Nordhaus-Gaddum-type results for the generalized edge-connectivity of graphs*

Xueliang Li [1], Yaping Mao [1,2]

[1]Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, China

[2]Department of Mathematics, Qinghai Normal
University, Xining, Qinghai 810008, China

E-mails: lxl@nankai.edu.cn; maoyaping@ymail.com

Abstract

For a graph G and a set S of vertices of G, let $\lambda(S)$ denote the maximum number ℓ of pairwise edge-disjoint Steiner trees $T_1, T_2, \cdots, T_\ell$ in G such that $S \subseteq V(T_i)$ for every $1 \leq i \leq \ell$. For an integer k with $2 \leq k \leq n$, where n is the order of G, the generalized k-edge-connectivity $\lambda_k(G)$ of G is defined as $\lambda_k(G) = \min\{\lambda(S) \,|\, S \subseteq V(G) \textit{ and } |S| = k\}$. In this paper, we consider the Nordhaus-Gaddum-type results for the parameter $\lambda_k(G)$. We obtain sharp upper and lower bounds of $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$ for a graph G of order n, as well as a graph G of order n and size m. Some graph classes attaining these bounds are also given.

Keywords: edge-connectivity; Steiner tree; edge-disjoint Steiner trees; generalized edge-connectivity; packing; complementary graph; Nordhaus-Gaddum-type result.

AMS subject classification 2010: 05C40, 05C05, 05C75, 05C76.

1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to the book [4] for graph theoretical notation and terminology not described here. For a graph $G(V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (*a Steiner tree* for short) is a subgraph $T(V', E')$ of G that is a tree with $S \subseteq V'$. Two Steiner trees T and T' connecting S are *edge-disjoint* if $E(T) \cap E(T') = \varnothing$. The *Steiner Tree Packing Problem* for a given graph $G(V, E)$ and $S \subseteq V(G)$ asks to find a set of maximum number of edge-disjoint S-Steiner trees in G. This problem has obtained wide attention and many results have been obtained, see [7, 8, 10, 11, 24, 26]. The problem for $S = V(G)$ is called the *Spanning Tree Packing Problem*. For any graph G of order n, the *spanning tree packing number* or *STP number*, is the maximum number of edge-disjoint spanning trees contained in G. For the STP number, we refer the reader to Palmer's survey [23].

*Supported by NSFC No.11071130 and the "973" program.

Recently, we introduced the concept of the generalized edge-connectivity of a graph G in [21]. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local edge-connectivity* $\lambda(S)$ is the maximum number of edge-disjoint Steiner trees connecting S in G. Note that when $|S| = 2$ a minimum Steiner tree connecting S is just a path connecting S. For an integer k with $2 \leq k \leq n$, where n is the order of G, the *generalized k-edge-connectivity* $\lambda_k(G)$ of a graph G is defined as $\lambda_k(G) = \min\{\lambda(S) : S \subseteq V(G) \text{ and } |S| = k\}$. Clearly, when $|S| = 2$, $\lambda_2(G)$ is nothing new but the edge-connectivity $\lambda(G)$ of G, that is, $\lambda_2(G) = \lambda(G)$, which is the reason why we address $\lambda_k(G)$ as the generalized k-edge-connectivity of G. Obviously, the STP number of a graph G is just $\lambda_n(G)$. By convention, for a connected graph G with less than k vertices, we set $\lambda_k(G) = 1$, and set $\lambda_k(G) = 0$ when G is disconnected. $\lambda_k(G)$ is called the generalized k-edge-connectivity also because it is a natural counterpart of the concept of the generalized (vertex) connectivity, introduced by Chartrand et al. [5] in 1984. Results on the generalized connectivity can be seen in [12, 13, 14, 15, 17, 18, 19, 20, 21].

Let $\mathcal{G}(n)$ denote the class of simple graphs of order n ($n \geq 2$) and $\mathcal{G}(n, m)$ the subclass of $\mathcal{G}(n)$ in which every graph has n vertices and m edges. Give a graph parameter $f(G)$ and a positive integer n, the *Nordhaus-Gaddum(**N-G**) Problem* is to determine sharp bounds for (1) $f(G) + f(\overline{G})$ and (2) $f(G) \cdot f(\overline{G})$, as G ranges over the class $\mathcal{G}(n)$, and characterize the extremal graphs, i.e., graphs that achieve the bounds. The Nordhaus-Gaddum type relations have received wide attention; see a recent survey paper [3] by Aouchiche and Hansen.

In this paper, we study the above problem on the generalized edge-connectivity. The paper is organized as follows. In Section 2, we study $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$ for the parameter $\lambda_k(G)$ where $G \in \mathcal{G}(n)$, and get the following result.

Theorem 1. *Let $G \in \mathcal{G}(n)$ and let k be an integer with $3 \leq k \leq n$. Then*

(1) $1 \leq \lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil k/2 \rceil$;

(2) $0 \leq \lambda_k(G) \cdot \lambda_k(\overline{G}) \leq \left[\frac{n - \lceil k/2 \rceil}{2}\right]^2$.

Moreover, the upper and lower bounds are sharp.

In Section 3, we focus our attention on the graph class $\mathcal{G}(n, m)$ and obtain the sharp bounds of $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$.

Theorem 2. *Let $G \in \mathcal{G}(n, m)$ and let k be an integer with $3 \leq k \leq n$. For $n \geq 6$, we have*

(1) $L(n, m) \leq \lambda_k(G) + \lambda_k(\overline{G}) \leq M(n, m)$;

(2) $0 \leq \lambda_k(G) \cdot \lambda_k(\overline{G}) \leq N(n, m)$,

where $L(n, m), M(n, m), N(n, m)$ are defined in Lemmas 8 and 9.

Moreover, the upper and lower bounds are sharp.

The following theorem and corollary will be used in Section 3 and Section 2, respectively.

Theorem 3. *(Nash-Williams [22], Tutte [25]) A multigraph G contains a system of ℓ edge-disjoint spanning trees if and only if*

$$\|G/\mathscr{P}\| \geq \ell(|\mathscr{P}| - 1)$$

holds for every partition $\mathscr{P}$ of $V(G)$, where $\|G/\mathscr{P}\|$ denotes the number of crossing edges in G, i.e., edges between distinct parts of $\mathscr{P}$.

Corollary 1. *Every 2ℓ-edge-connected graph contains a system of ℓ edge-disjoint spanning trees.*

2 Nordhaus-Gaddum-type results in $\mathcal{G}(n)$

All graphs considered in this section are of order n. The following observation is obvious.

Observation 1. *Let G be a graph of order n, and let k be an integer with $3 \leq k \leq n$.*

(1) *If G is a connected graph, then $1 \leq \lambda_k(G) \leq \lambda(G) \leq \delta(G)$.*

(2) *If H is a spanning subgraph of G, then $\lambda_k(H) \leq \lambda_k(G)$.*

(3) *Let G be a connected graph with minimum degree δ. If G has two adjacent vertices of degree δ, then $\lambda_k(G) \leq \delta - 1$.*

Alavi and Mitchem in [2] considered Nordhaus-Gaddum-type results for the connectivity and edge-connectivity parameters. We are concerned with analogous inequalities involving the generalized k-edge-connectivity.

To start with, let us recall the definition of Harary graph $H_{n,d}$:

Case 1. d even. Let $d = 2r$. Then $H_{n,2r}$ is constructed as follows. It has vertices $0, 1, \cdots, n-1$ and two vertices i and j are jointed if $i - r \leq j \leq i + r$ (where addition is taken modulo n).

Case 2. d odd, n even. Let $d = 2r + 1$. Then $H_{n,2r+1}$ is constructed by first drawing $H_{n,2r}$ and then adding edges joining vertex i to vertex $i + \frac{n}{2}$ for $1 \leq i \leq \frac{n}{2}$.

Case 3. d odd, n even. Let $d = 2r + 1$. Then $H_{n,2r+1}$ is constructed by first drawing $H_{n,2r}$ and then adding edges joining vertex 0 to vertices $\frac{n-1}{2}$ and $\frac{n+1}{2}$ and i to vertex $i + \frac{n+1}{2}$ for $1 \leq i \leq \frac{n-1}{2}$.

Observe that the Harary graph $H_{n,d}$ is constructed by arranging the n vertices in a circular order and spreading the d edges around the boundary in a nice way, keeping the chords as short as possible. They have the maximum connectivity for their size and $\kappa(H_{n,d}) = \lambda(H_{n,d}) = \delta(H_{n,d}) = d$. Palmer [23] gave the STP number of some special graph classes.

Lemma 1. [23] (1) *The STP number of a complete bipartite graph $K_{a,b}$ is $\lfloor \frac{ab}{a+b-1} \rfloor$.*

(2) *The STP number of a Harary graph $H_{n,d}$ is $\lfloor d/2 \rfloor$.*

According to (1) of Observation 1, we can obtain a sharp lower bound for the generalized k-edge-connectivity by Corollary 1. Actually, a λ-edge-connected graph G contains $\lfloor \frac{1}{2}\lambda(G) \rfloor$ edge-disjoint spanning trees, each of which is also a Steiner tree connecting S. So the following proposition is immediate.

Proposition 1. *For a connected graph G of order n and $3 \leq k \leq n$, $\lambda_k(G) \geq \lfloor \frac{1}{2}\lambda(G) \rfloor$. Moreover, the lower bound is sharp.*

For the sharpness of this lower bound when $k = n$, we consider the Harary graph $H_{n,2r}$. Clearly, $\lambda(G) = 2r$. From (2) of Lemma 1, $H_{n,2r}$ contains exactly r spanning trees, that is, $\lambda_n(H_{n,2r}) = r$. So $\lambda_n(H_{n,2r}) = \lfloor \frac{1}{2}\lambda(G) \rfloor$. For a general k ($3 \leq k \leq n$), one can check that the cycle C_n can attain the lower bound since $\frac{1}{2}\lambda(C_n) = 1 = \lambda_k(C_n)$.

The following proposition indicates that the monotone properties of λ_k, that is, $\lambda_n \leq \lambda_{n-1} \leq \cdots \lambda_4 \leq \lambda_3 \leq \lambda$, is true for $2 \leq k \leq n$.

