-1)$-sets in $V(H)\setminus C$, every such $(k-1)$-set can form a terminal edge with every vertex in U. Let R_U be the set of all the common $(k-1)$-sets associated with U in $V(H)\setminus C$. Then we have

$$|R_U| \geq \frac{n^{k-1}}{(k-1)\,\ell_1\,(k-1)!}\binom{(k-1)\,\ell_1}{\lfloor \frac{\ell_1+1}{2} \rfloor}^{-1} + O(n^{k-2}). \tag{5}$$

Let m_U denote the number of edges incident to vertices in U. Then,

$$m_U \leq \left\lfloor \frac{\ell_1+1}{2} \right\rfloor \binom{n - \lfloor \frac{\ell_1+1}{2} \rfloor}{k-1} + m',$$

where m' is the number of edges which contain at least two vertices in U. With some calculations, we have

$$\begin{aligned} & f(n,k,\{\ell_1,\dots,\ell_r\}) - f(n - \left\lfloor \frac{\ell_1+1}{2} \right\rfloor, k, \{\ell_2,\dots,\ell_r\}) - m_U \\ = & \binom{n-1}{k-1} + \binom{n-2}{k-1} + \cdots + \binom{n - \lfloor \frac{\ell_1+1}{2} \rfloor}{k-1} - m_U \\ \geq & \left[\binom{n-1}{k-1} - \binom{n - \lfloor \frac{\ell_1+1}{2} \rfloor}{k-1}\right] + \left[\binom{n-2}{k-1} - \binom{n - \lfloor \frac{\ell_1+1}{2} \rfloor}{k-1}\right] \\ & + \cdots + \left[\binom{n - \lfloor \frac{\ell_1+1}{2} \rfloor + 1}{k-1} - \binom{n - \lfloor \frac{\ell_1+1}{2} \rfloor}{k-1}\right] - m'. \end{aligned}$$

It is not difficult to deduce that the last expression is nonnegative (consider the combinatorial meaning of that expression). Hence, we can derive

$$\begin{aligned} E(H \setminus U) &= m - m_U > f(n, k, \{\ell_1, \dots, \ell_r\}) - m_U \\ &\geq f(n - \left\lfloor \frac{\ell_1+1}{2} \right\rfloor, k, \{\ell_2, \dots, \ell_r\}). \end{aligned}$$

Thus by the induction hypothesis, there exists a member of $\mathcal{S}(\ell_2, \dots, \ell_r)$ with vertex set W in $V(H) \setminus U$. Also, we have

$$|W| \leq (k-1) \sum_{i=2}^{r} \ell_i. \tag{6}$$

Now we focus on finding a k-uniform minimal ℓ_1-cycle disjoint from W. Considering the $(k-1)$-uniform hypergraph H_0 with vertex set $V(H) \setminus U$ and edge set R_U, we will prove the following claim:

Claim 1 *There are $\lfloor \frac{\ell_1}{2} \rfloor$ pairs of $(k-1)$-edges in H_0, say $\{a_i, b_i\}$, $i = 1, \dots, \lfloor \frac{\ell_1}{2} \rfloor$, such that for every i, a_i and b_i have exactly one common vertex, and for any $j \neq i$, $\{a_i, b_i\}$ and $\{a_j, b_j\}$ are vertex disjoint, moreover, all these $(k-1)$-edges disjoint from W.*

Proof. The number of $(k-1)$-edges incident with some vertices in W is at most $|W| \cdot \binom{n-1}{k-2}$. With the aid of (5) and (6), in R_U the number of $(k-1)$-edges disjoint from W is at least

$$\frac{n^{k-1}}{(k-1)\,\ell_1\,(k-1)!} \binom{(k-1)\,\ell_1}{\lfloor \frac{\ell_1+1}{2} \rfloor}^{-1} + O(n^{k-2}) - (k-1) \sum_{i=2}^{r} \ell_i \binom{n-1}{k-2} > \binom{n - \lfloor \frac{\ell_1+1}{2} \rfloor}{k-2}.$$

By Theorem 4, we can find a pair $\{a_1, b_1\}$ of $(k-1)$-edges with exactly one common vertex. Let $p = \lfloor \frac{\ell_1}{2} \rfloor (2k-3)$. Since $\frac{n^{k-1}}{(k-1)\ell_1(k-1)!} \binom{(k-1)\ell_1}{\lfloor \frac{\ell_1+1}{2} \rfloor}^{-1} + O(n^{k-2}) - (k-1) \sum_{i=2}^{r} \ell_i \binom{n-1}{k-2} - p\binom{n-1}{k-2} > \binom{n-\lfloor \frac{\ell_1+1}{2} \rfloor}{k-2}$, we can repeat the argument above to find $\{a_2, b_2\}$, $\dots, \{a_{\lfloor \frac{\ell_1}{2} \rfloor}, b_{\lfloor \frac{\ell_1}{2} \rfloor}\}$ satisfying the properties described in Claim 1. □

Let $U = \{u_1, \dots, u_{\lfloor \frac{\ell_1+1}{2} \rfloor}\}$. To form the required minimal ℓ_1-cycle, we need to consider the following two cases:

Case 1. ℓ_1 is even.

