Quadratic Forms and Congruences for ℓ-Regular Partitions Modulo $3, 5$ and 7

Qing-Hu Hou[a], Lisa H. Sun[b] and Li Zhang[b]

[a]Center for Applied Mathematics
Tianjin University, Tianjin 300072, P. R. China

[b]Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

Emails: hou@nankai.edu.cn, sunhui@nankai.edu.cn, zhangli427@mail.nankai.edu.cn

Abstract. Let $b_\ell(n)$ be the number of ℓ-regular partitions of n. We show that the generating functions of $b_\ell(n)$ with $\ell = 3, 5, 6, 7$ and 10 are congruent to the products of two items of Ramanujan's theta functions $\psi(q), f(-q)$ and $(q;q)_\infty^3$ modulo 3, 5 and 7. So we can express these generating functions as double summations in q. Based on the properties of binary quadratic forms, we obtain vanishing properties of the coefficients of these series. This leads to several infinite families of congruences for $b_\ell(n)$ modulo 3, 5 and 7.

Keywords: ℓ-regular partition, quadratic form, congruence

MSC(2010): 05A17, 11P83

1 Introduction

An ℓ-regular partition of n is a partition of n such that none of its parts is divisible by ℓ. Denote the number of ℓ-regular partitions of n by $b_\ell(n)$. The arithmetic properties, the divisibility and the distribution of $b_\ell(n)$ have been widely studied in recent years.

Alladi [2] studied the 2-adic behavior of $b_2(n)$ and $b_4(n)$ from a combinatorial point of view and obtained the divisibility results for small powers of 2. Lovejoy [13] proved the divisibility and distribution properties of $b_2(n)$ modulo primes $p \geq 5$ by using the theory of modular forms. Gordon and Ono [8] proved the divisibility properties of $b_\ell(n)$ modulo powers of the prime divisors of ℓ. Later, Ono and Penniston [15] studied the 2-adic behavior of $b_2(n)$. And Penniston [17] derived the behavior of p^a-regular partitions modulo p^j using the theory of modular forms.

The arithmetic properties of $b_\ell(n)$ modulo 2 have been widely investigated. Andrews, Hirschhorn and Sellers [3] derived some infinite families of congruences for $b_4(n)$ modulo 2. By applying the 2-dissection of the generating function of $b_5(n)$, Hirschhorn and Sellers [9] obtained many Ramanujan-type congruences for $b_5(n)$ modulo 2. Xia and Yao [18] established several infinite families of congruences for $b_9(n)$ modulo 2. Cui and Gu [5] derived congruences for $b_\ell(n)$ modulo 2 with $\ell = 2, 4, 5, 8, 13, 16$ by employing the p-dissection formulas of Ramanujan's theta functions $\psi(q)$ and $f(-q)$.

As for the arithmetic properties of $b_\ell(n)$ modulo 3, Cui and Gu [6] and Keith [10] and Xia and Yao [19] studied respectively the congruences for $b_9(n)$ modulo 3. Lin and Wang [12] showed that 9-regular partitions and 3-cores satisfy the same congruences modulo 3 and further generalized Keith's conjecture and derived a stronger result. Furcy and Penniston [7] obtained congruences for $b_\ell(n)$ modulo 3 with $\ell = 4, 7, 13, 19, 25, 34, 37, 43, 49$ by using the theory of modular forms.

Notice that all the above congruences for $b_\ell(n)$ were proven by using modular forms or elementary q-series manipulations. In this paper, we take a different approach which is based on the properties of binary quadratic forms. Lovejoy and Osburn [14] generalized the congruences modulo 3 for four types of partitions by employing the representations of numbers as certain quadratic forms. Employing the arithmetic properties of quadratic forms, Kim [11] proved that the number of overpartition pairs of n is almost always divisible by 2^8.

We derive infinite families of congruence relations for ℓ-regular partitions with $\ell = 3, 5, 6, 7, 10$ modulo 3, 5 and 7 by establishing a general method (see Proposition 2.1). Our method is based on a bivariate extension of Cui and Gu's approach [5].