Proposition 2. *For two integers k and n with $2 \leq k \leq n-1$, and a connected graph G, $\lambda_{k+1}(G) \leq \lambda_k(G)$.*

Proof. Assume $3 \leq k \leq n-1$. Set $\lambda_{k+1}(G) = \ell$. For each $S \subseteq V(G)$ with $|S| = k$, we let $S' = S \cup \{u\}$, where $u \in V(G)$ but $u \notin S$. Since $\lambda_{k+1}(G) = \ell$, there exist ℓ edge-disjoint trees connecting S'. These trees are also ℓ edge-disjoint trees connecting S. So $\lambda_k(G) \geq \ell$ and $\lambda_{k+1}(G) \leq \lambda_k(G)$. Combining this with (1) of Observation 1, we get that $\lambda_{k+1}(G) \leq \lambda_k(G)$ for $2 \leq k \leq n-1$. □

Now we give the lower bounds of $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$.

Lemma 2. *Let $G \in \mathcal{G}(n)$ and let k be an integer with $3 \leq k \leq n$. Then*

(1) $\lambda_k(G) + \lambda_k(\overline{G}) \geq 1$;

(2) $\lambda_k(G) \cdot \lambda_k(\overline{G}) \geq 0$.

Moreover, the two lower bounds are sharp.

Proof. (1) If $\lambda_k(G) + \lambda_k(\overline{G}) = 0$, then $\lambda_k(G) = \lambda_k(\overline{G}) = 0$, that is, both G and $\overline{G}$ are disconnected, which is impossible, and so $\lambda_k(G) + \lambda_k(\overline{G}) \geq 1$.

(2) By definition, $\lambda_k(G) \geq 0$ and $\lambda_k(\overline{G}) \geq 0$, and so $\lambda_k(G) \cdot \lambda_k(\overline{G}) \geq 0$. □

The following observation gives the graphs attaining the lower bound of (2) in Lemma 2.

Observation 2. $\lambda_k(G) \cdot \lambda_k(\overline{G}) = 0$ *if and only if G or $\overline{G}$ is disconnected.*

In [21] we obtained the exact value of the generalized k-edge-connectivity of a complete graph K_n.

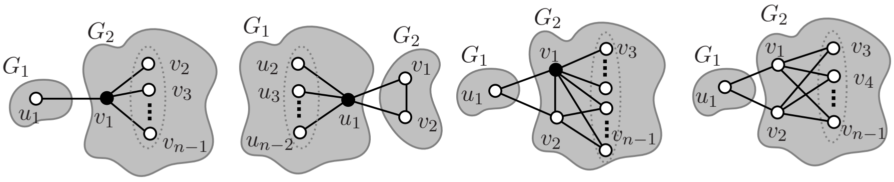
Lemma 3. [21] *For two integers n and k with $2 \leq k \leq n$, $\lambda_k(K_n) = n - \lceil k/2 \rceil$.*

For a connected graph G of order n, we know that $1 \leq \lambda_k(G) \leq \lambda_k(K_n) = n - \lceil k/2 \rceil$. In [21] we characterized the graphs attaining the upper bound.

Lemma 4. [21] *For a connected graph G of order n with $3 \leq k \leq n$, $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$ if and only if $G = K_n$ for k even; $G = K_n \setminus M$ for k odd, where M is an edge set such that $0 \leq |M| \leq \frac{k-1}{2}$.*

Now we want to characterize the graphs that attain the lower bound 1 of $\lambda_k(G)+\lambda_k(\overline{G})$. Before doing so, we give some graph classes (each graph of the classes has order n).

For $n \geq 5$, $\mathcal{G}_n^1$ is a graph class as shown in Figure 1 (a), each graph G of which satisfies that $\lambda(G) = 1$ and $d_G(v_1) = n-1$, where $v_1 \in V(G)$; $\mathcal{G}_n^2$ is a graph class as shown in



(a) $\mathcal{G}_n^1$ (b) $\mathcal{G}_n^2$ (c) $\mathcal{G}_n^3$ (d) $\mathcal{G}_n^4$

Figure 1. Graphs for Proposition 3 (The degree of a black vertex is $n-1$).

Figure 1 (b), each graph G of which satisfies that $\lambda(G) = 2$ and $d_G(u_1) = n-1$, where $u_1 \in V(G)$; $\mathcal{G}_n^3$ is a graph class as shown in Figure 1 (c), each graph G of which satisfies that $\lambda(G) = 2$ and $d_G(v_1) = n-1$, where $v_1 \in V(G)$; $\mathcal{G}_n^4$ is a graph class as shown in Figure 1 (d), each graph G of which satisfies $\lambda(G) = 2$.

The following observation and lemma are preparations for Proposition 3.

For $n \geq 5$, let $K_{2,n-2}^{+}$ be the graph obtained from the complete bipartite graph $K_{2,n-2}$ by adding one edge on the part having $n-2$ vertices and let $K_{2,n-2}^{++}$ denote any of the two graphs which are obtained from $K_{2,n-2}$ by adding two edges on the part having $n-2$ vertices.

Observation 3. *Let n be an integer with $n \geq 5$. Then*

(1) $\lambda_n(K_{2,n-2}^{++}) \geq 2$;

(2) $\lambda_{n-1}(K_{2,n-2}^{+}) \geq 2$, $\lambda_n(K_{2,n-2}^{+}) = 1$;

(3) $\lambda_{n-2}(K_{2,n-2}) \geq 2$, $\lambda_n(K_{2,n-2}) = \lambda_{n-1}(K_{2,n-2}) = 1$.

Proof. (1) As shown in Figure 2 (a), we have $\lambda_n(K_{2,n-2}^{++}) \geq 2$.

(2) As shown in Figure 2 (b), we have $\lambda_{n-1}(K_{2,n-2}^{+}) \geq 2$. Since $|E(K_{2,n-2}^{+})| = 2(n-2)+1$ and $\lambda_n(K_{2,n-2}^{+}) \leq \lfloor \frac{2(n-2)+1}{n-1} \rfloor$, then $\lambda_n(K_{2,n-2}^{+}) \leq 1$. Since $K_{2,n-2}^{+}$ is connected, then $\lambda_n(K_{2,n-2}^{+}) = 1$.

(3) As shown in Figure 2 (c), it follows that $\lambda_{n-2}(K_{2,n-2}) \geq 2$. Let $U = \{u_1, u_2\}$ and $W = \{w_1, w_2, \cdots, w_{n-2}\}$ be the two parts of the complete bipartite graph $K_{2,n-2}$. Choose $S = \{u_1, u_2, w_1, w_2, \cdots, w_{n-3}\}$. If there exists an S-tree containing the vertex w_{n-2}, then this tree will use $n-1$ edges of $E(K_{2,n-2})$, which implies that $\lambda_{n-1}(K_{2,n-2}) \leq 1$ since $|E(K_{2,n-2})| = 2(n-2)$. Suppose that any S-tree does not contain the vertex w_{n-2}. Pick up such a tree, say T. Then there exists a vertex with degree 2 in T, which implies that there is no other S-tree in $K_{2,n-2}$. So $\lambda_{n-1}(K_{2,n-2}) \leq 1$. Since $K_{2,n-2}$ is connected, $\lambda_{n-1}(K_{2,n-2}) = 1$. From Proposition 2, $\lambda_n(K_{2,n-2}) = 1$. □

Lemma 5. *Let G be a connected graph of order n, and let k be an integer with $3 \leq k \leq n$. If $\lambda(G) = 3$ and there exists a vertex $u \in V(G)$ such that $d_G(u) = n-1$, then $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$.*

Proof. Let $G_1, \cdots, G_r$ be the connected components of $G \setminus u$. Since $\lambda(G) = 3$, it follows that $\delta(G_i) \geq 2$ $(1 \leq i \leq r)$. Let $|V(G_i)| = n_i$ $(1 \leq i \leq r)$ and $V(G_i) = \{v_{i,1}, v_{i,2}, \cdots, v_{i,n_i}\}$. Then there exists an edge, without loss of generality, say $e_i = v_{i,1}v_{i,2} \in E(G_i)$ such

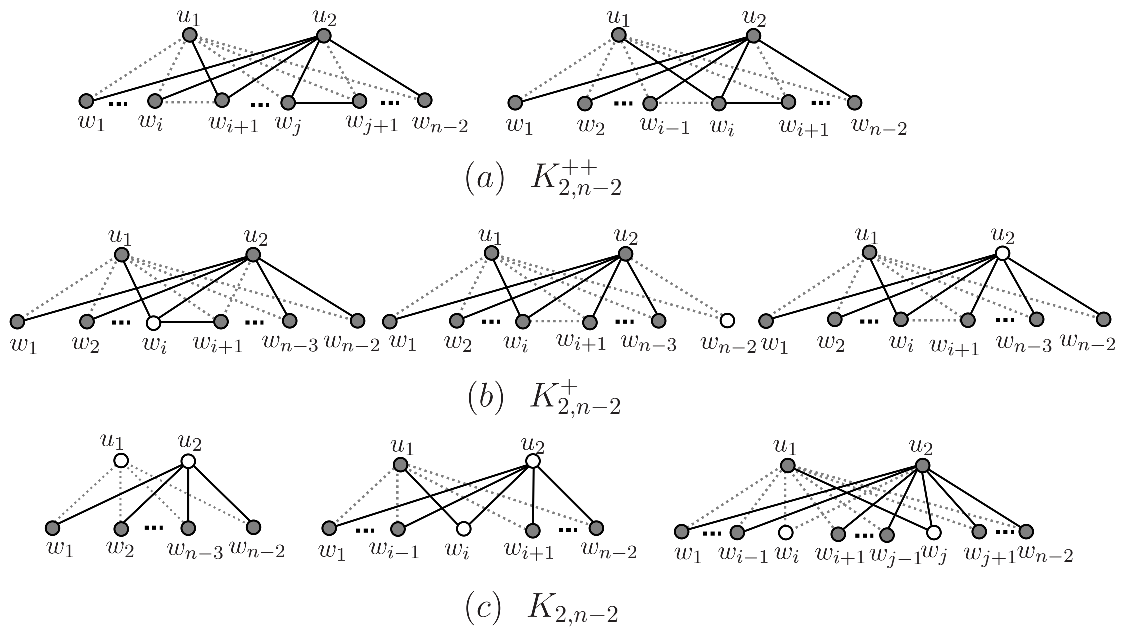


Figure 2. Graphs for Observation 3.

