Approximating longest cycles in graphs with bounded degrees

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Abstract

Jackson and Wormald conjecture that if G is a 3-connected n-vertex graph with maximum degree $d \geq 4$ then G has a cycle of length $\Omega(n^{\log_{d-1} 2})$. We show that this conjecture holds when $d-1$ is replaced by $\max\{64, 4d+1\}$. Our proof implies a cubic algorithm for finding such a cycle.

1 Introduction

From the point of view of approximation algorithms, finding a longest cycle in a graph is one of the "harder" NP-hard problems. There is no known polynomial time algorithm which guarantees an approximation ratio better than $n/\text{polylog}(n)$. For graphs with a cycle of length k, it was shown in [1] that one can find in polynomial time a cycle of length $\Omega((\log k)^2/\log\log k)$. Gabow [6] showed how to find in polynomial time a cycle of length $\exp(\Omega(\sqrt{\log k/\log\log k}))$ through a given vertex v in a graph that contains a cycle of length k through v. Recently, Feder and Motwani [5] obtained a cubic algorithm which, given a graph with maximum degree d and containing a k-vertex 3-cyclable minor, finds a cycle of length $k^{1/(2c\log d)}$ for some $c \geq 2$. A consequence of their result improves Gabow's result in certain situations.

Karger, Motwani, and Ramkumar [10] showed that unless $\mathcal{P} = \mathcal{NP}$ it is impossible to find, in polynomial time, a path of length $n - n^{\epsilon}$ in an n-vertex Hamiltonian graph for any $\epsilon < 1$. They conjecture that it is as hard even for graphs with bounded degrees. On the other hand, Feder, Motwani, and Subi [4] showed that there is a polynomial time algorithm for finding a cycle of length at least $n^{(\log_3 2)/2}$ in any 3-connected cubic n-vertex graph. They also proposed the problem for 3-connected graphs with bounded

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degrees. For a graph G, let $\Delta(G)$ denote its maximum degree. Jackson and Wormald [9] proved that every 3-connected n-vertex graph G with $\Delta(G) \le d$ has a cycle of length at least $\frac{1}{2}n^{\log_b 2} + 1$, where $b = 6d^2$. Recently, Chen, Xu, and Yu [3] gave a cubic algorithm that, given a 3-connected n-vertex graph G with $\Delta(G) \le d$, finds a cycle of length at least $n^{\log_b 2}$, where $b = 2(d-1)^2+1$. It was conjectured in 1993 by Jackson and Wormald [9] that for $d \ge 4$ the right value for b should be $d-1$. The main result of this paper shows that this conjecture holds for a linear function b of d. (This result appears in the extended abstract [2].)

(1.1) Theorem. *Let $n \ge 4$ and $d \ge 4$ be integers. Let G be a 3-connected graph with n vertices and $\Delta(G) \le d$. Then G contains a cycle of length at least $\frac{1}{2}n^{\log_b 2} + 3$, where $b = \max\{64, 4d+1\}$.*

For 3-connected graphs, this improves the above-mentioned result of Feder and Motwani [5]. Our proof of Theorem (1.1) implies a cubic algorithm for finding a cycle of length at least $\frac{1}{2}n^{\log_b 2} + 3$. The multiplicative constant 1/2 and the additive constant 3 are for induction purpose. As in [3], we prove the following three statements simultaneously.

(1.2) Theorem. *Let $n \ge 5$ and $d \ge 4$ be integers, let $b = \max\{64, 4d+1\}$ and $r = \log_b 2$, and let G be a 3-connected graph with n vertices. Then the following statements hold.*

(a) Let $xy \in E(G)$ and $z \in V(G) - \{x,y\}$, and let t denote the number of neighbors of z distinct from x and y. Assume $\Delta(G) \le d+1$, and that every vertex of degree $d+1$ (if any) is incident with edge zx or zy. Then there is a cycle C through xy in $G - z$ such that $|C| \ge \frac{1}{2}(\frac{(d-1)n}{dt})^r + 2$.

(b) Suppose $\Delta(G) \le d$. Then for any distinct $e, f \in E(G)$, there is a cycle C through e and f in G such that $|C| \ge \frac{1}{2}(\frac{n}{d})^r + 3$.

(c) Suppose $\Delta(G) \le d$. Then for any $e \in E(G)$, there is a cycle C through e in G such that $|C| \ge \frac{1}{2}n^r + 3$.

Note the degree condition in (a): zx and zy need not be edges of G, but if x (respectively, y) has degree $d+1$ then zx (respectively, zy) must be an edge of G, and if z has degree $d+1$ then zx or zy must be an edge of G. This condition is due to the addition of edges in order to maintain 3-connectivity.

When $n \ge 5$, Theorem (1.2)(c) clearly implies Theorem (1.1). When $n = 4$, Theorem (1.1) is obvious. The next result says that Theorem (1.2) holds for graphs with bounded size, which will enable us to avoid dealing with small graphs in inductive proofs. We omit its proof, since it is rather straightforward.

(1.3) Lemma. *Let G, n, d, b, r be the same as in Theorem (1.2). If $n \le 4d+1$ then Theorem (1.2)(a) and (b) hold, and if $n \le (4d+1)^2$ then Theorem (1.2)(c) holds.*

To prove Theorem (1.2), we need to deal with graphs obtained from 3-connected graphs by deleting a vertex (such as $G - z$ in (a)), and such graphs need not be 3-connected. By using a result of Tutte [11] and an algorithm of Hopcroft and Tarjan [7], we can decompose such graphs into "3-connected components". We then find long paths through certain 3-connected components and use properties of the function $x^{\log_b 2}$ to account for the unused 3-connected components. (For a brief outline of our approach, the reader is referred to the Algorithm in section 6.) Our approach is similar to that in [3], but here we prove stronger properties of the function $x^{\log_b 2}$ and analyze the 3-connected components in a more sophisticated way.

We organize this paper as follows. In section 2, we recall notation of Hopcroft and Tarjan [7] concerning the decomposition result of Tutte [11] of 2-connected graphs into 3-connected components. We then define cycle chains of 3-connected components, and prove several results on paths in cycle chains. We prove in section 3 several useful properties of the function $f(x) = x^{\log_b 2}$. We also define block chains of 3-connected components, and prove lemmas concerning paths in block chains. Theroem (1.2) will be shown inductively. So in sections 4 and 5, we show how to reduce Theorem (1.2) to smaller graphs. In Section 6, we complete the proof of our main result, and outline a cubic algorithm for finding a long cycle in a 3-connected graph with bounded degree.

For graphs G and H, we use $G \cong H$ (respectively $G \ncong H$) to mean that G is isomorphic to (respectively, not isomorphic to) H. Let G be a graph, H a subgraph of G, and $S := \{v_1, \ldots, v_k, x_1y_1, \ldots, x_py_p\}$, where v_i, x_j, y_j are vertices of G and $\{x_1, y_1, \ldots, x_p, y_p\} \subseteq \{v_1, \ldots, v_k\} \cup V(H)$. Then $H + S$ denotes the simple graph with $V(H + S) := V(H) \cup \{v_1, \ldots, v_k\}$ and $E(H + S) = E(H) \cup \{x_1y_1, \ldots, x_py_p\}$.

2 Paths in cycle chains

For convenience, we recall the decomposition of a 2-connected graph into 3-connected components. A detailed description can be found in [3] and [7].

Let G be a 2-connected graph. We allow multiple edges for the description of this decomposition. Then, $E(G)$ in this section is treated as a multi-set. We say that $\{a, b\} \subseteq V(G)$ is a *separation pair* in G if there are subgraphs G_1, G_2 of G such that $G_1 \cup G_2 = G$, $V(G_1 \cap G_2) = \{a, b\}$, $E(G_1 \cap G_2) = \emptyset$, and $|E(G_i)| \geq 2$ for $i = 1, 2$. Let $G_i' := (V(G_i), E(G_i) \cup \{ab\})$ for $i = 1, 2$. Then G_1' and G_2' are called *split graphs* of G with respect to the separation pair $\{a, b\}$, and the new edge ab added to G_i is called a *virtual* edge. It is easy to see that, since G is 2-connected, G_i' is 2-connected or G_i' consists of two vertices and at least three multiple edges between them.

Suppose a multigraph is split, and the split graphs are split, and so on, until no more splits are possible. Then each remaining graph is called a *split component*. No split component contains a separation pair and, therefore, each split component must be one of the following: a triangle, a *triple bond* (two vertices and three multiple edges between them), or a 3-connected graph.

It is not hard to see that if a split component of a 2-connected graph is 3-connected then it is uniquely determined. It is also easy to see that, for any two split components G_1, G_2 of a 2-connected graph, we have $|V(G_1 \cap G_2)| \leq 2$, and if $|V(G_1 \cap G_2)| = 2$ then either G_1 and G_2 share a virtual edge between the vertices in $V(G_1 \cap G_2)$ or there is a sequence of triple bonds such that the first shares a virtual edge with G_1, any two consecutive triple bonds in the sequence share a virtual edge, and the last triple bond shares a virtual edge with G_2.