Notice that the generating function of $b_\ell(n)$ is given by

$$B_\ell(q) = \sum_{n=0}^{\infty} b_\ell(n) q^n = \frac{(q^\ell; q^\ell)_\infty}{(q; q)_\infty},$$

where

$$(q; q)_\infty = \prod_{i=1}^{\infty} (1 - q^i)$$

is the standard notation in q-series. Let $\psi(q)$ and $f(-q)$ be Ramanujan's theta functions given by

$$\psi(q) = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \frac{(q^2, q^2)_\infty}{(q; q^2)_\infty} \quad \text{and} \quad f(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = (q; q)_\infty.$$

Denote $f(-q)^3$ by $g(q)$. By Jacobi's identity [4, Theorem 1.3.9], we have

$$g(q) = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{\binom{n+1}{2}}.$$

It is known that for any prime p,

$$(q^p; q^p)_\infty \equiv (q; q)_\infty^p \pmod{p}.$$

We thus derive the following congruences

$$B_3(q) \equiv f(-q)^2 \pmod 3, \tag{1.1}$$

$$B_6(q) \equiv f(-q^2)\psi(q) \pmod 3, \tag{1.2}$$

$$B_5(q) \equiv f(-q)g(q) \pmod 5, \tag{1.3}$$

$$B_{10}(q) \equiv g(q^2)\psi(q) \pmod 5, \tag{1.4}$$

and

$$B_7(q) \equiv g(q)^2 \pmod 7. \tag{1.5}$$

Note that the right hand side of the above congruences can be written in the following form

$$F(q) = \sum_{k,l=-\infty}^{\infty} c(k,l) q^{\theta(k,l)}, \tag{1.6}$$

where $\theta(k,l)$ is quadratic in k and l. By investigating the quadratic residues, we find that, for a certain prime p, there exists an integer $0 \le a \le p-1$ such that the congruence $\theta(k,l) \equiv a \pmod p$ has a unique solution $k \equiv r \pmod p$ and $l \equiv s \pmod p$. Then by considering the coefficients of q^n in $F(q)$ with $n \equiv a \pmod p$, we deduce a recursion and a vanishing property on the coefficients of $F(q)$. This leads to several infinite families of congruence relations for $b_\ell(n)$ with $\ell = 3, 5, 6, 7$ and 10.

As an example, when $\ell = 3$, let α, n be nonnegative integers and $p_i \ge 5$ be primes such that $p_i \equiv 3 \pmod 4$. By the vanishing property, we have

$$\sum_{n=0}^{\infty} b_3\left(p_1^2 \cdots p_\alpha^2 p_{\alpha+1} n + \frac{p_1^2 \cdots p_{\alpha+1}^2 - 1}{12}\right) q^n \equiv f(-q^{p_{\alpha+1}})^2 \pmod 3.$$

Thus for any integer $j \not\equiv 0 \pmod{p_{\alpha+1}}$, we have

$$b_3\left(p_1^2 \cdots p_{\alpha+1}^2 n + \frac{p_1^2 \cdots p_\alpha^2 p_{\alpha+1}(12j + p_{\alpha+1}) - 1}{12}\right) \equiv 0 \pmod 3.$$

Specially, when $\alpha = 0$, $p_1 = 7$ and $j \not\equiv 1 \pmod 7$, the above congruence reduces to

$$b_3(49n + 7j - 3) \equiv 0 \pmod 3.$$

This paper is organized as follows. In Section 2, we give a vanishing property on the coefficients of the formal power series in the form of (1.6) and derive congruence relations for $b_\ell(n)$ in general form. Then in Section 3, we give some explicit examples of these congruences.

2 The vanishing property and congruences of $b_\ell(n)$

Let

$$F(q) = \sum_{n=0}^{\infty} a(n)q^n = \sum_{k,l=-\infty}^{\infty} c(k,l)q^{\theta(k,l)}$$

be a formal power series in q. In this section, we first give a vanishing property on $a(n)$ by investigating the congruence of $\theta(k,l)$. Meanwhile, we also get a recursion of $a(n)$. As corollaries, we derive the vanishing properties of the products of $\psi(q), f(-q)$ and $g(q)$. Finally, combining the congruences (1.1)–(1.5), we obtain several infinite families of congruence relations for $b_\ell(n)$.

The following proposition gives a vanishing property and a recursion on the coefficients $a(n)$ of $F(q)$, which plays a key role in finding the congruences of $b_\ell(n)$.