that $G_i \setminus e_i$ is connected for $1 \leq i \leq r$. Thus $G_i \setminus e_i$ contains a spanning tree, say T_i $(1 \leq i \leq r)$. The tree T induced by the edges in $\{uv_{1,1}, uv_{2,1}, \cdots, uv_{r,1}\} \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_r)$ and the tree T' induced by the edges in $\{v_{1,1}v_{1,2}, uv_{1,2}, \cdots, uv_{1,n_1}\} \cup \{v_{2,1}v_{2,2}, uv_{2,2}, \cdots, uv_{2,n_2}\} \cup \cdots \cup \{v_{r,1}v_{r,2}, uv_{r,2}, \cdots, uv_{r,n_r}\}$ are two spanning trees of G, and hence $\lambda_n(G) \geq 2$. Combining this with Proposition 2, we get $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$. □

Proposition 3. *Let G be a graph of order n, and let k be an integer with $3 \leq k \leq n$. $\lambda_k(G) + \lambda_k(\overline{G}) = 1$ if and only if G (symmetrically, $\overline{G}$) satisfies one of the following conditions:*

(1) $G \in \mathcal{G}_n^1$ or $G \in \mathcal{G}_n^2$;

(2) $G \in \mathcal{G}_n^3$ and there exists a component G_i of $G \setminus v_1$ such that G_i is a tree and $|V(G_i)| < k$;

(3) $G \in \{K_{2,n-2}^+, K_{2,n-2}\}$ for $k = n$ and $n \geq 5$, or $G \in \{P_3, C_3\}$ for $k = n = 3$, or $G \in \{C_4, K_4 \setminus e\}$ for $k = n = 4$, or $G = K_{3,3}$ for $k = n = 6$, or $G = K_{2,n-2}$ for $k = n - 1$ and $n \geq 5$, or $G = C_4$ for $k = n - 1 = 3$.

Proof. Sufficiency. Let G be a graph satisfying one of the conditions of (1), (2) and (3). One can see that G is connected and its complement $\overline{G}$ is disconnected. Thus $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(G)$ and $\lambda_k(G) \geq 1$. We only need to show that $\lambda_k(G) \leq 1$ for each graph G satisfying one of the conditions of (1), (2) and (3). For $G \in \mathcal{G}_n^1$, since $\delta(G) = 1$ we have $\lambda_k(G) \leq 1$ by (1) of Observation 1. For $G \in \mathcal{G}_n^2$, it follows that $\lambda_k(G) \leq \delta(G) - 1 = 1$ by (3) of Observation 1 since $d_G(v_1) = d_G(v_2) = \delta(G) = 2$. Suppose that $G \in \mathcal{G}_n^3$ and there exists a connected component G_i of $G \setminus v_1$ such that G_i is a tree and $|V(G_i)| < k$. Set $V(G_i) = \{v_{i,1}, v_{i,2}, \cdots, v_{i,n_i}\}$. We choose $S \subseteq V(G)$ such that $V(G_i) \cup \{v_1\} = S' \subseteq S$. Then $|E(G[S'])| = 2n_i - 1$. Since every spanning tree of $G[S']$ uses n_i edges of $E(G[S'])$, there exists at most one spanning tree in $G[S']$, which implies that there is at most one tree connecting S in G. So $\lambda_k(G) \leq 1$. For $G = K^{+}_{2,n-2}$, $\lambda_n(G) = 1$ by (2) of Observation 3. For $G = K_{2,n-2}$, by (3) of Observation 3, we have $\lambda_n(K_{2,n-2}) = \lambda_{n-1}(K_{2,n-2}) = 1$. For $G = K_{3,3}$, $\lambda_n(G) \leq \lfloor \frac{|E(G)|}{n-1} \rfloor = \lfloor \frac{9}{5} \rfloor = 1$. For $G \in \{P_3, C_3, C_4, K_4 \setminus e\}$, one can check that $\lambda_k(G) \leq 1$ for $k = n$ or $k = n-1$. From these together with $\lambda_k(G) \geq 1$, we have $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(G) = 1$.

Necessity. Suppose $\lambda_k(G) + \lambda_k(\overline{G}) = 1$. Then $\lambda_k(G) = 1$ and $\lambda_k(\overline{G}) = 0$, or $\lambda_k(\overline{G}) = 1$ and $\lambda_k(G) = 0$. By symmetry, without loss of generality, let $\lambda_k(G) = 1$ and $\lambda_k(\overline{G}) = 0$. From these together with Proposition 1, $\lambda(\overline{G}) = 0$ and $1 \leq \lambda(G) \leq 3$. So we have the following three cases to consider.

Case 1. $\lambda(G) = 1$.

For $n = 3$, one can check that $G = P_3$ satisfies $\lambda(G) = 1$ but $\lambda(\overline{G}) = 0$. Now we assume $n \geq 4$. Since $\lambda(G) = 1$, there exists a cut edge in G, say $e = u_1v_1$. Let G_1 and G_2 be the two connected components of $G \setminus e$ such that $u_1 \in V(G_1)$ and $v_1 \in V(G_2)$. Set $V(G_1) = \{u_1, u_2, \cdots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \cdots, v_{n_2}\}$, where $n_1 + n_2 = n$. Suppose $n_i \geq 2$ $(i = 1, 2)$. For any $u_i, u_j \in V(G_1)$, u_i and u_j are connected in $\overline{G}$ since there exists a path $u_iv_2u_j$ in $\overline{G}$; for any $v_i, v_j \in V(G_2)$, v_i and v_j are connected in $\overline{G}$ since there exists a path $v_iu_2v_j$ in $\overline{G}$; for any $u_i \in V(G_1)$ and $v_j \in V(G_2)$ $(i \neq 1$ or $j \neq 1)$, $v_iv_j \in E(\overline{G})$. Clearly, the path $u_1v_2u_2v_1$ connects u_1 and v_1 in $\overline{G}$. So $\overline{G}$ is connected, a contradiction. Thus $n_1 = 1$ or $n_2 = 1$. Without loss of generality, let $n_1 = 1$. Then $V(G_1) = \{u_1\}$ and $V(G_2) = \{v_1, v_2, \cdots, v_{n-1}\}$. Clearly, G is a graph obtained from G_2 by attaching the edge $e = u_1v_1$. Since $u_1v_j \notin E(G)$ $(1 < j \leq n-1)$, $u_1v_j \in E(\overline{G})$. If $d_G(v_1) \leq n-2$, then there exists a vertex v_j such that $v_1v_j \in E(\overline{G})$, which results in $\lambda(\overline{G}) \geq 1$, a contradiction. So $d_G(v_1) = n-1$ and $G \in \mathcal{G}^1_n$; see Figure 1 (a).

Case 2. $\lambda(G) = 2$.

For $n = 3, 4$, the graph $G \in \{C_3, C_4, K_4 \setminus e\}$ satisfies that $\lambda(G) = 2$ and $\lambda(\overline{G}) = 0$. Since $\lambda_3(C_3) = 1$, $\lambda_3(C_4) = 1$, $\lambda_4(C_4) = 1$, $\lambda_3(K_4 \setminus e) = 2$ and $\lambda_4(K_4 \setminus e) = 1$, we have $G = C_3$ for $k = n = 3$; $G \in \{C_4, K_4 \setminus e\}$ for $k = n = 4$; $G = C_4$ for $k = n-1 = 3$. Now we assume $n \geq 5$. Since $\lambda(G) = 2$, there exists an edge cut M such that $|M| = 2$. Let G_1 and G_2 be the two connected components of $G \setminus M$, $V(G_1) = \{u_1, \cdots, u_{n_1}\}$ and $V(G_2) = \{v_1, \cdots, v_{n_2}\}$, where $n_1 + n_2 = n$. Clearly, $G[M] = 2K_2$ or $G[M] = P_3$.

At first, we consider the case $G[M] = 2K_2$. Without loss of generality, let $M = \{u_1v_1, u_2v_2\}$. Since $n \geq 5$, $n_1 \geq 3$ or $n_2 \geq 3$. Without loss of generality, let $n_1 \geq 3$. Clearly, any two vertices $v_i, v_j \in V(G_2)$ are connected in $\overline{G}$ since there exists a path $v_iu_3v_j$ in $\overline{G}$. Furthermore, for any $u_i \in V(G_1)$, $u_iv_1 \in E(\overline{G})$ or $u_iv_2 \in E(\overline{G})$. So $\overline{G}$ is connected and $\lambda(\overline{G}) \geq 1$, a contradiction.

Next, we consider the case $G[M] = P_3$. Without loss of generality, let $P = v_1u_1v_2$ be the path of order 3. Since $n \geq 5$, there exist at least two vertices in $G \setminus \{u_1, v_1, v_2\}$. If $n_1 \geq 2$ and $n_2 \geq 3$, then we can check that $\overline{G}$ is connected, a contradiction. So we assume $n_1 = 1$ or $n_2 = 2$, that is, $V(G_2) = \{v_1, v_2\}$ or $V(G_1) = \{u_1\}$.

For the former, $V(G_1) = \{u_1, u_2, \cdots, u_{n-2}\}$. Since $\lambda(G) = 2$, $v_1v_2 \in E(G)$. Clearly, $v_1u_j, v_2u_j \notin E(G)$ $(2 \leq j \leq n-2)$, which implies that $v_1u_j, v_2u_j \in E(\overline{G})$. Therefore, $u_1u_j \notin E(\overline{G})$ $(2 \leq j \leq n-2)$ since $\overline{G}$ is disconnected. Thus $u_1u_j \in E(G)$ for each

j $(2 \le j \le n-2)$. So $d_G(u_1) = n-1$ and $G \in \mathcal{G}_n^2$; see Figure 1 (b).