In order to make such decomposition unique, some triple bonds and triangles need to be merged. Let $G_i' = (V_i', E_i')$, $i = 1, 2$, be two split components, both containing a virtual edge ab. Let $G' = (V_1' \cup V_2', (E_1' - \{ab\}) \cup (E_2' - \{ab\}))$. The graph G' is called the *merge graph* of G_1' and G_2'. Clearly, a merge of triple bonds gives a graph consisting of two vertices and multiple edges, which is called a *bond*. Also a merge of triangles gives a cycle, and a merge of cycles gives a cycle as well.

Let $\mathcal{D}$ denote the set of those 3-connected split components of a 2-connected graph G. We merge the split components of G not in $\mathcal{D}$ as follows: the bonds are merged as much as possible to give a set of bonds $\mathcal{B}$, and the cycles are merged as much as possible to give a set of cycles $\mathcal{C}$. Then $\mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ is the set of the *3-connected components* of G. Note that any two 3-connected components either are edge disjoint or share exactly one virtual edge. The following theorem is a combination of a result of Tutte [11] and an algorithm of Hopcroft and Tarjan [7].

(2.1) Theorem. *The 3-connected components of any 2-connected graph are unique and can be found in $O(E)$ time.*

If we define a graph whose vertices are the 3-connected components of G and two vertices are adjacent whenever the corresponding 3-connected components share a virtual edge, then this graph is a tree, which we call the *block-bond tree* of G. For convenience, 3-connected components that are not bonds are called *3-blocks*. An *extreme* 3-block is a 3-block that contains at most one virtual edge. That is, either it is the only 3-connected component (in which case G is 3-connected), or it corresponds to a degree one vertex in the block-bond tree.

A *cycle chain* in a 2-connected graph G is a sequence $C_1C_2 \dots C_k$ of 3-blocks of G such that each C_i is a cycle and there exist bonds (possibly empty) $B_1, B_2, \dots, B_{k-1}$ in G such that $C_1B_1C_2B_2 \dots B_{k-1}C_k$ is a path in the block-bond tree of G. For convenience, we sometimes write $H := C_1 \dots C_k$ for a cycle chain, and view H as the simple graph obtained from the union of C_i ($1 \le i \le k$) by identifying virtual edges between the vertices of $C_i \cap C_{i+1}$ ($1 \le i \le k-1$). The following is a direct consequence of the definition of a cycle chain.

(2.2) Proposition. *Let G be a 2-connected graph and $H := C_1 \dots C_k$ be a cycle chain in G. Then deleting all edges of H with both ends in $V(C_i \cap C_{i+1})$, $1 \le i \le k-1$, results in a cycle.*

The next result finds a path linking two edges in a cycle chain.

(2.3) Proposition. *Let G be a 2-connected graph, let $H := C_1 \dots C_k$ be a cycle chain in G, let $uv \in E(C_1)$ with $\{u, v\} \ne V(C_1 \cap C_2)$ when $k \ne 1$, and let $ab \in E(C_k)$ with $\{a, b\} \ne V(C_{k-1} \cap C_k)$ when $k \ne 1$. Then there is a path in $H - \{v, ab\}$ from u to $\{a, b\}$ and containing $V(\bigcup_{i=1}^{k-1}(C_i \cap C_{i+1})) - (\{a, b\} \cup \{u, v\})$.*

Proof. We apply induction on k. The result holds trivially for $k = 1$. So assume $k \ge 2$. Let $H' := C_2 \dots C_k$ and $V(C_1 \cap C_2) = \{u_1, v_1\}$. Without loss of generality, we may assume that $C_1 - \{v, v_1\}$ contains a path P from u to u_1. Suppose $v_1 = v$. By induction, we find a path Q in $H' - \{v_1, ab\}$ from u_1 to $\{a, b\}$ and containing $V(\bigcup_{i=2}^{k-1}(C_i \cap C_{i+1})) - (\{a, b\} \cup \{u_1, v_1\})$. Then $P \cup Q$ gives the desired path. Now assume $v_1 \ne v$. By induction, we find a path Q' in $H' - \{u_1, ab\}$ from v_1 to $\{a, b\}$ and containing $V(\bigcup_{i=2}^{k-1}(C_i \cap C_{i+1})) - (\{a, b\} \cup \{u_1, v_1\})$. Now $(P \cup Q') + u_1v_1$ gives the desired path. □

Remark. The path, say R, found in Proposition (2.3), may use edges between the vertices of $C_i \cap C_{i+1}$ ($1 \le i \le k-1$). However either G also has an edge between the vertices of $C_i \cap C_{i+1}$, or $C_i \cap C_{i+1}$ is contained in a 3-block of G not in $\mathcal{H}$. Hence, from R we can produce a path in G by replacing virtual edges in R with appropriate paths in G, and this new path is at least as long as R. This observation applies to the next three results as well, and will be frequently used.

A similar argument establishes the following result, which finds a path in a cycle chain between two vertices and avoiding a specific vertex.

(2.4) Proposition. *Let G be a 2-connected graph, let $H := C_1 \ldots C_k$ be a cycle chain in G, let $uv \in E(C_1)$ with $\{u, v\} \ne V(C_1 \cap C_2)$ when $k \ne 1$, and let $x \in V(C_k)$ with $x \ne v$ when $k = 1$ and $x \notin V(C_{k-1} \cap C_k)$ when $k \ne 1$. Then there is a path in $H - v$ from u to x and containing $V(\bigcup_{i=1}^{k-1}(C_i \cap C_{i+1})) - \{v\}$.*

It is clear that the paths and cycle in the above three propositions can be found in $O(V)$ time. The following two results are Propositions (2.7) and (2.8) in [3], which find in $O(V)$ time paths through a given edge in a cycle chain.

(2.5) Proposition. *Let G be a 2-connected graph, let $H := C_1 \ldots C_k$ be a cycle chain in G, let $uv \in E(C_1)$ with $\{u, v\} \ne V(C_1 \cap C_2)$ when $k \ne 1$, $ab \in E(C_k)$ with $\{a, b\} \ne V(C_{k-1} \cap C_k)$ when $k \ne 1$, and $cd \in E(\bigcup_{i=1}^{k} C_i) - \{ab\}$. Suppose $ab \ne uv$ when $k = 1$. Then there is a path P in $H - ab$ from $\{a, b\}$ to $\{c, d\}$ such that $uv \in E(P)$, $cd \notin E(P)$ unless $cd = uv$, and $V(\bigcup_{i=1}^{k-1}(C_i \cap C_{i+1})) \subseteq V(P)$.*

(2.6) Proposition. *Let G be a 2-connected graph, let $H := C_1 \ldots C_k$ be a cycle chain in G, let $uv \in E(C_1)$ with $\{u, v\} \ne V(C_1) \cap V(C_2)$ when $k \ne 1$, $x \in V(C_k)$ with $x \notin V(C_{k-1} \cap C_k)$ when $k \ne 1$, and $cd \in E(\bigcup_{i=1}^{k} C_i)$. Then there is a path P in H from x to $\{c, d\}$ such that $uv \in E(P)$, $cd \notin E(P)$ unless $cd = uv$, and $V(\bigcup_{i=1}^{k-1}(C_i \cap C_{i+1})) \subseteq V(P)$.*

We conclude this section by recalling from [3] two graph operations and three lemmas. Let G be a graph and let e, f be distinct edges of G. An *H-transform* of G *at* $\{e, f\}$ is an operation that subdivides e and f by vertices x and y respectively and then adds the edge xy. Let $x \in V(G)$ such that x is not incident with e. A *T-transform* of G *at* $\{x, e\}$ is an operation that subdivides e with a vertex y and then adds the edge xy. If there is no need to specify e, f, x, we simply speak of an H-transform or a T-transform. The following result is Lemma (3.3) in [3].

(2.7) Lemma. *Let $d \ge 3$ be an integer and let G be a 3-connected graph with $\Delta(G) \le d$. Let G' be a graph obtained from G by an H-transform or a T-transform. Then G' is a 3-connected graph, the vertex of G involved in the T-transform has degree at most $d+1$, and all other vertices of G' has degree at most d.*

The next two results are Lemmas (3.6) and (3.7) in [3], where it is shown that the path P can be found in $O(V)$ time.

(2.8) Lemma. *Let G be a 3-connected graph, let $f \in E(G)$, let $ab, cd, vw \in E(G) - \{f\}$, and assume that $\{c, d\} \ne \{v, w\}$. Then there exists a path P in G from $\{a, b\}$ to some $z \in \{c, d\} \cup \{v, w\}$ such that (i) $f \in E(P)$, (ii) $cd \in E(P)$ or $vw \in E(P)$, (iii) if $cd \in E(P)$ then $z \in \{v, w\}$ and $vw \notin E(P)$, and (iv) if $vw \in E(P)$ then $z \in \{c, d\}$ and $cd \notin E(P)$.*

(2.9) Lemma. *Let G be a 3-connected graph, let $f \in E(G)$, let $x \in V(G)$ such that x is not incident with f, let $cd, vw \in E(G) - \{f\}$, and assume that $\{c, d\} \neq \{v, w\}$. Then there exists a path P in G from x to some $z \in \{c, d\} \cup \{v, w\}$ such that (i) $f \in E(P)$, (ii) $cd \in E(P)$ or $vw \in E(P)$, (iii) if $cd \in E(P)$ then $z \in \{v, w\}$ and $vw \notin E(P)$, and (iv) if $vw \in E(P)$ then $z \in \{c, d\}$ and $cd \notin E(P)$.*

3 Paths in block chains

We first prove four lemmas concerning the function $x^{\log_b 2}$. These lemmas will then be used to find long paths in block chains. First, we recall Lemma (3.1) in [3].