Find $\frac{\ell_1}{2}$ pairs of $(k-1)$-edges in H_0 as described in Claim 1, still denote them by $\{a_i, b_i\}$, $i = 1, \dots, \frac{\ell_1}{2}$. Construct a k-uniform minimal ℓ_1-cycle in H with edges:

$$a_1 \cup \{u_1\}, b_1 \cup \{u_2\}, a_2 \cup \{u_2\}, \dots, b_{\frac{\ell_1}{2}-1} \cup \{u_{\frac{\ell_1}{2}}\}, a_{\frac{\ell_1}{2}} \cup \{u_{\frac{\ell_1}{2}}\}, b_{\frac{\ell_1}{2}} \cup \{u_1\}.$$

Case 2. ℓ_1 is odd.

Find $\frac{\ell_1-3}{2}$ pairs of $(k-1)$-edges in H_0 as described in Claim 1. Similar to the proof of Claim 1. Let Q be the union of W and the set of vertices in all these $\frac{\ell_1-3}{2}$ pairs of $(k-1)$-edges. Hence, $|Q| = \frac{\ell_1-3}{2}(2k-3)+|W|$. By Theorem 1, $ex_{k-1}\left(n-\lfloor\frac{\ell_1+1}{2}\rfloor; \mathbb{P}_3^{(k-1)}\right) = \frac{1}{(k-2)!}n^{k-2}+O(n^{k-3})$, for sufficiently large n. In H_0, the number of $(k-1)$-edges disjoint from Q is at least $\frac{n^{k-1}}{(k-1)\ell_1(k-1)!}\binom{(k-1)\ell_1}{\lfloor\frac{\ell_1+1}{2}\rfloor}^{-1}+O(n^{k-2})-|Q|\binom{n-1}{k-2} > \frac{1}{(k-2)!}n^{k-2}+O(n^{k-3})$. That implies that in H_0 we can find a $\mathbb{P}_3^{(k-1)}$ in the remaining $(k-1)$-edges disjoint from Q. Let x, y, z be the three consecutive $(k-1)$-edges in $\mathbb{P}_3^{(k-1)}$. Then, in H we can form a k-uniform minimal ℓ_1-cycle with edges:

$$a_1\cup\{u_1\}, b_1\cup\{u_2\}, a_2\cup\{u_2\}, \ldots, a_{\frac{\ell_1-3}{2}}\cup\{u_{\frac{\ell_1-3}{2}}\},$$
$$b_{\frac{\ell_1-3}{2}}\cup\{u_{\frac{\ell_1-1}{2}}\}, x\cup\{u_{\frac{\ell_1-1}{2}}\}, y\cup\{u_{\frac{\ell_1+1}{2}}\}, z\cup\{u_1\}.$$

Moreover, it is easy to see that this k-uniform minimal ℓ_1-cycle is not only minimal, but also linear, no matter when ℓ_1 is even or odd. Thus, we have constructed r disjoint k-uniform minimal cycles. So, the hypergraph which contains no member of $\mathcal{S}(\ell_1,\ldots,\ell_r)$ can not have more than $f(n,k,\{\ell_1,\ldots,\ell_r\})$ edges. The proof is thus complete. ∎

3 Proof of Theorem 6

Let $g(n,k,\{\ell_1,\ldots,\ell_r\}) = \begin{pmatrix} n \\ k \end{pmatrix} - \begin{pmatrix} n-t \\ k \end{pmatrix} + J$. Firstly, we point out that the hypergraph on n vertices that has every edge incident to some fixed t-set S, along with all the k-edges disjoint from S containing some two fixed elements not in S when all of $\ell_1,\ldots,\ell_r$ are even, has exactly $g(n,k,\{\ell_1,\ldots,\ell_r\})$ edges and does not contain a copy of any member of $\mathbb{C}_{\ell_1}^{(k)}\cup\ldots\cup\mathbb{C}_{\ell_r}^{(k)}$.

Hence, it suffices to show that $ex_k\left(n;\mathbb{C}_{\ell_1}^{(k)},\ldots,\mathbb{C}_{\ell_r}^{(k)}\right) \le g(n,k,\{\ell_1,\ldots,\ell_r\})$. We proceed by induction on r again since the case $r=1$ is provided by Theorem 3. Let H be a hypergraph on n vertices with $m > g(n,k,\{\ell_1,\ldots,\ell_r\})$ edges. If one of $\ell_1,\ldots,\ell_r$ is even, rearrange the sequence to make sure ℓ_1 is even.