For the latter, let $V(G_2) = \{v_1, v_2, \cdots, v_{n-1}\}$. First we consider the case $v_1v_2 \in E(G)$. Since $u_1v_j \notin E(G)$ $(3 \le j \le n-1)$, we have $u_1v_j \in E(\overline{G})$. If $3 \le d_G(v_1) \le n-2$ and $3 \le d_G(v_2) \le n-2$, then there exist two vertices v_i and v_j such that $v_1v_i, v_2v_j \in E(\overline{G})$ $(3 \le i, j \le n-1)$, which implies that $\overline{G}$ is connected, a contradiction. So $d_G(v_1) = n-1$ or $d_G(v_2) = n-1$. Without loss of generality, let $d_G(v_1) = n-1$. Thus $G \in \mathcal{G}_n^3$; see Figure 1 (c).

Now we focus on the graph $G \setminus v_1$. Let $G_1, G_2, \cdots, G_r$ be the connected components of $G \setminus v_1$ and $V(G_i) = \{v_{i,1}, v_{i,2}, \cdots, v_{i,n_i}\}$ $(1 \le i \le r)$, where $\sum_{i=1}^{r} n_i = n-1$. If there exists some connected component G_i such that $G_i = K_2$, then $G \in \mathcal{G}_n^2$; see Figure 1 (b). So we assume $n_i \ge 3$. Then we show the following claim and get a contradiction.

Claim 1. For each connected component G_i of $G \setminus v_1$, if $n_i \ge k$, or $n_i \le k-1$ and $|E(G_i)| \ge n_i$, then $\lambda_k(G) \ge 2$ for $3 \le k \le n$.

Proof of Claim 1. For an arbitrary $S \subseteq V(G)$ with $|S| = k$, we only prove that $\lambda(S) \ge 2$ for $v_1 \notin S$. The case $v_1 \in S$ can be proved similarly. If there exists some connected component G_i such that $S = V(G_i)$, then $n_i = k$ and G_i has a spanning tree, say T_i. It is also a Steiner tree connecting S. Since the tree T_i' induced by the edges in $\{v_1v_{i,1}, v_1v_{i,2}, \cdots, v_1v_{i,n_i}\}$ is another Steiner tree connecting S and T_i, T_i' are two edge-disjoint trees, it follows that $\lambda(S) \ge 2$. Assume now $S \ne V(G_i)$ for $n_i \ge k$ $(1 \le i \le r)$. Let $S_i = S \cap V(G_i)$ $(1 \le i \le r)$ and $|S_i| = k_i$. It is clear that $\bigcup_{i=1}^{r} S_i = S$ and $\sum_{i=1}^{r} k_i = k$. Thus $S_i \subset V(G_i)$ for each connected component G_i such that $n_i \ge k$, and $S_j \subseteq V(G_j)$ for each connected component G_j such that $n_j \le k-1$ and $|E(G_j)| \ge n_j$. We will show that there are two edge-disjoint Steiner trees connecting $S_i \cup \{v_1\}$ in $G[S_i \cup \{v_1\}]$ for each i $(1 \le i \le r)$ so that we can combine these trees to form two edge-disjoint Steiner trees connecting S in G. Suppose that G_i is a connected component such that $n_i \ge k$. Note that $V(G_i) = \{v_{i,1}, v_{i,2}, \cdots, v_{i,n_i}\}$. Since $S_i \subset V(G_i)$, there exists a vertex, without loss of generality, say $v_{i,1}$, such that $v_{i,1} \notin S_i$. Clearly, G_i contains a spanning tree, say $T_{i,1}'$. Thus $T_{i,1} = v_1v_{i,1} \cup T_{i,1}'$ is a Steiner tree connecting $S_i \cup \{v_1\}$ in $G[G_i \cup \{v_1\}]$. Since the tree $T_{i,2}$ induced by the edges in $\{v_1v_{i,2}, v_1v_{i,3}, \cdots, v_1v_{i,n_i}\}$ is another Steiner tree connecting $S_i \cup \{v_1\}$. Clearly, $T_{i,1}$ and $T_{i,2}$ are edge-disjoint. Assume that G_j is a connected component such that $n_j \le k-1$ and $|E(G_j)| \ge n_j$. Note that $V(G_j) = \{v_{j,1}, v_{j,2}, \cdots, v_{j,n_j}\}$. Then there exists an edge, without loss of generality, say $e_j = v_{j1}v_{j2} \in E(G_j)$ such that $G_j \setminus e_j$ contains a spanning tree of G_j, say $T_{j,1}'$. Thus the tree $T_{j,1}$ induced by the edges in $\{v_1v_{j,1}\} \cup E(T_{j,1}')$ and the tree $T_{j,2}$ induced by the edges in $\{v_{j,1}v_{j,2}, v_1v_{j,2}, \cdots, v_1v_{j,n_j}\}$ are two edge-disjoint Steiner trees connecting $S_j \cup \{v_1\}$. Now we combine these small trees connecting $S_i \cup \{v_1\}$ $(1 \le i \le r)$ by the vertex v_1 to form two big trees connecting S. It is clear that the tree T_1 induced by the edges in $E(T_{1,1}) \cup E(T_{2,1}) \cup \cdots \cup E(T_{r,1})$ and the tree T_2 induced by the edges in $E(T_{1,2}) \cup E(T_{2,2}) \cup \cdots \cup E(T_{r,2})$ are our desired trees, and hence $\lambda(S) \ge 2$. From the arbitrariness of S, we have $\lambda_k(G) \ge 2$. □

By Claim 1, we know that $G \in \mathcal{G}_n^3$ and there exists a connected component G_i of $G \setminus \{v_1\}$ such that $n_i \le k-1$ and G_i is a tree.

We next consider the case $v_1v_2 \notin E(G)$; see Figure 1 (d). Thus $v_1v_2 \in E(\overline{G})$. Since $u_1v_j \notin E(G)$ $(3 \le j \le n-1)$, $u_1v_j \in E(\overline{G})$, which results in $v_1v_j, v_2v_j \notin E(\overline{G})$ since $\overline{G}$ is disconnected. Thus $v_1v_j, v_2v_j \in E(G)$ for each j $(3 \le j \le n-1)$. Let $R = \{v_j | 3 \le j \le n-1\}$. If $|E(G[R])| \ge 2$, then G contains a subgraph $K_{2,n-2}^{++}$, which implies that $\lambda_n(G) \ge 2$ by (1) of Observation 3. Combining this with Proposition 2, $\lambda_k(G) \ge 2$ for $3 \le k \le n$, a contradiction. If $|E(G[R])| < 2$, then $G = K_{2,n-2}$ and $K_{2,n-2}^{+}$. From Observation 3 and Proposition 2, we have $\lambda_k(K_{2,n-2}^{+}) \ge 2$ for $3 \le k \le n-1$ and $\lambda_k(K_{2,n-2}) \ge 2$ for $3 \le k \le n-2$, a contradiction. So $G = K_{2,n-2}^{+}$ for $k = n$, or $G = K_{2,n-2}$ for $k = n$, or $G = K_{2,n-2}^{+}$ for $k = n-1$.

Case 3. $\lambda(G) = 3$.

For $n = 4$, $G = K_4$, $\lambda_3(G) = \lambda_4(G) = 2$ by Lemma 3, that is, $\lambda_k(G) \ge 2$, a contradiction. Assume $n \ge 5$. Since $\lambda(G) = 3$, there exists an edge cut M such that $|M| = 3$. Let

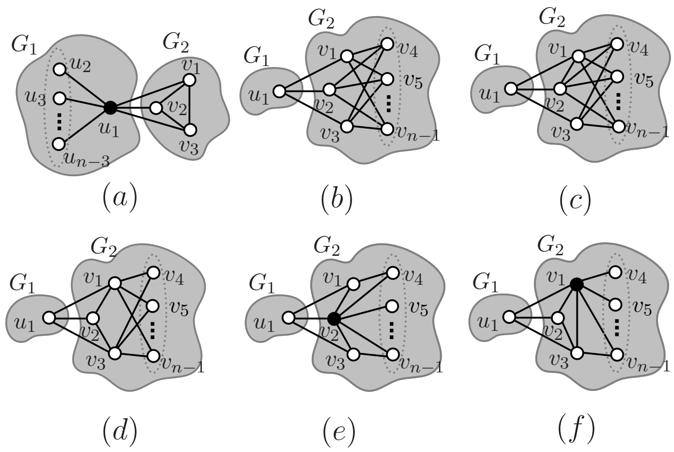


Figure 3. Graphs for Case 3 of Proposition 3.

G_1 and G_2 be the two connected components of $G \setminus M$, $V(G_1) = \{u_1, u_2, \cdots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \cdots, v_{n_2}\}$, where $n_1 + n_2 = n$. Clearly, $G[M] = P_4$ or $G[M] = P_3 \cup K_2$ or $G[M] = 3K_2$ or $G[M] = K_{1,3}$. For the former three cases, $n_i \ge 3$ $(i = 1, 2)$ and $n \ge 6$ since $\lambda(G) = 3$. To shorten the discussion, we only show $\lambda(\overline{G}) \ge 1$ for $G[M] = P_4$ and get a contradiction among the former three cases. Without loss of generality, let $G[M] = P_4 = u_1v_1u_2v_2$. For any $u_i, u_j \in V(G_1)$ $(1 \le i \le n_1)$, u_i and u_j are connected in $\overline{G}$ since there exists a path $u_iv_3u_j$ in $\overline{G}$; for any $v_i, v_j \in V(G_2)$ $(1 \le i \le n_2)$, v_i and v_j are connected in $\overline{G}$ since there exists a path $v_iu_3v_j$ in $\overline{G}$; for any $u_i \in V(G_1)$ and $v_j \in V(G_2)$ $(i \ne 3$ and $j \ne 3)$, u_i and v_j are connected in $\overline{G}$ since there exists a path $u_iv_3u_3v_j$ in $\overline{G}$. Since $u_3v_j \in E(\overline{G})$ $(1 \le j \le n_2)$ and $v_3u_i \in E(\overline{G})$ $(1 \le i \le n_1)$, $\overline{G}$ is connected, as desired.