Proposition 2.1 (Vanishing Property)**.** *Let p be a prime and*

$$F(q) = \sum_{n=0}^{\infty} a(n)q^n = \sum_{k,l=-\infty}^{\infty} c(k,l)q^{\theta(k,l)}.$$

Suppose that there exist integers θ_0, r, s and an invertible transformation $\sigma\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ satisfying the following three conditions

(a) *the congruence $\theta(k,l) \equiv \theta_0 \pmod p$ has a unique solution $k \equiv r \pmod p$ and $l \equiv s \pmod p$ in $\mathbb{Z}_p^2$;*

(b) $\theta(pk+r, pl+s) = p^2\theta(\sigma(k,l)) + \theta_0$*;*

(c) $c(pk+r, pl+s) = \lambda(p) \cdot c(\sigma(k,l))$*, where $\lambda(p)$ is a constant independent of k and l.*

Then the following two assertions hold.

(1) *For any integer n, we have*

$$a(p^2n + \theta_0) = \lambda(p) \cdot a(n).$$

(2) *For any integer n with $p \nmid n$, we have*

$$a(pn + \theta_0) = 0. \tag{2.1}$$

Proof. It is obvious to see that

$$\begin{aligned} &\{(k,l)\colon \theta(k,l) = p^2n + \theta_0\} \\ &\quad = \{(k,l)\colon k = pk' + r, l = pl' + s, \theta(k,l) = p^2n + \theta_0\} && \big(\text{by (a)}\big) \\ &\quad = \{(k,l)\colon k = pk' + r, l = pl' + s, \theta(\sigma(k',l')) = n\}. && \big(\text{by (b)}\big) \end{aligned}$$

Therefore, by Condition (c), we derive that

$$a(p^2n+\theta_0)=\sum_{(k,l)\colon\, \theta(k,l)=p^2n+\theta_0} c(k,l)=\sum_{(k',l')\colon\, \theta(\sigma(k',l'))=n} \lambda(p)\cdot c(\sigma(k',l'))=\lambda(p)\cdot a(n).$$

By Conditions (a) and (b), we have

$$\theta(k,l)\equiv\theta_0 \pmod p \quad\Longrightarrow\quad \theta(k,l)\equiv\theta_0 \pmod{p^2}.$$

The vanishing property (2.1) holds immediately. ∎

Now we apply the above property to the products of $\psi(q)$, $f(-q)$ and $g(q)$ to derive the congruence relations of ℓ-regular partitions.

Theorem 2.2. *Let α, n be nonnegative integers and $p_i \geq 5$ be primes such that $p_i \equiv 3 \pmod 4$. Then we have*

$$\sum_{n=0}^{\infty} b_3\left(p_1^2\cdots p_\alpha^2 p_{\alpha+1} n+\frac{p_1^2\cdots p_{\alpha+1}^2-1}{12}\right) q^n \equiv f(-q^{p_{\alpha+1}})^2 \pmod 3. \tag{2.2}$$

In particular, for any integer $j \not\equiv 0 \pmod{p_{\alpha+1}}$, we have

$$b_3\left(p_1^2\cdots p_{\alpha+1}^2 n+\frac{p_1^2\cdots p_\alpha^2 p_{\alpha+1}(12j+p_{\alpha+1})-1}{12}\right)\equiv 0 \pmod 3. \tag{2.3}$$

Proof. We have

$$\sum_{n=0}^{\infty} b_3(n)q^n=\frac{(q^3,q^3)_\infty}{(q;q)_\infty}\equiv (q;q)_\infty^2=f(-q)^2 \pmod 3.$$

Assume that $f(-q)^2=\sum_{n=0}^{\infty} a(n)q^n$. To prove (2.2), it suffices to show that

$$\sum_{n=0}^{\infty} a\left(p_1^2\cdots p_\alpha^2 p_{\alpha+1} n+\frac{p_1^2\cdots p_{\alpha+1}^2-1}{12}\right) q^n = f(-q^{p_{\alpha+1}})^2. \tag{2.4}$$

By the summation expression of $f(-q)$, we have

$$f(-q)^2=\sum_{k,l=-\infty}^{\infty} c(k,l)q^{\theta(k,l)},$$

where

$$c(k,l)=(-1)^{k+l} \quad\text{and}\quad \theta(k,l)=\frac{k(3k+1)}{2}+\frac{l(3l+1)}{2}.$$