(3.1) Lemma. *Let $b \geq 4$ be an integer, and let $m \geq n$ be positive integers. Then $m^{\log_b 2} + n^{\log_b 2} \geq (m + (b-1)n)^{\log_b 2}$.*

When m is sufficiently larger than n, we have the following result.

(3.2) Lemma. *Let $b \geq 9$ be an integer, let m and n be positive integers, and assume $m \geq \frac{b(b-1)}{4}n$. Then $m^{\log_b 2} + n^{\log_b 2} \geq (m + \frac{b(b-1)}{4}n)^{\log_b 2}$.*

Proof. By dividing $m^{\log_b 2}$ to the above inequality, we see what we need to prove is equivalent to the statement: for any $0 \leq s \leq \frac{4}{b(b-1)}$, $1 + s^{\log_b 2} \geq (1 + \frac{b(b-1)}{4}s)^{\log_b 2}$.

Let $f(s) = 1 + s^{\log_b 2} - (1 + \frac{b(b-1)}{4}s)^{\log_b 2}$. Clearly, $f(0) = 0$. Note that $b(b-1) > 4(b-1)$ when $b \geq 5$. Hence $f(1) = 2 - (1 + \frac{b(b-1)}{4})^{\log_b 2} < 2 - b^{\log_b 2} = 0$. Taking derivative about s, we have $f'(s) = (\log_b 2)(s^{(\log_b 2)-1} - \frac{b(b-1)}{4}(1 + \frac{b(b-1)}{4}s)^{(\log_b 2)-1})$. A simple calculation shows that $f'(s) = 0$ has a unique solution. Therefore, if $f(c) > 0$ for some $0 < c < 1$, then $f(s) \geq 0$ for all $0 \leq s \leq c$.

Note that $0 < \frac{4}{b(b-1)} < 1$ and $f(\frac{4}{b(b-1)}) > 1 + (\frac{1}{b^2})^{\log_b 2} - 2^{\log_b 2} \geq 1.25 - 2^{\log_9 2} = 0.005587\ldots > 0$. Therefore, we have $f(s) \geq 0$ for all $s \in [0, \frac{4}{b(b-1)}]$. □

When m is not sufficiently larger than n, we have the following complementary result.

(3.3) Lemma. *Let $b \geq 64$ be an integer, let $m \geq n$ be positive integers, and assume $m \leq \frac{b(b-1)}{4}n$. Then $m^{\log_b 2} + n^{\log_b 2} \geq (4m)^{\log_b 2}$.*

Proof. The statement of Lemma (3.3) is equivalent to $1 + s^{\log_b 2} \geq 4^{\log_b 2}$ for all $\frac{4}{b(b-1)} \leq s \leq 1$. Therefore, it suffices to show $1 + (\frac{4}{b(b-1)})^{\log_b 2} \geq 4^{\log_b 2}$. This is true because $1 + (\frac{4}{b(b-1)})^{\log_b 2} \geq 1 + (\frac{4}{b^2})^{\log_b 2} = 1 + \frac{4^{\log_b 2}}{4} > 4^{\log_b 2}$ (since $b \geq 64$). □

We shall also use the following observations in the proof of Theorem (1.2).

(3.4) Lemma. *Let m be an integer, $d \geq 3$, and $b \geq d+1$. If $m \geq 4$ then $m \geq \frac{1}{2}m^{\log_b 2} + 3$. If $m \geq 3$ then $m > \frac{1}{2}(\frac{m}{d})^{\log_b 2} + 2$. If $m \geq 2$ then $m > \frac{1}{2}(\frac{m}{d})^{\log_b 2} + 1$.*

Proof. Let $f(x) = x - \frac{1}{2}x^{\log_b 2}$. We can show that $f'(x) > 0$ for $x \geq 1$. Hence $f(x)$ is an increasing function when $x \geq 1$. Thus, when $x \geq 4$, we have $f(x) \geq f(4) = 4 - \frac{1}{2}4^{\log_b 2} \geq 3$ (since $b \geq 4$). The first inequality holds.

Let $f(x) = x - \frac{1}{2}(\frac{x}{d})^{\log_b 2}$; then $f(x)$ is increasing when $x \geq 1$. The second inequality follows from $f(3) > 2$, and the third inequality follows from $f(2) > 1$. $\square$

We now turn to paths in block chains. Let G be a 2-connected graph. A *block chain* in G is a sequence $H_1 \ldots H_h$ for which (1) each H_i is either a cycle chain in G or a 3-connected 3-block of G, (2) for any $1 \leq s \leq h-1$, H_s or H_{s+1} is 3-connected, and (3) there exist bonds (possibly empty) $B_1, \ldots, B_{h-1}$ such that $H_1 B_1 H_2 B_2 \ldots B_{h-1} H_h$ form a path in the block-bond tree of G (by also including the tree paths corresponding to H_i when H_i is a cycle chain). A detailed description with examples can be found in [3]. For convenience, we sometimes write $\mathcal{H} := H_1 \ldots H_h$ for a block chain and view $\mathcal{H}$ as the simple graph obtained from $\bigcup_{i=1}^n H_i$ by identifying edges between the vertices in $H_i \cap H_{i+1}$ $(1 \leq i \leq n-1)$. The edges of $\mathcal{H}$ between the vertices of $H_i \cap H_{i+1}$ are called *separating edges* of $\mathcal{H}$. Such edges are to be avoided when we find paths in block chains.

Let $H_1 \ldots H_h$ be a block chain and let $V(H_s \cap H_{s+1}) = \{x_s, y_s\}$, $1 \leq s \leq h-1$. For each $1 \leq s \leq h$, we define $A(H_s)$ as follows. If H_s is 3-connected then $A(H_s) := V(H_s)$. If $H_s = C_1 \ldots C_k$ is a cycle chain then let

- $A(H_s) := V(\bigcup_{i=1}^{k-1}(C_i \cap C_{i+1})) - (\{x_{s-1}, y_{s-1}\} \cup \{x_s, y_s\})$ when $1 < s < h$,
- $A(H_s) := V(\bigcup_{i=1}^{k-1} C_i \cap C_{i+1})$ when $s = 1 = h$, $A(H_s) := V(\bigcup_{i=1}^{k-1}(C_i \cap C_{i+1})) - \{x_s, y_s\}$ when $s = 1 < h$, and
- $A(H_s) := V(\bigcup_{i=1}^{k-1}(C_i \cap C_{i+1})) - \{x_{s-1}, y_{s-1}\}$ when $1 < s = h$.

We write $\sigma(\mathcal{H}) := \sum_{s=1}^h |A(H_s)|$ and $|\mathcal{H}| := |V(\bigcup_{i=1}^h H_i)|$. For convenience, we define $B_1(\mathcal{H}) = \{H_i : H_i \text{ is 3-connected or } |A(H_i)| \leq 1\}$ and $B_2(\mathcal{H}) = \{H_i : H_i \text{ is a cycle chain and } |A(H_i)| \geq 2\}$.

In the remainder of this section, we show how to find long paths in block chains (in terms of $\sigma(\mathcal{H})$). All proofs imply $O(V)$ algorithms that reduce the problem of finding a path to Theorem (1.2) for smaller graphs.

(3.5) Lemma. *Let $n \geq 6$ be an integer and assume Theorem (1.2) holds for graphs with at most $n-1$ vertices. Let $\mathcal{H} := H_1 H_2 \cdots H_h$ be a block chain in a 2-connected graph such that $|\mathcal{H}| < n$ and $\Delta(H_i) \leq d$ for $1 \leq i \leq h$. Let $uv \in E(H_1)$ such that $\{u, v\}$ is not a cut of H_1, and if $h \geq 2$ then $\{u, v\} \neq V(H_1 \cap H_2)$. Then there is a path P in $\mathcal{H}$ from u to v such that $|E(P)| \geq \frac{1}{2}(\frac{(d-1)\sigma(\mathcal{H})}{d})^r + 2$ and P contains no separating edge of $\mathcal{H}$.*

Proof. When $h \geq 2$, we use a, b to denote the vertices in $V(H_1 \cap H_2)$. Suppose $|A(H_1)| \geq \frac{(d-1)\sigma(\mathcal{H})}{d}$. First assume H_1 is a cycle chain or $H_1 \cong K_4$. Then there is a Hamilton path P_1 in H_1 from u to v (by Proposition (2.2) when H_1 is a cycle chain). If $|H_1| = 3$ then $|A(H_1)| = 0$, and hence, $|E(P_1)| \geq \frac{1}{2}|A(H_1)|^r + 2$. If $|H_1| \geq 4$ then $|E(P_1)| \geq 3$, and by Lemma (3.4), $|E(P_1)| \geq \frac{1}{2}|H_1|^r + 2 \geq \frac{1}{2}|A(H_1)|^r + 2$. Now assume H_1 is 3-connected and $H_1 \not\cong K_4$. Then by Theorem (1.2)(c), H_1 has a cycle C_1 through uv such that $|E(C_1)| \geq \frac{1}{2}|H_1|^r + 3 = \frac{1}{2}|A(H_1)|^r + 3$. Let $P_1 := C_1 - uv$. If $h = 1$ or $ab \notin E(P_1)$ then $P := P_1$ gives the desired path. If $h \geq 2$ and $ab \in E(P_1)$ then, by replacing ab with a path in $H_2 \ldots H_h$ between a and b and not containing any separating edge of $\mathcal{H}$, we obtain the desired path P.