Now we consider the graph G such that $G[M] = K_{1,3}$. Assume $n_1 \ge 2$. If $n_2 \ge 4$, then we can check that $\overline{G}$ is connected and get a contradiction. Therefore, $n_2 = 3$, $V(G_2) = \{v_1, v_2, v_3\}$ and $V(G_1) = \{u_1, u_2 \cdots, u_{n-3}\}$. Since $\lambda(G) = 3$, it follows that $v_1v_2, v_2v_3, v_1v_3 \in E(G)$. Since $v_iu_j \notin E(G)$ $(1 \le i \le 3,\ 2 \le j \le n-3)$, we have $v_iu_j \in E(\overline{G})$. If there exists some vertex u_j $(2 \le j \le n-3)$ such that $u_1u_j \in E(\overline{G})$, then $\overline{G}$ is connected, a contradiction. So $u_1u_j \in E(G)$ for $2 \le j \le n-3$. Thus $d_G(u_1) = n-1$ (See Figure 3 (a)). From Lemma 5, $\lambda_k(G) \ge 2$ for $3 \le k \le n$ since $\lambda(G) = 3$, a contradiction.

Now assume $n_1 = 1$. Then $V(G_1) = \{u_1\}$ and $V(G_2) = \{v_1, v_2 \cdots, v_{n-1}\}$. If

$G[\{v_1, v_2, v_3\}] = 3K_1$ or $G[\{v_1, v_2, v_3\}] = K_1 \cup K_2$, then we have $u_1v_j \in E(\overline{G})$ since $u_1v_j \notin E(G)$ $(4 \leq j \leq n-1)$. From this together with the fact that $\overline{G}$ is disconnected and $v_1v_3, v_2v_3 \in E(\overline{G})$, $v_iv_j \notin E(\overline{G})$ $(1 \leq i \leq 3,\ 4 \leq j \leq n-1)$, we have $v_iv_j \in E(G)$ $(1 \leq i \leq 3,\ 4 \leq j \leq n-1)$. Thus G contains a complete bipartite graph $K_{3,n-3}$ as its subgraph; see Figure 3 (b) and (c). From (1) of Lemma 1, $\lambda_n(G) = \lfloor \frac{3(n-3)}{n-1} \rfloor \geq 2$ for $n \geq 7$, which implies that $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$ and $n \geq 7$. Since $\lambda(G) = 3$, $n \geq 6$. So we only need to consider the case $n = 6$. Thus $G = H_i$ $(1 \leq i \leq 4)$ (See Figure 4). If $G = H_i$ $(2 \leq i \leq 4)$, then $\lambda_n(G) \geq 2$ for $k = n = 6$; see Figure 4 $(b), (c)$ and (d). Therefore $\lambda_k(G) \geq 2$ for $3 \leq k \leq 6$. If $G = H_1$, then $\lambda_n(G) \leq \lfloor \frac{|E(G)|}{n-1} \rfloor = \lfloor \frac{9}{5} \rfloor = 1$ for $k = n = 6$. For $k = 5$, we can check that $\lambda_3(G) \geq \lambda_4(G) \geq \lambda_5(G) \geq 2$; see Figure 4 (e). So $G = K_{3,3}$ for $k = n = 6$.

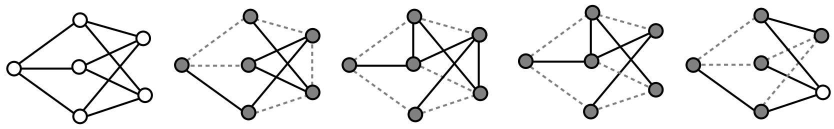


Figure 4. Graphs for Case 3 of Proposition 3.

Suppose $G[\{v_1, v_2, v_3\}] = P_3$. Without loss of generality, let $v_1v_2, v_2v_3 \in E(G)$. If $3 \leq d_G(v_2) \leq n-2$ (see Figure 3 (d)), then there exists at least one vertex v_j such that $v_2v_j \in E(\overline{G})$, which results in $v_1v_j, v_3v_j \notin E(\overline{G})$ $(4 \leq j \leq n-1)$ since $u_1v_j \in E(\overline{G})$ $(4 \leq j \leq n-1)$, $v_1v_3 \in E(\overline{G})$ and $\overline{G}$ is disconnected. Thus $v_1v_t, v_3v_t \in E(G)$ for each t $(4 \leq t \leq n-1)$. Since $d(v_4) \geq \delta(G) \geq \lambda(G) = 3$, we have $v_4v_2 \in E(G)$ or there exists some vertex v_j $(5 \leq j \leq n-1)$ such that $v_4v_j \in E(G)$, which implies that G contains a subgraph $K_{2,n-2}^{++}$ and so $\lambda_n(G) \geq 2$ by (1) of Observation 3. From Proposition 2, $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$, a contradiction. If $d_G(v_2) = n-1$ (See Figure 3 (e)), then $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$ by Lemma 5 since $\lambda(G) = 3$, a contradiction.

Assume that $G[\{v_1, v_2, v_3\}] = K_3$. Without loss of generality, let $v_1v_2, v_1v_3, v_2v_3 \in E(G)$. If $d_G(v_1) = n-1$ or $d_G(v_2) = n-1$ or $d_G(v_3) = n-1$ (see Figure 3 (f)), then by Lemma 5 $\lambda_k(G) \geq 2$ for $3 \leq k \leq n$ since $\lambda(G) = 3$, a contradiction. If $3 \leq d_G(v_i) \leq n-2$ $(1 \leq i \leq 3)$, then $\overline{G}$ is connected, another contradiction. □

Now we turn to studying the upper bounds of $\lambda_k(G) + \lambda_k(\overline{G})$ and $\lambda_k(G) \cdot \lambda_k(\overline{G})$.

Lemma 6. *Let $G \in \mathcal{G}(n)$, and let k be an integer with $3 \leq k \leq n$. Then*

(1) $\lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil k/2 \rceil$.

(2) $\lambda_k(G) \cdot \lambda_k(\overline{G}) \leq \left[\frac{n - \lceil k/2 \rceil}{2}\right]^2$.

Moreover, the two upper bounds are sharp.

Proof. (1) Since $G \cup \overline{G} = K_n$, $\lambda_k(G) + \lambda_k(\overline{G}) \leq \lambda_k(K_n)$. Combining this with Lemma 3, $\lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil \frac{k}{2} \rceil$.

(2) The conclusion holds by (1). □

Consider (1) of Lemma 6. If one of G and $\overline{G}$ is disconnected, we can characterize the graphs attaining the upper bound by Lemma 4.

Proposition 4. *Let G be a graph of order n, and let k be an integer with $3 \leq k \leq n$. If G is disconnected, then $\lambda_k(G) + \lambda_k(\overline{G}) = n - \lceil \frac{k}{2} \rceil$ if and only if $\overline{G} = K_n$ for k even; $\overline{G} = K_n \setminus M$ for k odd, where M is an edge set such that $0 \leq |M| \leq \frac{k-1}{2}$.*

If both G and $\overline{G}$ are connected, we can obtain a property of the graphs attaining the upper bound.

Proposition 5. *Let G be a graph of order n, and let k be an integer with $3 \leq k \leq n$. If $\lambda_k(G) + \lambda_k(\overline{G}) = n - \lceil \frac{k}{2} \rceil$, then $\Delta(G) - \delta(G) \leq \lceil \frac{k}{2} \rceil - 1$.*

Proof. Assume that $\Delta(G) - \delta(G) \geq \lceil \frac{k}{2} \rceil$. Since $\lambda_k(\overline{G}) \leq \delta(\overline{G}) = n - 1 - \Delta(G)$, $\lambda_k(G) + \lambda_k(\overline{G}) \leq \delta(G) + n - 1 - \Delta(G) \leq n - 1 - \lceil \frac{k}{2} \rceil$, a contradiction. □

The next example shows that for $k = n$ the two upper bounds in Lemma 6 are sharp.

Example 1. Let n, r be two positive integers such that $n = 4r + 1$. From (1) of Lemma 1, we know that the STP number of the complete bipartite graph $K_{2r,2r+1}$ is $\lfloor \frac{2r(2r+1)}{2r+(2r+1)-1} \rfloor = r$, that is, $\lambda_n(K_{2r,2r+1}) = r$. Let $\mathcal{E}$ be the set of the edges of these r spanning trees in $K_{2r,2r+1}$. Then there remain $2r(2r + 1) - 4r^2 = 2r$ edges in $K_{2r,2r+1}$ except the edges in $\mathcal{E}$. Let M be the set of these $2r$ edges. Set $G = K_{2r,2r+1} \setminus M$. Then $\lambda_n(G) = r$, $M \subseteq E(\overline{G})$ and $\overline{G}$ is a graph obtained from two cliques K_{2r} and K_{2r+1} by adding $2r$ edges of M between them, that is, one endpoint of each edge belongs to K_{2r} and the other endpoint belongs to K_{2r+1}. Note that $E(\overline{G}) = E(K_{2r}) \cup M \cup E(K_{2r+1})$. Now we show that $\lambda_n(\overline{G}) \geq r$. As we know, K_{2r} contains r Hamiltonian paths, say $P_1, P_2, \cdots, P_r$, and so does K_{2r+1}, say $P'_1, P'_2, \cdots, P'_r$. Pick up r edges from M, say $e_1, e_2, \cdots, e_r$, and let T_i be the tree induced by the edges in $E(P_i) \cup E(P'_i) \cup \{e_i\}$ $(1 \leq i \leq r)$. Then $T_1, T_2, \cdots, T_r$ are r spanning trees in $\overline{G}$, thus, $\lambda_n(\overline{G}) \geq r$. Since $|E(\overline{G})| = \binom{2r}{2} + \binom{2r+1}{2} + 2r = 4r^2 + 2r$ and each spanning tree uses $4r$ edges, these edges can form at most $\lfloor \frac{4r^2+2r}{4r} \rfloor = r$ spanning trees, and hence $\lambda_n(\overline{G}) \leq r$. So $\lambda_n(\overline{G}) = r$. Clearly, $\lambda_n(G) + \lambda_n(\overline{G}) = 2r = \frac{n-1}{2} = n - \lceil \frac{n}{2} \rceil$ and $\lambda_n(\overline{G}) \cdot \lambda_n(\overline{G}) = r^2 = \left[\frac{n - \lceil n/2 \rceil}{2} \right]^2$, which implies that the upper bounds of Lemma 6 are sharp.

Combining Lemmas 2 and 6, we complete the proof of Theorem 1.