Notice that

$$\theta(k,l)=\frac{3}{2}\left(\left(k+\frac{1}{6}\right)^2+\left(l+\frac{1}{6}\right)^2\right)-\frac{1}{12}.$$

For any $1 \le i \le \alpha+1$, we have

$$\theta(k,l) \equiv -\frac{1}{12} \pmod{p_i} \quad \Leftrightarrow \quad \left(k+\frac{1}{6}\right)^2 + \left(l+\frac{1}{6}\right)^2 \equiv 0 \pmod{p_i}$$

Since $p_i \equiv 3 \pmod 4$, -1 is not a quadratic residue modulo p_i. Hence

$$\left(k+\frac{1}{6}\right)^2 \equiv -\left(l+\frac{1}{6}\right)^2 \pmod{p_i} \quad \Leftrightarrow \quad k \equiv -\frac{1}{6} \quad \& \quad l \equiv -\frac{1}{6} \pmod{p_i}.$$

If $p_i \equiv 7 \pmod{12}$, we have $k \equiv \frac{p_i-1}{6} \pmod{p_i}$ and $l \equiv \frac{p_i-1}{6} \pmod{p_i}$. Hence, we have

$$\theta\left(kp_i + \frac{p_i-1}{6}, lp_i + \frac{p_i-1}{6}\right) = p_i^2\theta(k,l) + \frac{p_i^2-1}{12}$$

and

$$c\left(kp_i + \frac{p_i-1}{6}, lp_i + \frac{p_i-1}{6}\right) = (-1)^{\frac{p_i-1}{3}}(-1)^{p_i(k+l)} = c(k,l).$$

If $p_i \equiv 11 \pmod{12}$, we have $k \equiv \frac{-p_i-1}{6} \pmod{p_i}$ and $l \equiv \frac{-p_i-1}{6} \pmod{p_i}$. Thus we obtain that

$$\theta\left(kp_i + \frac{-p_i-1}{6}, lp_i + \frac{-p_i-1}{6}\right) = p_i^2\theta(-k,-l) + \frac{p_i^2-1}{12}$$

and

$$c\left(kp_i + \frac{-p_i-1}{6}, lp_i + \frac{-p_i-1}{6}\right) = (-1)^{\frac{-p_i-1}{3}}(-1)^{p_i(k+l)} = c(-k,-l).$$

We thus deduce from Proposition 2.1 (1) the recursion

$$a\left(p_i^2 n + \frac{p_i^2-1}{12}\right) = a(n). \tag{2.5}$$

Iteratively using recursion (2.5), we obtain

$$\begin{aligned} &a\left(p_1^2\cdots p_\alpha^2 p_{\alpha+1} n + \frac{p_1^2\cdots p_{\alpha+1}^2-1}{12}\right) \\ &\quad = a\left(p_1^2\left(p_2^2\cdots p_\alpha^2 p_{\alpha+1} n + \frac{p_2^2\cdots p_{\alpha+1}^2-1}{12}\right) + \frac{p_1^2-1}{12}\right) \\ &\quad = a\left(p_2^2\cdots p_\alpha^2 p_{\alpha+1} n + \frac{p_2^2\cdots p_{\alpha+1}^2-1}{12}\right) \\ &\quad = \cdots \\ &\quad = a\left(p_{\alpha+1} n + \frac{p_{\alpha+1}^2-1}{12}\right). \end{aligned}$$

By Proposition 2.1 (2), $a\left(p_{\alpha+1}n+\frac{p_{\alpha+1}^2-1}{12}\right)\neq 0$ only when $p_{\alpha+1}\mid n$. Therefore,

$$\sum_{n=0}^{\infty} a\left(p_{\alpha+1}n+\frac{p_{\alpha+1}^2-1}{12}\right)q^n=\sum_{n'=0}^{\infty} a\left(p_{\alpha+1}^2n'+\frac{p_{\alpha+1}^2-1}{12}\right)q^{p_{\alpha+1}n'}.$$

Using recursion (2.5) once again, the above sum reduces to

$$\sum_{n'=0}^{\infty} a(n')q^{p_{\alpha+1}n'}=f(-q^{p_{\alpha+1}})^2,$$

which completes the proof of (2.4).