So we may assume $|A(H_1)| < \frac{(d-1)\sigma(\mathcal{H})}{d}$. In particular, $h \geq 2$. If H_1 is a cycle chain or $H_1 \cong K_4$ then, as in the above paragraph, we find a Hamilton path P_1 from u to v in H_1 through ab such that $|E(P_1)| \geq \frac{1}{2}|A(H_1)|^r + 2$. Now assume H_1 is 3-connected and $H_1 \ncong K_4$. Then by Theorem (1.2)(b), H_1 has a cycle C_1 through uv and ab such that $|E(C_1)| \geq \frac{1}{2}(\frac{|A(H_1)|}{d})^r + 3$; let $P_1 := C_1 - uv$.

By induction, we find a path P' in $\mathcal{H}' := H_2 \dots H_h$ from a to b and containing no separating edges of $\mathcal{H}'$ such that $|E(P')| \geq \frac{1}{2}(\frac{(d-1)\sigma(\mathcal{H}')}{d})^r + 2$. Let $P := (P_1 - ab) \cup P'$. Since $\sigma(\mathcal{H}) \leq A(H_1) + \sigma(\mathcal{H}')$ and $|A(H_1)| < \frac{(d-1)\sigma(\mathcal{H})}{d}$, $\frac{|A(H_1)|}{d} < \frac{(d-1)\sigma(\mathcal{H}')}{d}$. Hence by Lemma (3.2),

$$
\begin{aligned}
|E(P)| &> \frac{1}{2}(\frac{|A(H_1)|}{d})^r + \frac{1}{2}(\frac{(d-1)\sigma(\mathcal{H}')}{d})^r + 2 \\
&\geq \frac{1}{2}((b-1)\frac{|A(H_1)|}{d} + \frac{(d-1)\sigma(\mathcal{H}')}{d})^r + 2 \\
&> \frac{1}{2}(\frac{(d-1)\sigma(\mathcal{H})}{d})^r + 2.
\end{aligned}
$$

So P gives the desired path. □

For the next two lemmas, we define uv and x in a block chain $\mathcal{H} := H_0 H_1 \cdots H_h$ (in a 2-connected graph). Suppose $h = 0$. If H_0 is 3-connected or H_0 is a cycle then let $uv \in E(H_0)$ and $x \in V(H_0) - \{u, v\}$, and if $H_0 = C_1 \dots C_k$ is a cycle chain with $k \geq 2$ then let $uv \in E(C_1)$ with $\{u, v\} \neq V(C_1 \cap C_2)$ and let $x \in V(C_k) - V(C_{k-1})$. Now assume $h \geq 1$. If H_0 is 3-connected or H_0 is a cycle then let $uv \in E(H_0)$ with $\{u, v\} \neq V(H_0 \cap H_1)$, if $H_0 = C_1 \dots C_k$ is a cycle chain with $k \geq 2$ and $V(H_0 \cap H_1) = V(C_k \cap H_1)$ then let $uv \in E(C_1)$ with $\{u, v\} \neq V(C_1 \cap C_2)$, if H_h is a cycle or H_h is 3-connected then let $x \in V(H_h) - V(H_{h-1})$, and if $H_h = C_1 \dots C_k$ is a cycle chain with $k \geq 2$ and $V(H_{h-1} \cap H_h) = V(H_{h-1} \cap C_1)$ then let $x \in V(C_k) - V(C_{k-1})$.

(3.6) Lemma. *Let $n \geq 6$ be an integer and assume Theorem (1.2) holds for graphs with at most $n-1$ vertices. Let $\mathcal{H} := H_0 H_1 \cdots H_h$, uv, x be defined as above, and assume $|\mathcal{H}| < n$, $\Delta(H_i) \leq d$ for $0 \leq i \leq h$, and the degree of x in H_h is at most $d-1$. Then there exists a path P in $\mathcal{H} - v$ from u to x and containing no separating edge of $\mathcal{H}$ such that*

(i) $|E(P)| \geq \frac{1}{2}(\sum_{i=0}^{h}(\frac{|A(H_i)|}{d})^r) + 1 \geq \frac{1}{2}(\frac{\sigma(\mathcal{H})}{d})^r + 1$, and

(ii) $|E(P)| \geq \frac{1}{2}(\sum\{(\frac{|A(H_i)|}{d})^r : H_i \in B_1(\mathcal{H})\}) + (\sum\{\max\{1, |A(H_i)| - 2\} : H_i \in B_2(\mathcal{H})\}) + 1$.

Proof. We apply induction on h. Suppose $h = 0$. If H_0 is 3-connected and $H_0 \ncong K_4$, then by assumption and because x has degree at most $d-1$, Theorem (1.2)(a) holds for $H_0 + \{vx, ux\}$. Hence, $H_0 - v$ contains a path P from u to x such that $|E(P)| \geq \frac{1}{2}(\frac{|A(H_0)|}{d})^r + 1$. If $H_0 \cong K_4$, then we can find a path P from u to x in $H_0 - v$ such that $|E(P)| = 2 \geq \frac{1}{2}(\frac{|A(H_0)|}{d})^r + 1$. If H_0 is a cycle chain, then by Proposition (2.4), there is a path P from u to x in $H_0 - v$ containing $A(H_0) - \{v\}$. Note that $x \notin A(H_0)$ and if $v \in A(H_0)$ then $u \notin A(H_0)$. Thus, $|E(P)| \geq |A(H_0)|$. Because $|E(P)| \geq 1$ and since

$|A(H_0)| = 0$ or $|A(H_0)| \geq 2$, we have $|E(P)| \geq \frac{1}{2}(\frac{|A(H_0)|}{d})^r + 1$ (by Lemma (3.4)). Clearly, $|E(P)| \geq \max\{1, |A(H_0)| - 2\} + 1$ when $H_0 \in B_2(\mathcal{H})$.

Now assume $h \geq 1$. Let $V(H_0 \cap H_1) = \{u_0, v_0\}$, and assume the notation is chosen so that $u_0 \notin \{u, v\}$. By the above argument for $h = 0$, if H_0 is a cycle chain or $H_0 \cong K_4$ then $H_0 - v$ has a path P_0 from u to u_0 such that $|E(P_0)| \geq \frac{1}{2}(\frac{|A(H_0)|}{d})^r + 1$, and $|E(P_0)| \geq \max\{1, |A(H_0)| - 2\} + 1$ when $H_0 \in B_2(\mathcal{H})$. (Note in the case H_0 is a cycle chain, $u_0 \notin A(H_0)$ because $h \geq 1$.) Now assume H_0 is 3-connected and $|H_0| \geq 5$. If $v = v_0$ then we apply Theorem (1.2)(a) to find a path P_0 from u to u_0 in $(H_0 + uu_0) - v$ such that $|E(P_0| \geq \frac{1}{2}(\frac{|A(H_0)|}{d})^r + 1$. If $v \neq v_0$ then let H_0' be obtained from H_0 by a T-transform at $\{v, u_0v_0\}$ and let u' denote the new vertex. By Theorem (1.2)(a), we find a path P_0^* in $(H_0' + uu') - v$ from u to u' such that $|E(P_0^*)| \geq \frac{1}{2}(\frac{|A(H_0)|}{d})^r + 1$; and let $P_0 := P_0^* - u'$ (in this case $u_0v_0 \notin E(P_0)$.

Let $P_0' := P_0$ if $u_0v_0 \notin E(P_0)$; otherwise, let $P_0' := P_0 - u_0$. Then P_0' is a path in $H_0 - \{v, u_0v_0\}$ from u to $\{u_0, v_0\}$ such that $|E(P_0')| \geq \frac{1}{2}(\frac{|A(H_0)|}{d})^r$, and $|E(P_0')| \geq \max\{1, |A(H_0)| - 2\}$ when $H_0 \in B_2(\mathcal{H})$. Without loss of generality, we may assume that P_0' is from u_0 to u.

By applying induction to $\mathcal{H}' := H_1 \ldots H_h$, there is a path P_1 from u_0 to x in $\mathcal{H}' - v_0$ containing no separating edge of $\mathcal{H}'$ such that $|E(P_1)| \geq \frac{1}{2}(\sum_{i=1}^{h}(\frac{|A(H_i)|}{d})^r) + 1 \geq \frac{1}{2}(\frac{\sigma(\mathcal{H}')}{d})^r + 1$ and $|E(P_1)| \geq \frac{1}{2}(\sum\{(\frac{|A(H_i)|}{d})^r : H_i \in B_1(\mathcal{H}) \text{ and } i \neq 0\}) + (\sum\{\max\{1, |A(H_i)| - 2\} : H_i \in B_2(\mathcal{H}) \text{ and } i \neq 0\}) + 1$.