3 Nordhaus-Gaddum-type results in $\mathcal{G}(n, m)$

Achthan et al. [1] restricted their attention to the subclass of $\mathcal{G}(n, m)$ consisting of graphs with n vertices and m edges. They investigated the edge-connectivity, diameter and chromatic number parameters. For the edge-connectivity $\lambda(G)$, they showed that $\lambda(G) + \lambda(\overline{G}) \geq max\{1, n - 1 - m\}$. In this section, we consider a similar problem on the generalized edge-connectivity.

Lemma 7. *If $M \subseteq E(K_n)$ such that $0 \leq m = |M| \leq \lfloor \frac{n}{3} \rfloor$, then $G = K_n \setminus M$ contains ℓ edge-disjoint spanning trees, where $\ell = \min\{n - 2m - 1, \lfloor \frac{n}{2} - \frac{2m}{n-1} \rfloor\}$.*

Proof. Let $\mathscr{P}=\bigcup_{i=1}^{p} V_i$ be a partition of $V(G)$ with $|V_i|=n_i$ $(1\leq i\leq p)$, and $\mathcal{E}_p$ be the set of edges between distinct parts of $\mathscr{P}$ in G. It suffices to show that $|\mathcal{E}_p|\geq \ell(|\mathscr{P}|-1)$ so that we can use the Nash-Williams-Tutte Theorem.

The case $p=1$ is trivial, and thus we assume $2\leq p\leq n$. Then $|\mathcal{E}_p|\geq \binom{n}{2}-\sum_{i=1}^{p}\binom{n_i}{2}-|M|\geq \binom{n}{2}-\sum_{i=1}^{p}\binom{n_i}{2}-m$. We will show that $\binom{n}{2}-\sum_{i=1}^{p}\binom{n_i}{2}-m\geq \ell(p-1)$, that is, $\frac{n(n-1)}{2}-m-\ell(p-1)\geq \sum_{i=1}^{p}\binom{n_i}{2}$. We only need to prove that $\frac{n(n-1)}{2}-m-\ell(p-1)\geq \max\{\sum_{i=1}^{p}\binom{n_i}{2}\}$. Since $f(n_1,n_2,\cdots,n_p)=\sum_{i=1}^{p}\binom{n_i}{2}$ achieves its maximum value when $n_1=n_2=\cdots=n_{p-1}=1$ and $n_p=n-p+1$, we need the inequality $\frac{n(n-1)}{2}-m-\ell(p-1)\geq \binom{1}{2}(p-1)+\binom{n-p+1}{2}$, that is, $\frac{n(n-1)}{2}-m-\frac{(n-p+1)(n-p)}{2}\geq \ell(p-1)$. Actually, $\ell\leq \frac{n(n-1)-(n-p+1)(n-p)-2m}{2(p-1)}$ is our required inequality, namely, $\ell\leq n-\frac{1}{2}-(\frac{p-1}{2}+\frac{2m}{p-1})$. Since $f(x)=\frac{x}{2}+\frac{2m}{x}$ achieves its maximum value $\max\{2m+\frac{1}{2},\frac{n-1}{2}+\frac{2m}{n-1}\}$ when $1\leq x\leq n-1$, we need $\ell\leq \min\{n-2m-1,\frac{n}{2}-\frac{2m}{n-1}\}$. Since this inequality holds for $0\leq m\leq \lfloor\frac{n}{3}\rfloor$, we have $|\mathcal{E}_p|\geq \binom{n}{2}-\sum_{i=1}^{p}\binom{n_i}{2}-|M|\geq \ell(p-1)$. From Theorem 1, we know that G has ℓ edge-disjoint spanning trees. □

Lemma 8. *Let $G\in\mathcal{G}(n,m)$, and let k be an integer with $3\leq k\leq n$. For $n\geq 6$, we have*

(1) $\lambda_k(G)+\lambda_k(\overline{G})\geq L(n,m)$, *where*

$$L(n,m)=\begin{cases}\max\{1,\lfloor\frac{1}{2}(n-2-m)\rfloor\}, & if\ \lfloor\frac{n}{3}\rfloor+1\leq m\leq\binom{n}{2};\\ \min\{n-2m-1,\lfloor\frac{n}{2}-\frac{2m}{n-1}\rfloor\}, & if\ 0\leq m\leq\lfloor\frac{n}{3}\rfloor.\end{cases}$$

(2) $\lambda_k(G)\cdot\lambda_k(\overline{G})\geq 0$.

Moreover, the above lower bounds are sharp.

Proof. (1) Since at least one of G and $\overline{G}$ must be connected, we have $\lambda_k(G)+\lambda_k(\overline{G})\geq 1$. For $m<n-1$, $\lambda_k(G)+\lambda_k(\overline{G})\geq \lfloor\frac{1}{2}\lambda(G)\rfloor+\lfloor\frac{1}{2}\lambda(\overline{G})\rfloor\geq \lfloor\frac{1}{2}(\lambda(G)+\lambda(\overline{G})-1)\rfloor\geq \lfloor\frac{1}{2}(\max\{1,n-1-m\}-1)\rfloor\geq \lfloor\frac{1}{2}(n-2-m)\rfloor$ by Proposition 1. So $\lambda_k(G)+\lambda_k(\overline{G})\geq \max\{1,\lfloor\frac{1}{2}(n-2-m)\rfloor\}$. In particular, for $0\leq m\leq\lfloor\frac{n}{3}\rfloor$, we can give a better lower bound of $\lambda_k(G)+\lambda_k(\overline{G})$ by Lemma 7, that is, $\lambda_k(G)+\lambda_k(\overline{G})=\lambda_k(\overline{G})\geq\lambda_n(\overline{G})\geq \min\{n-2m-1,\lfloor\frac{n}{2}-\frac{2m}{n-1}\rfloor\}$.

To show the sharpness of the above lower bound for $\lfloor\frac{n}{3}\rfloor+1\leq m\leq\binom{n}{2}$, we consider the graph $G=K_{1,n-2}\cup K_1$. Then $m=n-2$ and $\overline{G}$ is a graph obtained from a complete graph K_{n-1} by attaching a pendant edge. Clearly, $\lambda_k(G)=0$ and $\lambda_k(\overline{G})=1$. So $\lambda_k(G)+\lambda_k(\overline{G})=1=\max\{1,\lfloor\frac{1}{2}(n-2-m)\rfloor\}$. To show the sharpness of the above lower bound for $0\leq m\leq\lfloor\frac{n}{3}\rfloor$, we consider the graph $G=nK_1$. Thus $m=0$ and $\overline{G}=K_n$. Since $\lambda_n(G)+\lambda_n(\overline{G})=0+\lfloor\frac{n}{2}\rfloor=\min\{n-2\cdot 0-1,\lfloor\frac{n}{2}-\frac{2\cdot 0}{n-1}\rfloor\}$, that is, the lower bound is sharp for $k=n$.

(2) The inequality follows from Theorem 1.

To show the sharpness of the above lower bound for $0\leq m\leq\binom{n-1}{2}$, we consider the graph $G=G'\cup K_1$, where G' is a graph of order $n-1$ and size m. Observe that G is disconnected. Thus, $\lambda_k(G)=0$ and hence $\lambda_k(G)\cdot\lambda_k(\overline{G})=0$. To show the sharpness of the above lower bound for $\binom{n-1}{2}+1\leq m\leq\binom{n}{2}$, we consider a graph G of order $n-1$ and size m. Note that $|E(\overline{G})|\leq\binom{n}{2}-\binom{n-1}{2}-1=n-2$. Therefore, $\lambda_k(\overline{G})=0$ and hence $\lambda_k(G)\cdot\lambda_k(\overline{G})=0$. □

It was pointed out by Harary [9] that given the number of vertices and edges of a graph, the largest connectivity possible can also be read out of the inequality $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Theorem 4. [9] *For each n, m with $0 \leq n-1 \leq m \leq \binom{n}{2}$,*

$$\kappa(G) \leq \lambda(G) \leq \left\lfloor \frac{2m}{n} \right\rfloor,$$

where the maximum is taken over all graphs $G \in \mathcal{G}(n,m)$.

Corollary 2. *For any graph $G \in \mathcal{G}(n,m)$ and $3 \leq k \leq n$, $\lambda_k(G) \leq \lfloor \frac{2m}{n} \rfloor$ for $m \geq n-1$. Moreover, the upper bound is sharp.*

Proof. Since $m \geq n-1$, $\lambda_k(G) \leq \lambda(G) \leq \lfloor \frac{2m}{n} \rfloor$ by (1) of Observation 1 and Theorem 4. One can check that the complete bipartite graph $G = K_{r,r+1}$ satisfies that $\lambda_3(G) = r$, $m = e(G) = r(r+1)$ and $\lfloor \frac{2m}{n} \rfloor = \lfloor \frac{2r(r+1)}{2r+1} \rfloor = \lfloor r + \frac{r}{2r+1} \rfloor = r$. Thus $\lambda_3(G) = r = \lfloor \frac{2m}{n} \rfloor$ and so the upper bound is sharp. □

Although the above bound of $\lambda_k(G)$ is the same as $\lambda(G)$, the graphs attaining the upper bound seem to be very rare. Actually, we can obtain some properties of these graphs.

Proposition 6. *For any $G \in \mathcal{G}(n,m)$ and $3 \leq k \leq n$, if $\lambda_k(G) = \lfloor \frac{2m}{n} \rfloor$ for $m \geq n-1$, then*

(1) *$\frac{2m}{n}$ is not an integer;*

(2) *$\delta(G) = \lfloor \frac{2m}{n} \rfloor$;*

(3) *for $u, v \in V(G)$ such that $d_G(u) = d_G(v) = \lfloor \frac{2m}{n} \rfloor$, $uv \notin E(G)$.*

Proof. One can check that the conclusion holds for the case $m = n-1$. Assume $m \geq n$. We claim that $\frac{2m}{n}$ is not an integer; otherwise, let $r = \frac{2m}{n}$ be an integer. We will show that $\lambda_k(G) \leq r - 1 = \frac{2m}{n} - 1$ and get a contradiction. If G has at least one vertex v_i such that $d(v_i) > r$, then, since the average degree of G is exactly r, there must be a vertex v_j whose degree $d(v_j) < r$. From (1) of Observation 1, we have $\lambda_k(G) \leq \delta(G) \leq d(v_j) < r$, that is, $\lambda_k(G) \leq r-1$. If, on the other hand, G is a regular graph, then by (3) of Observation 1, $\lambda_k(G) \leq \delta(G) - 1 = r - 1$. So (1) holds.