Furthermore, since the right hand side of (2.4) contains only those terms of q^n with $p_{\alpha+1}\mid n$, congruence (2.3) follows immediately. ∎

By a similar discussion, we derive the following congruence relations for $b_6(n)$ modulo 3, $b_5(n)$ and $b_{10}(n)$ modulo 5, and $b_7(n)$ modulo 7. We only give the proofs for congruences (1.2)–(1.5) and certify Condition (a) in Proposition 2.1.

Theorem 2.3. *Let $\alpha,n\geq 0$ be nonnegative integers and let p_i be primes with $p_i\equiv 13,17,19$ or $23 \pmod{24}$. Then we have*

$$\sum_{n=0}^{\infty} b_6\Big(p_1^2\cdots p_\alpha^2 p_{\alpha+1}n+\frac{5(p_1^2\cdots p_{\alpha+1}^2-1)}{24}\Big)q^n \equiv (-1)^{\frac{\pm p_1-1}{6}+\cdots+\frac{\pm p_{\alpha+1}-1}{6}} f(-q^{2p_{\alpha+1}})\psi(q^{p_{\alpha+1}}) \pmod{3}, \quad (2.6)$$

where $\pm$ depends on the condition that $\frac{\pm p_i-1}{6}$ should be an integer for any $1\leq i\leq\alpha+1$. In particular, for any integer $j\not\equiv 0 \pmod{p_{\alpha+1}}$, we have

$$b_6\Big(p_1^2\cdots p_{\alpha+1}^2 n+\frac{p_1^2\cdots p_\alpha^2 p_{\alpha+1}(24j+5p_{\alpha+1})-5}{24}\Big)\equiv 0 \pmod{3}. \quad (2.7)$$

Proof. We have

$$\sum_{n=0}^{\infty} b_6(n)q^n=\frac{(q^6,q^6)_\infty}{(q;q)_\infty}\equiv\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty}\equiv (q^2;q^2)_\infty\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}=f(-q^2)\psi(q) \pmod{3}.$$

By the summation expressions of $f(-q)$ and $\psi(q)$, we have

$$f(-q^2)\psi(q)=\sum_{k,l=-\infty}^{\infty} c(k,l)q^{\theta(k,l)},$$

where

$$c(k,l)=\begin{cases}(-1)^k, & l\geq 0,\\ 0, & l<0,\end{cases} \quad\text{and}\quad \theta(k,l)=k(3k+1)+\frac{l(l+1)}{2}.$$

Notice that

$$\theta(k,l) = 3\left(k+\frac{1}{6}\right)^2 + \frac{1}{2}\left(l+\frac{1}{2}\right)^2 - \frac{5}{24}.$$

When $p \equiv 13, 17, 19$ or $23 \pmod{24}$, we have $\left(\frac{-6}{p}\right) = -1$, where $\left(\frac{\cdot}{p}\right)$ is the Jacobi symbol. Hence the congruence equation $\theta(k,l) \equiv -\frac{5}{24} \pmod p$ has a unique solution

$$k \equiv \frac{\pm p - 1}{6} \quad \text{and} \quad l \equiv \frac{p-1}{2} \pmod p,$$

where $\pm$ depends on the condition that $\frac{\pm p-1}{6}$ should be an integer. ∎

Theorem 2.4. *Let α, n be nonnegative integers and let $p_i \equiv -1 \pmod 6$ be primes. Then we have*

$$\sum_{n=0}^{\infty} b_5\left(p_1^2 \cdots p_\alpha^2 p_{\alpha+1} n + \frac{p_1^2 \cdots p_{\alpha+1}^2 - 1}{6}\right) q^n \equiv (-1)^{\alpha+1} p_1 \cdots p_{\alpha+1} f(-q^{p_{\alpha+1}}) g(q^{p_{\alpha+1}}) \pmod 5. \quad (2.8)$$

In particular, for any integer $j \not\equiv 0 \pmod{p_{\alpha+1}}$, we have

$$b_5\left(p_1^2 \ldots p_{\alpha+1}^2 n + \frac{p_1^2 \cdots p_\alpha^2 p_{\alpha+1}(6j + p_{\alpha+1}) - 1}{6}\right) \equiv 0 \pmod 5. \quad (2.9)$$