Let $P := P_0' \cup P_1$. Because $h \geq 1$, H_0 or H_1 is not a cycle chain, and hence, $\sigma(\mathcal{H}) \leq |A(H_0)| + \sigma(\mathcal{H}')$. It is easy to see that P satisfies (i) and (ii). Note that the second inequality in (i) follows from the first in (i) by applying Lemma (3.1). □

(3.7) Lemma. *Assume the same hypothesis of Lemma (3.6). Then for any $0 \leq t \leq h$ and for any $pq \in E(H_t)$ such that $|H_t| \leq n - 3$ when $h \geq 1$, there exists a path P in $\mathcal{H}$ from x to $\{p, q\}$ and containing no separating edge of $\mathcal{H}$ such that*

(i) $pq \notin E(P)$, and $|E(P)| \geq \frac{1}{2}|A(H_0)|^r + \frac{1}{2}(\sum\{(\frac{|A(H_i)|}{d})^r : H_i \in B_1(\mathcal{H}) \text{ and } i \neq 0\}) + (\sum\{\max\{1, |A(H_i)| - 2\} : H_i \in B_2(\mathcal{H}) \text{ and } i \neq 0\}) + 1$.

(ii) if we require $uv \in E(P)$, then $pq \notin E(P)$ unless $pq = uv$, and $|E(P)| \geq \frac{1}{2}(\sum\{(\frac{|A(H_i)|}{d})^r : H_i \in B_1(\mathcal{H})\}) + (\sum\{\max\{1, |A(H_i)| - 2\} : H_i \in B_2(\mathcal{H})\}) + 1 \geq \frac{1}{2}(\frac{\sigma(\mathcal{H})}{d})^r + 1$.

Proof. We apply induction on h. Note that the second inequality in (ii) follows from the first in (ii) by applying Lemma (3.1).

Case 1. $h = 0$.

First, assume H_0 is a cycle chain. Then by Proposition (2.6), there is a path P from x to $\{p, q\}$ in H_0 such that $uv \in E(P)$, $pq \notin E(P)$ unless $pq = uv$, and $A(H_0) \subseteq V(P)$. Because $x \notin A(H_0)$, $|E(P)| \geq |A(H_0)|$. Because $x \notin \{u, v\}$, $|E(P)| \geq 2$. So $|E(P)| \geq \max\{1, |A(H_0)| - 2\} + 1$. Moreover, if $|A(H_0)| \leq 3$ then $|E(P)| \geq 2 > \frac{1}{2}|A(H_0)|^r + 1$, and if $|A(H_0)| \geq 4$ then by Lemma (3.4) we have $|E(P)| \geq |A(H_0)| \geq \frac{1}{2}|A(H_0)|^r + 3$. Clearly (i) and (ii) hold.

Now assume $H_0 \cong K_4$. Let P denote a Hamilton path in H_0 from x to $\{p, q\}$ such that $uv \in E(P)$, and $pq \notin E(P)$ unless $pq = uv$. Then $|E(P)| = 3 > \frac{1}{2}|A(H_0)|^r + 1$ and (i) and (ii) hold.

Finally, assume H_0 is 3-connected and $H_0 \not\cong K_4$. Then $5 \le |H_0| < n$. If $x \in \{p, q\}$, then we apply Theorem (1.2)(c) (respectively, Theorem (1.2)(b)) to find a cycle C through pq (respectively, pq and uv) such that $|C| \ge \frac{1}{2}|A(H_0)|^r + 3$ (respectively, $|C| \ge \frac{1}{2}(\frac{|A(H_0)|}{d})^r + 3$). Now it is easy to see that (i) and (ii) hold with $P := C - pq$. So assume $x \notin \{p, q\}$. Then let H_0' be obtained from H_0 by a T-transform at $\{x, pq\}$ and let x' denote the new vertex. By Theorem (1.2)(c) (respectively, Theorem (1.2)(b)), we find a cycle C through xx' (respectively, xx' and uv) such that $|C| \ge \frac{1}{2}|H_0|^r + 3$ (respectively, $|C| \ge \frac{1}{2}(\frac{|H_0|}{d})^r + 3$). Now it is easy to see that (i) and (ii) hold with $P := C - x'$.

Case 2. $h \ge 1$.

Let $\{a, b\} = V(H_0 \cap H_1)$.

Suppose $pq \in \mathcal{H}' := H_1 \ldots H_h$. By applying induction to $\mathcal{H}'$ (with ab playing the role of uv), we find a path P' in $\mathcal{H}'$ from x to $\{p, q\}$ and containing no separating edge of $\mathcal{H}'$ such that $ab \in E(P')$, $pq \notin E(P')$ unless $pq = ab$, and $|E(P')| \ge \frac{1}{2}(\sum\{(\frac{|A(H_i)|}{d})^r : H_i \in B_1(\mathcal{H}) \text{ and } i \neq 0\}) + (\sum\{\max\{1, |A(H_i)| - 2\} : H_i \in B_2(\mathcal{H}) \text{ and } i \neq 0\}) + 1$. If H_0 is a cycle chain or $H_0 \cong K_4$, then H_0 has a Hamilton cycle C through ab and uv. If H_0 is 3-connected and $|H_0| \ge 5$, we apply Theorem (1.2)(c) (respectively, Theorem (1.2)(b)) to find a cycle C through ab (respectively, ab and uv) such that $|C| \ge \frac{1}{2}|H_0|^r + 3$ (respectively, $|C| \ge \frac{1}{2}(\frac{|H_0|}{d})^r + 3$). Then $P := (C - ab) \cup (P' - ab)$ gives the desired path for (i) and (ii).

Therefore, we may assume $pq \in H_0$ and $pq \neq ab$. Let H_0' be obtained from H_0 by an H-transform at $\{pq, ab\}$, and let a', p' denote the new vertices. By Theorem (1.2)(c) (respectively, Theorem (1.2)(b)) we find a cycle C in H_0' through $a'p'$ (respectively, $a'p'$ and uv) such that $|C| \ge \frac{1}{2}|H_0|^r + 3$ (respectively, $|C| \ge \frac{1}{2}(\frac{|H_0|}{d})^r + 3$). Let $P_0 := C - \{a', p'\}$ and, without loss of generality, let a be the end of P_0. By Lemma (3.6), we can find a path P' in $\mathcal{H}' - b$ from x to a and containing no separating edge of $\mathcal{H}'$ such that $|E(P')| \ge \frac{1}{2}(\sum\{(\frac{|A(H_i)|}{d})^r : H_i \in B_1(\mathcal{H}) \text{ and } i \neq 0\}) + (\sum\{\max\{1, |A(H_i)| - 2\} : H_i \in B_2(\mathcal{H}) \text{ and } i \neq 0\}) + 1$. Now $P := P_0 \cup P'$ gives the desired path, except for (ii) when $pq = uv$. In the exceptional case, we may assume $v \notin \{a, b\}$. Let H_0'' be obtained from H_0 by a T-transform at $\{v, ab\}$, with new vertex a''. We apply Theorem (1.2)(a) to find a cycle C in $(H_0'' + ua'') - v$ through ua'' such that $|C| \ge \frac{1}{2}(\frac{|H_0|}{d})^r + 2$. Without loss of generality, we may assume a is the end of $C - a'$. Let P' be found as above. Then $P := ((C - a'') \cup P') + uv$ gives the desired path for (ii). □

4 Cycles through two edges

We reduce Theorem (1.2)(a) and (b) to Theorem (1.2) for smaller graphs. Note that finding a long cycle in Theorem (1.2)(a) through xy avoiding z is equivalent to finding a long cycle through edges xz and yz. First, we reduce Theorem (1.2)(a); our proof implies an $O(E)$ time reduction.

(4.1) Lemma. *Let $n \ge 6$ be an integer, and assume that Theorem (1.2) holds for graphs with at most $n - 1$ vertices. Let G be a 3-connected graph with n vertices, let $xy \in E(G)$ and $z \in V(G) - \{x, y\}$, and let t denote the number of neighbors of z distinct from x and y. Assume $\Delta(G) \le d + 1$, and every vertex of degree $d + 1$ in G (if any) is* *incident with the edge zx or zy. Then there is a cycle C through xy in G − z such that* $|C| \geq \frac{1}{2}(\frac{(d-1)n}{dt})^r + 2$.

Proof. By Lemma (1.3), we may assume $n \geq 4d + 2$. Since G is 3-connected, $t \geq 1$.

Assume that $G - z$ is 3-connected. By assumption, $\Delta(G - z) \leq d$. Since $n \geq 6$, $|G - z| \geq 5$. So by Theorem (1.2)(c), $G - z$ contains a cycle C through xy such that $|C| \geq \frac{1}{2}(n-1)^r + 3$. By Lemma (3.1), $|C| \geq \frac{1}{2}n^r + 2 > \frac{1}{2}(\frac{(d-1)n}{dt})^r + 2$.