For a graph G such that $\frac{2m}{n}$ is not an integer, $\lfloor \frac{2m}{n} \rfloor = \lambda_k(G) \leq \delta(G) \leq \lfloor \frac{2m}{n} \rfloor$, that is, $\delta(G) = \lfloor \frac{2m}{n} \rfloor$. So (2) holds.

For $u, v \in V(G)$ such that $d_G(u) = d_G(v) = \lfloor \frac{2m}{n} \rfloor$, we claim that $uv \notin E(G)$; otherwise, $uv \in E(G)$. Since $d_G(u) = d_G(v) = \delta(G) = \lfloor \frac{2m}{n} \rfloor$, $\lambda_k(G) \leq \delta(G) - 1 = \lfloor \frac{2m}{n} \rfloor - 1$ by (3) of Observation 1, a contradiction. So (3) holds. □

Corollary 3. *For any graph G with n vertices and m edges, if $\frac{2m}{n}$ is an integer, then $\lambda_k(G) \leq \frac{2m}{n} - 1$.*

Lemma 9. *Let $G \in \mathcal{G}(n,m)$, and let k be an integer with $3 \leq k \leq n$. Then*

(1) *$\lambda_k(G) + \lambda_k(\overline{G}) \leq M(n,m)$, where*

$$M(n,m)=\begin{cases} n-\lceil\frac{k}{2}\rceil, & if\ m\geq n-1,\\ & \quad or\ k\ is\ even\ and\ m=0,\\ & \quad or\ k\ is\ odd\ and\ 0\leq m\leq \frac{k-1}{2};\\ n-\lceil\frac{k}{2}\rceil-1, & if\ k\ is\ even\ and\ 1\leq m<n-1,\\ & \quad or\ k\ is\ odd\ and\ \frac{k+1}{2}\leq m<n-1.\end{cases}$$

(2) $\lambda_k(G)\cdot\lambda_k(\overline{G})\leq N(n,m)$, where

$$N(n,m)=\begin{cases} 0, & if\ 0\leq m\leq n-2;\\ (\frac{2m}{n}-1)(n-2-\frac{2m}{n}), & if\ m\geq n-1\ and\ 2m\equiv 0(mod\ n);\\ \lfloor\frac{2m}{n}\rfloor(n-2-\lfloor\frac{2m}{n}\rfloor), & otherwise.\end{cases}$$

Moreover, these upper bounds are sharp.

Proof. From Theorem 1, (1) holds for $m\geq n-1$. We have given Example 1 to show that the upper bound is sharp. From Proposition 4, $\lambda_k(G)+\lambda_k(\overline{G})=\lambda_k(\overline{G})=n-\lceil\frac{k}{2}\rceil$ for k even and $m=0$, or k odd and $0\leq m\leq\frac{k-1}{2}$. So for k even and $1\leq m<n-1$, or k odd and $\frac{k+1}{2}\leq m<n-1$, $\lambda_k(G)+\lambda_k(\overline{G})\leq n-\lceil\frac{k}{2}\rceil-1$.

To prove the sharpness of the bound for k odd and $\frac{k+1}{2}\leq m<n-1$, we consider the graph $G=K_{1,\frac{k+1}{2}}\cup(n-\frac{k+3}{2})K_1$. Clearly, $\overline{G}$ is a graph obtained from the complete graph K_n by deleting all the edges of a star $K_{1,\frac{k+1}{2}}$. On one hand, by Lemma 4, it follows that $\lambda_k(\overline{G})\leq n-\frac{k+1}{2}-1$. On the other hand, by Lemma 4, we have $\lambda_k(\overline{G}+e)=n-\frac{k+1}{2}$ for any $e\notin E(\overline{G})$, which implies that $\lambda_k(\overline{G})\geq n-\frac{k+1}{2}-1$ (note that $\lambda_k(H\setminus e)\geq\lambda_k(H)-1$ for a connected graph H, where $e\in E(H)$). So $\lambda_k(G)+\lambda_k(\overline{G})=\lambda_k(\overline{G})=n-\frac{k+1}{2}-1$. By the same reason, for k even and $1\leq m<n-1$ one can check that the graph $G=K_2\cup(n-2)K_1$ satisfies that $\lambda_k(G)+\lambda_k(\overline{G})=\lambda_k(\overline{G})=n-\frac{k}{2}-1$.

(2) First, if $0\leq m\leq n-2$, then $G\in\mathcal{G}(n,m)$ is disconnected. So $\lambda_k(G)\cdot\lambda_k(\overline{G})=0$. Next, if $m\geq n-1$ and $\frac{2m}{n}=r$ is an integer, then $\frac{2e(\overline{G})}{n}=n-1-r$ is also an integer. From Corollary 3, we have $\lambda_k(G)\leq r-1$ and $\lambda_k(\overline{G})\leq n-2-r$. So $\lambda_k(G)\cdot\lambda_k(\overline{G})\leq(r-1)(n-2-r)=(\frac{2m}{n}-1)(n-2-\frac{2m}{n})$. Finally, if $2m=nr+\ell$ where $1\leq\ell\leq n-1$, then $\Delta(G)\geq r+1$. By (1) of Observation 1, $\lambda_k(\overline{G})\leq\delta(\overline{G})=n-1-\Delta(G)\leq n-2-r$. So $\lambda_k(G)\cdot\lambda_k(\overline{G})\leq r(n-2-r)=\lfloor\frac{2m}{n}\rfloor(n-2-\lfloor\frac{2m}{n}\rfloor)$.

To show the sharpness of the upper bound for $0\leq m\leq n-2$, we consider the graph G of size m. Clearly, $\lambda_k(G)=0$ and hence $\lambda_k(G)\cdot\lambda_k(\overline{G})=0$. For $m\geq n-1$ and $\frac{2m}{n}=r+\ell$ $(1\leq\ell\leq n-1)$, we let $G=P_4$. Then $\lambda_3(G)=1=\lfloor\frac{6}{4}\rfloor=\lfloor\frac{2m}{n}\rfloor$ and $\lambda_3(\overline{G})=\lambda_3(P_4)=1=4-2-\lfloor\frac{6}{4}\rfloor=n-2-\lfloor\frac{2m}{n}\rfloor$. So $\lambda_3(G)\cdot\lambda_3(\overline{G})=\lfloor\frac{2m}{n}\rfloor(n-2-\lfloor\frac{2m}{n}\rfloor)$. $\square$

To show the sharpness of the upper bound for $m\geq n-1$ and $2m\equiv 0$ $(mod\ n)$, we consider the following example.

Example 2. Let G be a cycle $C_n=w_1w_2\cdots w_nw_1(n\geq 9)$. Clearly, $\lambda_3(G)=1=\frac{2m}{n}-1$. Since $\frac{2m}{n}=2$ is an integer, it suffices to show that $\lambda_3(\overline{G})=n-2-\frac{2m}{n}=n-4$. First we show that $\lambda_3(\overline{G})\geq n-4$. For arbitrary $S=\{x,y,z\}\subseteq V(G)=V(C_n)$.