Proof. We have

$$\sum_{n=0}^{\infty} b_5(n) q^n = \frac{(q^5, q^5)_\infty}{(q;q)_\infty} \equiv (q;q)_\infty^4 = f(-q)g(q) \pmod 5.$$

By the summation expressions of $f(-q)$ and $g(q)$, we have

$$f(-q)g(q) = \sum_{k,l=-\infty}^{\infty} c(k,l) q^{\theta(k,l)},$$

where

$$c(k,l) = \begin{cases} (-1)^{k+l}(2l+1), & l \geq 0, \\ 0, & l < 0, \end{cases} \quad \text{and} \quad \theta(k,l) = \frac{k(3k+1)}{2} + \frac{l(l+1)}{2}.$$

Notice that

$$\theta(k,l) = \frac{3}{2}\left(k+\frac{1}{6}\right)^2 + \frac{1}{2}\left(l+\frac{1}{2}\right)^2 - \frac{1}{6}.$$

When $p \equiv -1 \pmod 6$, we have $\left(\frac{-3}{p}\right) = -1$ and hence the congruence equation $\theta(k,l) \equiv -\frac{1}{6} \pmod p$ has a unique solution

$$k \equiv \frac{-p-1}{6} \quad \text{and} \quad l \equiv \frac{p-1}{2} \pmod p. \quad ∎$$

Theorem 2.5. *Let α, n be nonnegative integers and let p_i be primes such that $p_i \equiv 5$ or $7 \pmod 8$. Then we have*

$$\sum_{n=0}^{\infty} b_{10}\Big(p_1^2 \cdots p_\alpha^2 p_{\alpha+1} n + \frac{3(p_1^2 \cdots p_{\alpha+1}^2 - 1)}{8}\Big) q^n$$

$$\equiv (-1)^{\frac{p_1+\cdots+p_{\alpha+1}-(\alpha+1)}{2}} p_1 \cdots p_{\alpha+1} g(q^{2p_{\alpha+1}})\psi(q^{p_{\alpha+1}}) \pmod 5. \quad (2.10)$$

In particular, for any integer $j \not\equiv 0 \pmod{p_{\alpha+1}}$, we have

$$b_{10}\Big(p_1^2 \ldots p_{\alpha+1}^2 n + \frac{p_1^2 \cdots p_\alpha^2 p_{\alpha+1}(8j + 3p_{\alpha+1}) - 3}{8}\Big) \equiv 0 \pmod 5. \quad (2.11)$$

Proof. We have

$$\sum_{n=0}^{\infty} b_{10}(n) q^n = \frac{(q^{10}, q^{10})_\infty}{(q;q)_\infty} \equiv \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty} \equiv (q^2;q^2)_\infty^3 \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = g(q^2)\psi(q) \pmod 5. \quad (2.12)$$

By the summation expressions of $g(q)$ and $\psi(q)$, we have

$$g(q^2)\psi(q) = \sum_{k,l=0}^{\infty} c(k,l) q^{\theta(k,l)},$$

where

$$c(k,l) = (-1)^k (2k+1) \quad \text{and} \quad \theta(k,l) = k(k+1) + \frac{l(l+1)}{2}.$$

Notice that

$$\theta(k,l) = \left(k + \frac{1}{2}\right)^2 + \frac{1}{2}\left(l + \frac{1}{2}\right)^2 - \frac{3}{8}.$$

When $p \equiv 5$ or $7 \pmod 8$, we have $\left(\frac{-2}{p}\right) = -1$ and hence the congruence equation $\theta(k,l) \equiv -\frac{3}{8} \pmod p$ has a unique solution

$$k \equiv \frac{p-1}{2} \quad \text{and} \quad l \equiv \frac{p-1}{2} \pmod p. \qquad \blacksquare$$

Theorem 2.6. *Let α, n be nonnegative integers and let p_j be primes such that $p_j \equiv 3 \pmod 4$. Then we have*

$$\sum_{n=0}^{\infty} b_7\left(p_1^2 \cdots p_\alpha^2 p_{\alpha+1} n + \frac{p_1^2 \cdots p_{\alpha+1}^2 - 1}{4}\right) q^n \equiv p_1^2 \cdots p_{\alpha+1}^2 g(q^{p_{\alpha+1}})^2 \pmod 7. \quad (2.13)$$