Therefore, we may assume that $G - z$ is not 3-connected. By Theorem (2.1), we decompose $G - z$ into 3-connected components. Let $\mathcal{H} := H_1 \dots H_h$ be a block chain in $G - z$ such that (i) H_h contains an extreme 3-block of $G - z$, (ii) $xy \in E(H_1)$ and $\{x, y\} \neq V(H_1) \cap V(H_2)$ when $h \neq 1$, and if $H_1 = C_1 \dots C_k$ is a cycle chain with $k \geq 2$ and $V(H_1 \cap H_2) = V(C_k \cap H_2)$ (when $h \neq 1$) then $xy \in E(C_1)$ and $\{x, y\} \neq V(C_1 \cap C_2)$, and (iii) subject to (i) and (ii), $\sigma(\mathcal{H})$ is maximum.

We claim that $\sigma(\mathcal{H}) \geq \frac{n-1-2t}{t}$. Since G is 3-connected, each extreme 3-block of $G - z$ distinct from H_1 contains a neighbor of z. Therefore, there are at most $2t$ degree 2 vertices in $G - z$ and at most t extreme 3-blocks of $G - z$ different from H_1. Note that the vertices of $G - z$ with degree at least 3 are counted in $\sigma(\mathcal{K})$ for some block chain $\mathcal{K}$ (defined as $\mathcal{H}$ above except condition (iii)). It then follows from (iii) that $\sigma(\mathcal{H}) \geq \frac{n-1-2t}{t}$.

Since $n > 4d + 1$ and $t \leq d$, $\sigma(\mathcal{H}) \geq 2$. By Lemma (3.5), there is a path P from x to y such that $|E(P)| \geq \frac{1}{2}(\frac{(d-1)\sigma(\mathcal{H})}{d})^r + 2$. Let $C^* := P + xy$. Then

$$\begin{aligned} |C^*| &= |E(P)| + 1 \\ &\geq \frac{1}{2}(\frac{(d-1)\sigma(\mathcal{H})}{d} + (b-1))^r + 2 \quad \text{(by Lemma (3.1))} \\ &\geq \frac{1}{2}(\frac{(d-1)(n-1-2t) + dt(b-1)}{dt})^r + 2 \\ &> \frac{1}{2}(\frac{(d-1)n}{dt})^r + 2 \quad (\text{since } b \geq 4d+1). \end{aligned}$$

The desired cycle C can now be obtained from C^* by replacing virtual edges in C^* with appropriate paths in G. □

We now reduce Theorem (1.2)(b); our proof implies an $O(E)$ time reduction.

(4.2) Lemma. *Let $n \geq 6$ be an integer, and assume that Theorem (1.2) holds for graphs with at most $n - 1$ vertices. Suppose G is a 3-connected graph on n vertices and $\Delta(G) \leq d$. Then for any $\{e, f\} \subseteq E(G)$, there is a cycle C through e, f in G such that $|C| \geq \frac{1}{2}(\frac{n}{d})^r + 3$.*

Proof. By Lemma (1.3), we may assume $n \geq 4d+2$. First, assume that e is incident with f. Let $e = xz$ and $f = yz$, and let $G' := G + xy$. Then G' is 3-connected, $\Delta(G') \leq d + 1$, and the possible vertices of degree $d + 1$ in G' are x and y. By applying Lemma (4.1) to G', xy, z, there is a cycle C' through xy in $G' - z$ such that $|C'| \geq \frac{1}{2}(\frac{(d-1)n}{dt})^r + 2$, where t is the number of neighbors of z in G' distinct from x and y. Since $zx, zy \in E(G)$, $t \leq d - 1$. Let $C := (C' - xy) + \{e, f\}$; then $|C| \geq \frac{1}{2}(\frac{(d-1)n}{dt})^r + 3 \geq \frac{1}{2}(\frac{n}{d})^r + 3$. So C gives the desired cycle in G.

Therefore, we may assume that e and f are not incident. Let $e = xy$; then $f \in E(G - y)$. Since G is 3-connected, $G - y$ is 2-connected.

Suppose $G-y$ is 3-connected. Let $y' \neq x$ be a neighbor of y. Then $G' := (G-y)+xy'$ is a 3-connected graph, $\Delta(G') \leq d$, and $5 \leq |G'| < n$. By Theorem (1.2)(b), there is a cycle C' through xy' and f in G' such that $|C'| \geq \frac{1}{2}(\frac{n-1}{d})^r+3$. Let $C := (C'-xy')+\{y, xy, yy'\}$. Then $|C| = |C'| + 1 \geq \frac{1}{2}(\frac{n-1}{d})^r + 4$. By Lemma (3.1), $|C| \geq \frac{1}{2}(\frac{n}{d})^r + 3$. So C gives the desired cycle in G.

Hence, we may assume that $G - y$ is not 3-connected. By Theorem (2.1), we decompose $G - y$ into 3-connected components. Let $\mathcal{H} := H_1 \dots H_h$ be a block chain in $G - y$ such that (a) $f \in E(H_1)$ and $x \in V(H_h)$, (b) if $h = 1$ and $H_1 = C_1 \dots C_k$ is a cycle chain with $k \geq 2$ then $x \in V(C_k) - V(C_{k-1})$, $f \in E(C_1)$, and f is not incident with both vertices in $V(C_1 \cap C_2)$, (c) if $h \geq 2$ then $x \in V(H_h) - V(H_{h-1})$, if $H_h = C_1 \dots C_k$ is a cycle chain with $k \geq 2$ and $V(H_{h-1} \cap H_h) = V(C_1 \cap H_{h-1})$ then $x \in V(C_k) - V(C_{k-1})$, $f \in E(H_1)$, f is not incident with both vertices in $V(H_1 \cap H_2)$, and if $H_1 = C_1 \dots C_k$ is a cycle chain with $k \geq 2$ and $V(H_1 \cap H_2) = V(C_k \cap H_2)$ then $f \in E(C_1)$ and f is not incident with both vertices in $V(C_1 \cap C_2)$. Define $V(H_s \cap H_{s+1}) = \{a_s, b_s\}$ for $1 \leq s \leq h - 1$.

Suppose $V(\mathcal{H}) = V(G - y)$. If $h = 1$ then $G - y$ is a cycle chain, and it is easy to see that G has a Hamilton cycle through e and f, and hence, Theorem (1.2)(b) holds. So assume $h \geq 2$. Let $x' \in V(H_1) - V(H_2)$ so that $yx' \in E(G)$, and in addition, if f has an end with degree 2 in $\mathcal{H}$ then choose x' to be that end (in this case, $yx' \in E(G)$). Let G' be obtained from $G - y$ by adding xx' and then suppressing all degree 2 vertices and deleting separating edges of $\mathcal{H}$. Now G' is 3-connected, $|G'| \geq n - 1 - (d - 2)$ (because degree of y in G is at most d), and $\Delta(G') \leq d$. Therefore, by Theorem (1.2)(b), G' has a cycle C' through f and xx' such that $|C'| \geq \frac{1}{2}(\frac{|G'|}{d})^r + 3$. By replacing edges in $(C' - xx') + \{y, yx, yx'\}$ but not in G with appropriate paths in G, we obtain a cycle C in G through e and f such that $|C| \geq |C'| + 1 \geq \frac{1}{2}(\frac{n-d+1}{d})^r + 4 \geq \frac{1}{2}(\frac{n}{d})^r + 3$, where the final inequality follows from Lemma (3.1). So C is the desired cycle.

We thus may assume that $\mathcal{H} \neq G - y$. Then there is a 2-cut $\{p, q\}$ of $G - y$ such that pq is a virtual edge in H_t for some $1 \leq t \leq h$. Define G_1 as the graph obtained from G by deleting those components of $(G - y) - \{p, q\}$ containing a vertex of $\mathcal{H}$. Note that $G_1 - \{p, q, y\}$ contains a neighbor of y. We choose $\{p, q\}$ so that $|G_1|$ is maximum. Because y has degree at most d in G and $yx \in E(G)$, and since all degree 2 vertices of $G - y$ are neighbors of y, we have (from the choice of G_1),

Observation 1. $|G_1| \geq \frac{n-\sigma(\mathcal{H})}{d-1}$.

If there is a 2-cut $\{v, w\}$ of $G-y$ such that $\{v, w\} \subseteq V(\mathcal{H} \cup G_1)$ and $(G-y) - \{v, w\}$ has a component not containing any vertex of $\mathcal{H} \cup G_1$, then let G_2 denote the graph obtained from G by deleting those components of $(G - y) - \{v, w\}$ containing a vertex of $\mathcal{H} \cup G_1$. If such a 2-cut does not exist, then let $G_2 = \emptyset$. From the definition of G_1, we see that $\{v, w\} \subseteq V(\mathcal{H})$, $\{v, w\} \neq \{p, q\}$, and $V(G_1 \cap G_2) \subseteq \{p, q, y\} \cap \{v, w, y\}$. Choose $\{v, w\}$ so that $|G_2|$ is maximum. By the same reason for Observation 1, we have the following two observations.

Observation 2. If $\sigma(\mathcal{H}) \geq |G_2|$ then $\sigma(\mathcal{H}) \geq \frac{n-|G_1|}{d-1}$.