By the definition of $\lambda_3(\overline{G})$, we need to show that $\lambda(S) \geq n-4$. If $d_{C_n}(x,y) = 1$ and $d_{C_n}(y,z) = 1$, without loss of generality, let $N_{C_n}(x) = \{x_1, y\}$ and $N_{C_n}(z) = \{y, z_2\}$, then the trees T_i induced by the edges in $\{xw_i, yw_i, zw_i\}$ together with the tree T_1 induced by the edges in $\{xz, zx_1, x_1y\}$ form $n-4$ edge-disjoint S-trees in $\overline{G}$ (see Figure 5 (a)) and hence $\lambda(S) \geq n-4$, where $\{w_1, w_2, \cdots, w_{n-5}\} = V(G) \setminus \{x, y, z, x_1, z_2\}$. If $d_{C_n}(x,y) = 2$ and $d_{C_n}(y,z) = 1$, without loss of generality, let $N_{C_n}(x) = \{x_1, y_1\}$ and $N_{C_n}(y) = \{y_1, z\}$ and $N_{C_n}(z) = \{y, z_2\}$, then the trees T_i induced by the edges in $\{xw_i, yw_i, zw_i\}$ together with the tree T_1 induced by the edges in $\{xy, xz\}$ and the tree T_2 induced by the edges in $\{z_2x, z_2y, z_2y_1, y_1z\}$ form $n-4$ edge-disjoint S-trees in $\overline{G}$ (see Figure 5 (b)) and hence $\lambda(S) \geq n-4$, where $\{w_1, w_2, \cdots, w_{n-6}\} = V(G) \setminus \{x, y, z, x_1, y_1, z_2\}$. If $d_{C_n}(x,y) \geq 3$ and $d_{C_n}(y,z) = 1$, without loss of generality, let $N_{C_n}(x) = \{x_1, x_2\}$ and $N_{C_n}(z) = \{y_1, z\}$ and $N_{C_n}(z) = \{y, z_2\}$, then the trees T_i induced by the edges in $\{xw_i, yw_i, zw_i\}$ together with the tree T_1 induced by the edges in $\{xy, xz\}$ and the tree T_2 induced by the edges in $\{z_2x, z_2y, z_2y_1, y_1z\}$ and the tree T_3 induced by the edges in $\{xy_1, y_1x_1, x_1y, x_1z\}$ form $n-4$ edge-disjoint S-trees in $\overline{G}$ (see Figure 5 (c)), and hence $\lambda(S) \geq n-4$, where $\{w_1, w_2, \cdots, w_{n-7}\} = V(G) \setminus \{x, y, z, x_1, x_2, y_1, z_2\}$. If $d_{C_n}(x,y) = 2$ and $d_{C_n}(y,z) = 2$, without loss of generality, let $N_{C_n}(x) = \{x_1, y_1\}$ and $N_{C_n}(y) = \{y_1, z_1\}$ and $N_{C_n}(z) = \{z_1, z_2\}$, then the trees T_i induced by the edges in $\{xw_i, yw_i, zw_i\}$ together with the tree T_1 induced by the edges in $\{xz, xy\}$ and the tree T_2 induced by the edges in $\{xz_2, yz_2, yz\}$ and the tree T_3 induced by the edges in $\{x_1y, x_1z, x_1z_1, xz_1\}$ form $n-4$ edge-disjoint S-trees in $\overline{G}$ (see Figure 5 (d)), and hence $\lambda(S) \geq n-4$, where $\{w_1, w_2, \cdots, w_{n-7}\} = V(G) \setminus \{x, y, z, x_1, y_1, z_1, z_2\}$. If $d_{C_n}(x,y) \geq 3$ and $d_{C_n}(y,z) = 2$, without loss of generality, let $N_{C_n}(x) = \{x_1, x_2\}$ and $N_{C_n}(y) = \{y_1, z_1\}$ and $N_{C_n}(z) = \{z_1, z_2\}$, then the trees T_i induced by the edges in $\{xw_i, yw_i, zw_i\}$ together with the tree T_1 induced by the edges in $\{xz, xy\}$ and the tree T_2 induced by the edges in $\{xz_2, z_2y, yz\}$ and the tree T_3 induced by the edges in $\{x_1y, x_1z, x_1y_1, xy_1\}$ and the tree T_4 induced by the edges in $\{x_2y, x_2z, x_2z_1, z_1x\}$ form $n-4$ edge-disjoint S-trees in $\overline{G}$ (see Figure 5 (e)), and thus $\lambda(S) \geq n-4$, where $\{w_1, w_2, \cdots, w_{n-8}\} = V(G) \setminus \{x, y, z, x_1, x_2, y_1, y_2, z_2\}$. Suppose that $d_{C_n}(x,y) \geq 3$ and $d_{C_n}(y,z) \geq 3$, without loss of generality, let $N_{C_n}(x) = \{x_1, x_2\}$ and $N_{C_n}(y) = \{y_1, y_2\}$ and $N_{C_n}(z) = \{z_1, z_2\}$. Then the trees T_i induced by the edges in $\{xw_i, yw_i, zw_i\}$ together with the tree T_1 induced by the edges in $\{xz, xy\}$ and the tree T_2 induced by the edges in $\{xz_2, yz_2, yz\}$ and the tree T_3 induced by the edges in $\{x_1y, x_1z, x_1y_1, y_1x\}$ and the tree T_3 induced by the edges in $\{xz_1, yz_1, y_2z_1, y_2z\}$ and the tree T_5 induced by the edges in $\{x_2y, x_2z, x_2y_2, y_2x\}$ form $n-4$ edge-disjoint S-trees in $\overline{G}$ (see Figure 5 (f)), and hence $\lambda(S) \geq n-4$, where $\{w_1, w_2, \cdots, w_{n-9}\} = V(G) \setminus \{x, y, z, x_1, x_2, y_1, y_2, z_1, z_2\}$. From the arbitrariness of S, we know that $\lambda_3(\overline{G}) \geq n-4$. We now prove that $\lambda_3(\overline{G}) \leq n-4$ for $\overline{G} = \overline{C_n}$. Choose $S = \{w_1, w_2, w_3\} \subseteq V(G) = V(C_n)$. Then $w_1w_n \in E(C_n)$ and $w_3w_4 \in E(C_n)$. Thus $|E(\overline{G}[S])| = 1$ and $|E_{\overline{G}}[S, \bar{S}]| = 3(n-3)-2$, which implies that $|E(\overline{G}[S]) \cup E_{\overline{G}}[S, \bar{S}]| = 3(n-3)-1$ (see Figure 5 (g)). One can see that each tree connecting S in $\overline{G}$ uses at least 3 edges from $E(\overline{G}[S]) \cup E_{\overline{G}}[S, \bar{S}]$. Therefore $\lambda_3(\overline{G}) \leq \frac{3(n-3)-1}{3} = n-3-\frac{1}{3}$, which results in $\lambda_3(\overline{G}) \leq n-4$ since $\lambda_3(\overline{G})$ is an integer. So $\lambda_3(\overline{G}) = n-4$ and $\lambda_3(G) \cdot \lambda_3(\overline{G}) = \lambda_3(C_n) \cdot \lambda_3(\overline{C_n}) = 1 \cdot (n-4) = (\frac{2m}{n}-1)(n-2-\frac{2m}{n})$. The upper bound is sharp.

Combining with Lemmas 8 and 9, we complete the proof of Theorem 2.

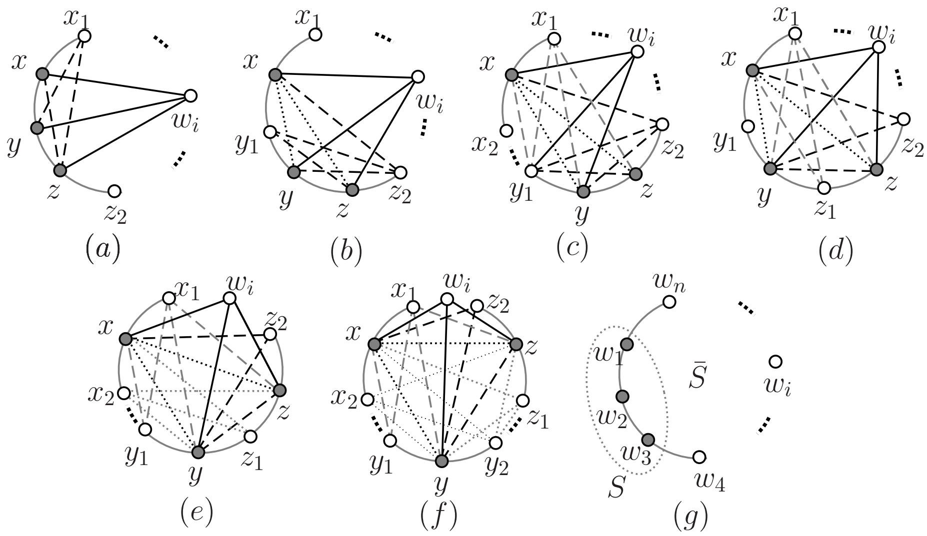


Figure 5. Graphs for Example 3.

Acknowledgement: The authors are very grateful to the editor and the referees' comments and suggestions, which helped to improve the presentation of the paper.

References

[1] N. Achuthan, N. R. Achuthan, L. Caccetta, *On the Nordhaus-Gaddum problems*, Australasian J. Combin. 2(1990), 5-27.

[2] Y. Alavi, J. Mitchem, *The connectivity and edge-connectivity of complementary graphs*, Lecture Notes in Math. 186(1971), 1-3.

[3] M. Aouchiche, P. Hansen, *A survey of Nordhaus-Gaddum type relations*, Discrete Appl. Math. 161(4-5)(2013), 466-546.

[4] J.A. Bondy, U.S.R. Murty, *Graph Theory*, GTM 244, Springer, 2008.

[5] G. Chartrand, S.F. Kappor, L. Lesniak, D.R. Lick, *Generalized connectivity in graphs*, Bull. Bombay Math. Colloq. 2(1984), 1-6.

[6] G. Chartrand, F. Okamoto, P. Zhang, *Rainbow trees in graphs and generalized connectivity*, Networks 55(4)(2010), 360-367.

[7] M. Grötschel, *The Steiner tree packing problem in $VLSI$ design*, Math. Program. 78(1997), 265-281.

[8] M. Grötschel, A. Martin, R. Weismantel, *Packing Steiner trees: A cutting plane algorithm and commputational results*, Math. Program. 72(1996), 125-145.

[9] F. Harary, *The maximum connectivity of a graph*, Proc. Nat. Acad. Sci. USA 48(1962), 1142-1146.

[10] M. Kriesell, *Edge-disjoint trees containing some given vertices in a graph*, J. Combin. Theory, Ser.B, 88(2003), 53-65.

[11] M. Kriesell, *Edge-disjoint Steiner trees in graphs without large bridges*, J. Graph Theory 62(2009), 188-198.

[12] H. Li, X. Li, Y. Mao, *On extremal graphs with at most two internally disjoint Steiner trees connecting any three vertices*, Bull. Malays. Math. Sci. Soc. (2)37(3)(2014), 747-756.

[13] H. Li, X. Li, Y. Mao, Y. Sun, *Note on the generalied connectivity*, Ars Combin. 114(2014), 193-202.

[14] H. Li, X. Li, Y. Sun, *The generalied 3-connectivity of Cartesian product graphs*, Discrete Math. Theor. Comput. Sci. 14(1)(2012), 43-54.

[15] S. Li, X. Li, Note on the hardness of generalized connectivity, J. Combin. Optim. 24(2012), 389-396.

[16] S. Li, W. Li, X. Li, *The generalized connectivity of complete equipartition 3-partite graphs*, Bull. Malays. Math. Sci. Soc. (2)37(1)(2014), 103-121.

[17] S. Li, X. Li, W. Zhou, *Sharp bounds for the generalized connectivity* $\kappa_3(G)$, Discrete Math. 310(2010), 2147-2163.

[18] X. Li, Y. Mao, *On extremal graphs with at most* ℓ *internally disjoint Steiner trees connecting any* $n-1$ *vertices*, accepted by Graphs & Combin.

[19] X. Li, Y. Mao, *The generalied 3-connectivity of lexigraphical product graphs*, Discrete Math. Theor. Comput. Sci. 16(1)(2014), 339-354.

[20] X. Li, Y. Mao, *The minimal size of a graph with given generalized 3-edge-connectivity*, Ars Combin. 118(2015).

[21] X. Li, Y. Mao, Y. Sun, *On the generalized (edge-)connectivity of graphs*, Australasian J. Combin. 58(2)(2014), 304-319.

[22] C.St.J.A, Nash-Williams, *Edge-disjonint spanning trees of finite graphs*, J. London Math. Soc. 36(1961), 445-450.

[23] E. Palmer, *On the spanning tree packing number of a graph: a survey*, Discrete Math. 230(2001), 13-21.

[24] N. Sherwani, *Algorithms for VLSI physical design automation*, 3rd Edition, Kluwer Acad. Pub., London, 1999.

[25] W. Tutte, *On the problem of decomposing a graph into* n *connected factors*, J. London Math. Soc. 36(1961), 221-230.

[26] D. West, H. Wu, *Packing Steiner trees and* S*-connectors in graphs*, J. Combin. Theory, Ser.B, 102(2012), 186-205.