In particular, for any integer $j \not\equiv 0 \pmod{p_{\alpha+1}}$, we have

$$b_7\left(p_1^2 \ldots p_{\alpha+1}^2 n + \frac{p_1^2 \cdots p_\alpha^2 p_{\alpha+1}(4j + p_{\alpha+1}) - 1}{4}\right) \equiv 0 \pmod 7. \quad (2.14)$$

Proof. We have

$$\sum_{n=0}^{\infty} b_7(n)q^n = \frac{(q^7,q^7)_\infty}{(q;q)_\infty} \equiv (q;q)_\infty^6 = g(q)^2 \pmod 7. \tag{2.15}$$

By the summation expression of $g(q)$, we have

$$g(q)^2 = \sum_{k,l=0}^{\infty} c(k,l)q^{\theta(k,l)},$$

where

$$c(k,l) = (-1)^{k+l}(2k+1)(2l+1) \quad \text{and} \quad \theta(k,l) = \frac{k(k+1)}{2} + \frac{l(l+1)}{2}.$$

Notice that

$$\theta(k,l) = \frac{1}{2}\left(k+\frac{1}{2}\right)^2 + \frac{1}{2}\left(l+\frac{1}{2}\right)^2 - \frac{1}{4}.$$

When $p \equiv 3 \pmod 4$, we have $\left(\frac{-1}{p}\right) = -1$ and hence the congruence equation $\theta(k,l) \equiv -\frac{1}{4} \pmod p$ has a unique solution

$$k \equiv \frac{p-1}{2} \quad \text{and} \quad l \equiv \frac{p-1}{2} \pmod p.$$

∎

Remark that, the above congruence relations obtained by using the vanishing property also can be derived by applying the Hecke operator on certain eigenforms.

3 Some examples

In this section, we give some specializations of the congruence relations in the previous section.

The first specialization is to set $\alpha = 0$ and $p_1 = 5$ in (2.8). We thus obtain

$$b_5(5n+4) \equiv 0 \pmod 5,$$

which can be easily derived from Ramanujan's congruence $p(5n+4) \equiv 0 \pmod 5$ for ordinary partitions. In a similar way, we obtain from (2.10) and (2.13) that

$$b_{10}(5n+4) \equiv 0 \pmod 5 \quad \text{and} \quad b_7(7n+5) \equiv 0 \pmod 7.$$

The second specialization is that setting all the primes $p_1, p_2, \ldots, p_{\alpha+1}$ to be equal to the same prime p. We thus derive the following infinite families of congruences for $b_\ell(n)$.

Let α be a positive integer, p be a prime and j be an integer with $p \nmid j$.

1. If $p \geq 5$ and $p \equiv 3 \pmod 4$, then we have

$$b_3\left(p^{2\alpha}n + p^{2\alpha-1}j + \frac{p^{2\alpha}-1}{12}\right) \equiv 0 \pmod 3.$$

2. If $p \equiv 13, 17, 19$ or $23 \pmod{24}$, then we have

$$b_6\left(p^{2\alpha}n + p^{2\alpha-1}j + \frac{5(p^{2\alpha}-1)}{24}\right) \equiv 0 \pmod 3.$$

3. If $p \equiv -1 \pmod 6$, then we have

$$b_5\left(p^{2\alpha}n + p^{2\alpha-1}j + \frac{p^{2\alpha}-1}{6}\right) \equiv 0 \pmod 5.$$

4. If $p \equiv 5$ or $7 \pmod 8$, then we have

$$b_{10}\left(p^{2\alpha}n + p^{2\alpha-1}j + \frac{3(p^{2\alpha}-1)}{8}\right) \equiv 0 \pmod 5.$$

5. If $p \equiv 3 \pmod 4$, then we have

$$b_7\left(p^{2\alpha}n + p^{2\alpha-1}j + \frac{p^{2\alpha}-1}{4}\right) \equiv 0 \pmod 7.$$

Now setting $\alpha = 1$ and taking some explicit primes in the above congruence relations, we obtain the following congruences.