Observation 3. If $|G_2| \geq \sigma(\mathcal{H})$ then $|G_2| \geq \frac{n-|G_1|}{d-1}$.

Case 1. $\sigma(\mathcal{H}) \geq |G_2|$.

We use $\mathcal{H}$ and G_1 to find the desired cycle. Choose t so that $\{p,q\} \neq \{a_t, b_t\}$. (Note that a_t, b_t are not defined when $t = h$.) Clearly, $|H_t| \leq n-3$ when $h \geq 2$. By Lemma (3.7)(ii), there is a path P from x to $\{p,q\}$ in $\mathcal{H}$ such that $f \in E(P)$, $pq \notin E(P)$ unless $pq = f$, and $|E(P)| \geq \frac{1}{2}(\frac{\sigma(\mathcal{H})}{d})^r + 1$. Assume the notation of $\{p,q\}$ is chosen so that P is from x to p.

Since G is 3-connected, $G_1' := G_1 + \{yp, yq, pq\}$ is 3-connected. If $G_1' \cong K_4$, then we can find a path Q in $G_1' - q$ from p to y such that $|E(Q)| = 2 \geq \frac{1}{2}(\frac{|G_1|}{d})^r + 1$. Now assume that $G_1' \ncong K_4$. Note that $\Delta(G_1') \leq d+1$, and y, p, q are the only possible vertices with degree $d+1$. By Theorem (1.2)(a), there is a cycle C_1 through py in $G_1' - q$ such that $|C_1| \geq \frac{1}{2}(\frac{(d-1)|G_1|}{dt_1})^r + 2$, where $t_1 \leq d-1$ is the number of neighbors of q in G_1' distinct from p and y. Hence, $|C_1| \geq \frac{1}{2}(\frac{|G_1|}{d})^r + 2$.

Let $C^* := (P \cup (C - py)) + xy$. Then C^* is a cycle through e and f and $|C^*| \geq \frac{1}{2}[(\frac{\sigma(\mathcal{H})}{d})^r + (\frac{|G_1|}{d})^r] + 3$. If $\sigma(\mathcal{H}) \leq |G_1|$, then

$$\begin{aligned} |C^*| &\geq \frac{1}{2}(\frac{(b-1)\sigma(\mathcal{H})}{d} + \frac{|G_1|}{d})^r + 3 \quad \text{(by Lemma (3.1))} \\ &> \frac{1}{2}(\frac{n-|G_1|}{d} + \frac{|G_1|}{d})^r + 3 \quad \text{(by Observation 2)} \\ &= \frac{1}{2}(\frac{n}{d})^r + 3. \end{aligned}$$

So we may assume $\sigma(\mathcal{H}) \geq |G_1|$. Then

$$\begin{aligned} |C^*| &\geq \frac{1}{2}(\frac{\sigma(\mathcal{H})}{d} + \frac{(b-1)|G_1|}{d})^r + 3 \quad \text{(by Lemma (3.1))} \\ &> \frac{1}{2}(\frac{\sigma(\mathcal{H})}{d} + \frac{n-\sigma(\mathcal{H})}{d})^r + 3 \quad \text{(by Observation 1)} \\ &= \frac{1}{2}(\frac{n}{d})^r + 3. \end{aligned}$$

The desired cycle C can be obtained from C^* by replacing virtual edges in C^* with appropriate paths in G.

Case 2. $\sigma(\mathcal{H}) < |G_2|$.

Then G_2 is non-empty. We use G_1 and G_2 to find the desired cycle. There exists some $1 \leq u \leq h$ such that $\{v,w\} \subseteq V(H_u)$, and we may choose u so that $\{v,w\} \neq \{a_{u-1}, b_{u-1}\}$. (Note that a_{u-1}, b_{u-1} are not defined when $u = 1$.) We may choose t so that $\{p,q\} \neq \{a_{t-1}, b_{t-1}\}$. Again, a_{t-1}, b_{t-1} are not defined when $t = 1$.

(1) We claim that there is a path P in $\mathcal{H}$ from x to some $z \in \{p,q\} \cup \{v,w\}$ and containing no separating edge of $\mathcal{H}$ such that (i) $f \in E(P)$, (ii) $pq \in E(P)$ or $vw \in E(P)$, (iii) if $pq \in E(P)$ then $z \in \{v,w\}$, and $vw \notin E(P)$ unless $vw = f$, and (iv) if $vw \in E(P)$ then $z \in \{p,q\}$, and $pq \notin E(P)$ unless $pq = f$.

We prove (1) for $t \leq u$; the case $t \geq u$ can be treated in the same way.

First, we define Q. When $t \neq 1$, we find a cycle Q' in $\bigcup_{s=1}^{t-1} H_s$ through $a_{t-1}b_{t-1}$ and f and containing no separating edge of $\mathcal{H}$ (except $a_{t-1}b_{t-1}$). Let $Q := Q' - a_{t-1}b_{t-1}$, which is a path from a_{t-1} to b_{t-1} through f. Let $Q = \emptyset$ when $t = 1$.

Suppose $t < u$. Since removing separating edges of $H_{t+1} \ldots H_s$ different from vw results in a 2-connected graph, we may choose the notation of $\{a_t, b_t\}$ so that

$(\bigcup_{s=t+1}^{h} H_s) - b_t$ contains a path X from a_t to x through vw and containing no separating edge of $\mathcal{H}$ (except possibly vw).

We claim that there is a path C_t in $H_t - a_tb_t$ from a_t to $\{p,q\}$ through $a_{t-1}b_{t-1}$ (or f when $t=1$), or a path C_t' in H_t from a_t to b_t through $a_{t-1}b_{t-1}$ (or f when $t=1$) and pq. If $\{p,q\}=\{a_t,b_t\}$, then the existence of C_t follows from 2-connectivity of H_t. So we may assume that $\{p,q\}\neq\{a_t,b_t\}$. Again by 2-connectivity of H_t there is a cycle D in H_t through pq and $a_{t-1}b_{t-1}$ (or f when $t=1$). If $a_tb_t\in E(D)$ then $C_t' := D - a_tb_t$ is as desired. So we may assume $a_tb_t\notin E(D)$. By 2-connectivity of H_t, there is a path A in H_t from a_t to D and internally disjoint from D. One can easily check that C_t exists in $A\cup D$.

If we find C_t, then let $P_t := C_t - a_{t-1}b_{t-1}$ when $t\neq 1$ and $P_t := C_t$ when $t=1$. In this case, $P := Q\cup P_t\cup X$ gives the desired path for (1). So assume that we find C_t'. Let $P_t := C_t'$ if $t=1$, and otherwise let $P_t := C_t' - a_{t-1}b_{t-1}$. Let $H := H_{t+1}\dots H_h$. If $x\in\{v,w\}$, then we find a cycle C' in H through a_tb_t and vw and containing no separating edge of $\mathcal{H}$ (except a_tb_t and vw), and $P := Q\cup P_t\cup(C'-\{a_tb_t,vw\})$ gives the desired path for (1). Therefore, we may assume $x\notin\{v,w\}$. Let H' be obtained from H by a T-transform at $\{x,vw\}$, let x' denote the new vertex, and let H'' be obtained from H' by deleting all separating edges of $\mathcal{H}$ different from a_tb_t. Then H'' is a 2-connected graph. So there is a cycle C'' in H'' through a_tb_t and xx'. Now $P := Q\cup P_t\cup(C''-\{x',a_tb_t\})$ gives the desired path for (1).

Therefore, we may assume $t=u$. We claim that there is a path Q_t in H_t from $\{a_t,b_t\}$ when $t\neq h$, or from x when $t=h$, to some $z\in\{p,q\}\cup\{v,w\}$ such that (i) $a_{t-1}b_{t-1}\in E(Q_t)$ (or $f\in E(Q_t)$ when $t=1$), (ii) $pq\in E(Q_t)$ or $vw\in E(Q_t)$, (iii) if $pq\in E(Q_t)$ then $z\in\{v,w\}$, and $vw\notin E(Q_t)$ unless $vw=f$, and (iv) if $vw\in E(Q_t)$ then $z\in\{p,q\}$, and $pq\notin E(Q_t)$ unless $pq=f$. This is easy to see if H_t is a cycle chain (because $pq\neq vw$). Otherwise, it follows from Lemma (2.8) or Lemma (2.9) when $f\notin\{pq,vw\}$, and follows from 3-connectivity of H_t when $f\in\{pq,vw\}$.

Assume without loss of generality that a_t is an end of Q_t. When $t\neq h$, we find a path R from a_t to x in $(H_{t+1}\dots H_h)-b_t$ containing no separating edge of $\mathcal{H}$. When $t=h$, let $R=\emptyset$. Let $P_t := Q_t$ when $t=1$, and otherwise let $P_t := Q_t - a_{t-1}b_{t-1}$. Then $P := Q\cup P_t\cup R$ gives the desired path for (1).

We may assume that $vw\in E(P)$ and p is an end of P; since the case $pq\in E(P)$ is similar.