As in the proof of Theorem 5, since $g(n,k,\{\ell_1,\ldots,\ell_r\}) > g(n,k,\ell_1)$ for sufficiently large n, there exists at least one k-uniform linear ℓ_1-cycle in H. Take one of them, denote its vertex set by C. Similarly, we have $|E(H\setminus C)| \le g(n-|C|,k,\{\ell_2,\ldots,\ell_r\})$. Still let m_C denote the number of edges in H incident to some vertices in C. With some calculations, we can get

$$m_C \geq \frac{\left\lfloor \frac{\ell_1+1}{2} \right\rfloor}{(k-1)!} n^{k-1} + O\left(n^{k-2}\right).$$

Again we define terminal edges, T_R, X, Y as before, we then can find the $\lfloor \frac{\ell_1+1}{2} \rfloor$-set U, too. Then by induction hypothesis, we can find a copy of $\mathbb{C}_{\ell_2}^{(k)} \cup \ldots \cup \mathbb{C}_{\ell_r}^{(k)}$ on vertex set W in $V(H) \setminus U$. Now we focus on finding a k-uniform linear ℓ_1-cycle disjoint from W. Again considering the $(k-1)$-uniform hypergraph H_0 with vertex set $V(H) \setminus U$ and edge set R_U, it is easy to see that the Claim 1 still holds. Thus, like Theorem 5, we have the terminal set $U = \{u_1, \ldots, u_{\lfloor \frac{\ell_1+1}{2} \rfloor}\}$. To form the required linear ℓ_1-cycle, we also need to consider the following two cases:

Case 1. ℓ_1 is even.

Find $\frac{\ell_1}{2}$ pairs of $(k-1)$-edges in H_0 as described in Claim 1, still denote them by $\{a_i, b_i\}$, $i = 1, \ldots, \frac{\ell_1}{2}$. Construct a k-uniform linear ℓ_1-cycle in H with edges:

$$a_1 \cup \{u_1\}, b_1 \cup \{u_2\}, a_2 \cup \{u_2\}, \ldots, b_{\frac{\ell_1}{2}-1} \cup \{u_{\frac{\ell_1}{2}}\}, a_{\frac{\ell_1}{2}} \cup \{u_{\frac{\ell_1}{2}}\}, b_{\frac{\ell_1}{2}} \cup \{u_1\}.$$

Case 2. ℓ_1 is odd.

Find $\frac{\ell_1-3}{2}$ pairs of $(k-1)$-edges in H_0 as described in Claim 1. Similar to the proof of Claim 1. Let Q be the union of W and the set of vertices in all these $\frac{\ell_1-3}{2}$ pairs of $(k-1)$-edges. Hence, $|Q| = \frac{\ell_1-3}{2}(2k-3) + |W|$. By Theorem 1, $ex_{k-1}\left(n - \lfloor \frac{\ell_1+1}{2} \rfloor; \mathbb{P}_3^{(k-1)}\right) = \frac{1}{(k-2)!} n^{k-2} + O(n^{k-3})$, for sufficiently large n. In H_0, the number of $(k-1)$-edges disjoint from Q is at least $\frac{n^{k-1}}{(k-1)\ell_1(k-1)!} \binom{(k-1)\ell_1}{\lfloor \frac{\ell_1+1}{2} \rfloor}^{-1} + O(n^{k-2}) - |Q|\binom{n-1}{k-2} > \frac{1}{(k-2)!} n^{k-2} + O(n^{k-3})$. That implies that in H_0 we can find a $\mathbb{P}_3^{(k-1)}$ in the remaining $(k-1)$-edges disjoint from Q. Let x, y, z be the three consecutive $(k-1)$-edges in $\mathbb{P}_3^{(k-1)}$. Then, in H we can form a k-uniform linear ℓ_1-cycle with edges:

$$a_1 \cup \{u_1\}, b_1 \cup \{u_2\}, a_2 \cup \{u_2\}, \ldots, a_{\frac{\ell_1-3}{2}} \cup \{u_{\frac{\ell_1-3}{2}}\},$$
$$b_{\frac{\ell_1-3}{2}} \cup \{u_{\frac{\ell_1-1}{2}}\}, x \cup \{u_{\frac{\ell_1-1}{2}}\}, y \cup \{u_{\frac{\ell_1+1}{2}}\}, z \cup \{u_1\}.$$

Since we construct this k-uniform linear ℓ_1-cycle avoiding the vertices in W, we know that the hypergraph containing no $\mathbb{C}_{\ell_1}^{(k)} \cup \ldots \cup \mathbb{C}_{\ell_r}^{(k)}$ can not have more than $g(n, k, \{\ell_1, \ldots, \ell_r\})$ edges. The proof is then complete. ∎

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