1. For $n \geq 0$ and $j \not\equiv 1 \pmod 7$, we have

$$b_3(49n + 7j - 3) \equiv 0 \pmod 3.$$

2. For $n \geq 0$ and $j \not\equiv 3 \pmod{13}$, we have

$$b_6(169n + 13j - 4) \equiv 0 \pmod 3.$$

3. For $n \geq 0$ and $j \not\equiv 2 \pmod{11}$, we have

$$b_5(121n + 11j - 2) \equiv 0 \pmod 5.$$

4. For $n \geq 0$ and $j \not\equiv 2 \pmod 5$, we have

$$b_{10}(25n + 5j - 1) \equiv 0 \pmod 5.$$

5. For $n \geq 0$, we have

$$b_7(9n+5) \equiv 0 \pmod 7, \qquad b_7(9n+8) \equiv 0 \pmod 7.$$

We conclude this paper with an example involving two primes. Setting $\alpha = 1$, $p_1 = 3$ and $p_2 = 7$ in (2.14), we obtain

$$b_7(441n + 63j + 110) \equiv 0 \pmod 7,$$

where $n \geq 0$ and $j \not\equiv 0 \pmod 7$.

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education and the National Science Foundation of China.

References

[1] S. Ahlgren and J. Lovejoy, The arithmetic of partitions into distinct parts, Mathematika, 48 (2001), 203–211.

[2] K. Alladi, Partition identities involving gaps and weights, Trans. Amer. Math. Soc., 349 (1997), 5001–5019.

[3] G.E. Andrews, M.D. Hirschhorn and J.A. Sellers, Arithmetic properties of partitions with even parts distinct, Ramanujan J., 23 (2010), 169–181.

[4] B.C. Berndt, Number Theory in the Spirit of Ramanujan, American Mathematical Society, Providence, 2004.

[5] S.-P. Cui and N.S.S. Gu, Arithmetic properties of ℓ-regular partitions, Adv. Appl. Math., 51 (2013), 507–523.

[6] S.-P. Cui and N.S.S. Gu, Congruences for 9-regular partitions modulo 3, Ramanujan J., 35 (2014), 157–164.

[7] D. Furcy and D. Penniston, Congruences for ℓ-regular partition functions modulo 3, Ramanujan J., 27 (2012), 101–108.

[8] B. Gordon and K. Ono, Divisibility of certain partition functions by powers of primes, Ramanujan J., 1 (1997), 25–34.

[9] M.D. Hirschhorn and J.A. Sellers, Elementary proofs of parity results for 5-regular partitions, Bull. Austral. Math. Soc., 81 (2010), 58–63.

[10] W.J. Keith, Congruences for 9-regular partitions modulo 3, Ramanujan J., 35 (2014), 157–164.

[11] B. Kim, Overpartition pairs modulo powers of 2, Discrete Math., 311 (2011), 835–840.

[12] B.L.S. Lin and A.Y.Z. Wang, Generalisation of Keith's conjecture on 9-regular partitions and 3-cores, Bull. Austral. Math. Soc., 90 (2014), 204–212.

[13] J. Lovejoy, Divisibility and distribution of partitions into distinct parts, Adv. Math., 158 (2001), 253–263.

[14] J. Lovejoy and R. Osburn, Quadratic forms and four partition functions modulo 3, Integers, 11 (2011), #A4.

[15] K. Ono and D. Penniston, The 2-adic behavior of the number of partitions into distinct parts, J. Combin. Theory, Ser. A, 92 (2000), 138–157.

[16] K. Ono, Distribution of the partition function modulo m, Ann. Math., 151 (1) (2000), 293–307.

[17] D. Penniston, The p^a-regular partition function modulo p^j, J. Number Theory, 94 (2002), 320–325.

[18] E.X.W. Xia and O.X.M. Yao, Parity results for 9-regular partitions, Ramanujan J., 34 (2014), 109–117.

[19] E.X.W. Xia and O.X.M. Yao, A proof of Keith's conjecture for 9-regular partitions modulo 3, Int. J. Number Theory, 3 (2014), 669–674.

[20] O.X.M. Yao, New congruences modulo powers of 2 and 3 for 9-regular partitions, J. Number Theory, 142 (2014), 89–101.