(2) Note that $G_1' := G_1+\{yp,yq,pq\}$ is 3-connected, $\Delta(G_1')\le d+1$, and y,p,q are the possible vertices of degree $d+1$ in G_1'. If $G_1'\cong K_4$, then we can find a path P_1 from p to y in $G_1'-q$ such that $|E(P_1)|=2\ge\frac{1}{2}(\frac{|G_1|}{d})^r+1$. If $G_1'\not\cong K_4$ then by Theorem (1.2)(a), there is a cycle C_1 through py in $G_1'-q$ such that $|C_1|\ge\frac{1}{2}(\frac{(d-1)|G_1|}{dt_1})^r+2$, where $t_1\le d-1$ is the number of neighbors of q in G_1' distinct from p and y. Let $P_1 := C_1-py$; then $|E(P_1)|\ge\frac{1}{2}(\frac{|G_1|}{d})^r+1$.

(3) Note that $G_2' := G_2+\{yv,yw,vw\}$ is 3-connected, $\Delta(G_2')\le d+1$, and y,v,w are the possible vertices of degree $d+1$ in G_2'. If $G_2'\cong K_4$, then we can find a path P_2 from v to w in $G_2'-y$ such that $|E(P_2)|=2\ge\frac{1}{2}(\frac{|G_2|}{d})^r+1$. If $G_2'\not\cong K_4$ then by Theorem (1.2)(a), there is a cycle C_2 through vw in $G_2'-y$ such that $|C_2|\ge\frac{1}{2}(\frac{(d-1)|G_2'|}{dt_2})^r+2$, where $t_2\le d-1$ is the number of neighbors of y in G_2' distinct from v and w. Let $P_2 := C_2-vw$; then $|E(P_2)|\ge\frac{1}{2}(\frac{|G_2|}{d})^r+1$.

Let $C^* := ((P - vw) \cup P_1 \cup P_2) + e$. Then C^* is a cycle through e and f and

$$\begin{aligned}
|C^*| &\geq |E(P_1)| + |E(P_2)| + 1 \\
&\geq \frac{1}{2}(\frac{|G_1|}{d})^r + \frac{1}{2}(\frac{|G_2|}{d})^r + 3 \quad \text{(by (2) and (3))} \\
&\geq \frac{1}{2}(\frac{|G_1|}{d} + \frac{(b-1)|G_2|}{d})^r + 3 \quad \text{(by Lemma (3.1) and since } |G_1| \geq |G_2|) \\
&> \frac{1}{2}(\frac{|G_1|}{d} + \frac{n-|G_1|}{d})^r + 3 \quad \text{(by Observation 3 and since } |G_2| \geq \sigma(\mathcal{H})) \\
&= \frac{1}{2}(\frac{n}{d})^r + 3.
\end{aligned}$$

As before, the desired cycle C can be obtained by modifying C^*. □

5 Cycles through one edge

We now reduce Theorem (1.2)(c); our proof implies an $O(E)$ time reduction. Here we use Lemmas (3.2) and (3.3), and we need $b = \max\{64, 4d+1\}$.

(5.1) Lemma. *Let $n \geq 6$ be an integer, and assume that Theorem (1.2) holds for graphs with at most $n-1$ vertices. Let G be a 3-connected graph with n vertices and $\Delta(G) \leq d$. Then for any $e \in E(G)$, there is a cycle C through e in G such that $|C| \geq \frac{1}{2}n^r + 3$.*

Proof. By Lemma (1.3), we may assume $n > (4d+1)^2$. Let $e = xy \in E(G)$. If $G - y$ is 3-connected, then let y' be a neighbor of y other than x. Clearly, $G' := (G - y) + xy'$ is 3-connected, $\Delta(G') \leq d$, and $5 \leq |G'| < n$. By Theorem (1.2)(c), there is a cycle C' through xy' in G' such that $|C'| \geq \frac{1}{2}(n-1)^r + 3$. Now let $C := (C' - xy') + \{y, xy, yy'\}$. Then C is a cycle through xy in G and, by Lemma (3.1),

$$|C| = |C'| + 1 \geq \frac{1}{2}(n-1)^r + 1 + 3 \geq \frac{1}{2}n^r + 3.$$

Therefore, we may assume that $G - y$ is not 3-connected. Since $G - y$ is 2-connected, we use Theorem (2.1) to decompose $G - y$ into 3-connected components.

Suppose all 3-blocks of $G - y$ are cycles. Let $\mathcal{L} = L_1 \dots L_\ell$ be a cycle chain in $G - y$ such that (i) $x \in V(L_1)$, (ii) L_ℓ is an extreme 3-block of $G - y$, and (iii) subject to (i) and (ii), $|\mathcal{L}|$ is maximum. Because G is 3-connected, each degree 2 vertex in $\mathcal{L}$ is a neighbor of y or is contained in a 3-block of $G - y$ not in $\mathcal{L}$. Hence, it is easy to see that there is some $y' \in V(\mathcal{L}) - \{x\}$ such that $\mathcal{L}$ contains a Hamilton path P from x to y' and G has a path Q from y' to y disjoint from $V(\mathcal{L}) - \{y'\}$. Let $C := (P \cup Q) + \{y, xy, yy'\}$, which is a cycle in G. Then $|C| \geq |\mathcal{L}| + 1$. If $V(G - y) = V(\mathcal{L})$ then $|C| = n \geq \frac{1}{2}n^r + 3$ (since $n \geq 5$ and by Lemma (3.4)). So we may assume $V(G - y) \neq V(\mathcal{L})$. Write $B := L_1$. Because $x \in V(L_1)$ and $xy \in E(G)$, it follows from (iii) that $|\mathcal{L}| \geq \frac{(n-1)-|B|}{t} + |B| = \frac{n+(t-1)|B|-1}{t}$, where t is the number of extreme 3-blocks of $G - y$ distinct from L_1. So $2 \leq t \leq d-1$ (because $V(G - y) \neq V(\mathcal{L})$). Then $|C| \geq |\mathcal{L}| + 1 \geq \frac{n+(t-1)|B|-1}{t} + 1$. Note that $|C| - 3 \geq \frac{n+(t-1)|B|-1}{t} - 2 \geq \frac{n+t-4}{t}$ (since $|B| \geq 3$). Using elementary calculus, we can show that the function $\frac{x+t-4}{t} - \frac{1}{2}x^r$ is increasing when $x \geq (4d+1)^2$. Hence $\frac{n+t-4}{t} \geq \frac{1}{2}n^r$ (because $t \leq d-1$ and $n \geq (4d+1)^2$). Therefore, $|C| \geq \frac{1}{2}n^r + 3$ and C gives the desired cycle in G.

Hence, we may assume that not all 3-blocks of $G-y$ are cycles. We choose a 3-connected 3-block H_0 of $G-y$ with $|H_0|$ maximum. Let $\mathcal{H} = H_0H_1H_2\cdots H_h$ be a block chain in $G-y$ such that either $h=0$ and $x \in V(H_0)$, or $h \geq 1$ and $x \in V(H_h)-V(H_{h-1})$, and if $H_h = C_1 \dots C_k$ is a cycle chain with $k \geq 2$ and $V(H_{h-1} \cap H_h) = V(C_1 \cap C_2)$ then $x \in V(C_k) - V(C_{k-1})$. For $0 \leq i \leq h-1$, let $V(H_i \cap H_{i+1}) = \{a_i, b_i\}$.

If $V(G-y) \neq V(\mathcal{H})$, there is a block chain $\mathcal{L} := L_1L_2\cdots L_\ell$ in $G-y$ such that $V(\mathcal{H} \cap \mathcal{L}) = V(\mathcal{H} \cap L_1)$ consists of two vertices c_0 and d_0, L_ℓ is (or contains) an extreme 3-block of $G-y$, and if $L_1 = C_1 \dots C_k$ is a cycle chain with $k \geq 2$ and $V(L_1 \cap L_2) = V(C_k \cap H_2)$ when $\ell \geq 2$ then $c_0d_0 \in E(C_1)$ and $\{c_0, d_0\} \neq V(C_1 \cap C_2)$. For $1 \leq i \leq \ell - 1$, let $V(L_i \cap L_{i+1}) = \{c_i, d_i\}$. If $\mathcal{L}$ exists, we choose $\mathcal{L}$ so that $\sigma(\mathcal{L})$ is maximum.

(1) We may assume $V(G-y) \neq V(\mathcal{H})$, and $\sigma(\mathcal{L}) + 2 \geq \frac{n-\sigma(\mathcal{H})-1}{d-1}$.

Suppose $V(G-y) = V(\mathcal{H})$. When $h=0$, let x' be a neighbor of y in $H_0 - x$, otherwise, let x' be a neighbor of y in $H_0 - V(H_1)$. Let G' be obtained from $H + xx'$ by suppressing all degree 2 vertices and deleting separating edges of $\mathcal{H}$. Then G' is 3-connected. By Theorem (1.2)(c), there is a cycle C' in G' through xx' such that $|C'| \geq \frac{1}{2}|G'|^r + 3$. Let $C^* := (C' - xx') + \{y, yx, yx'\}$. Since $\Delta(G) \leq d$, $|G'| \geq (n-1)-(d-2)$. Hence, $|C^*| = |C'| + 1 \geq \frac{1}{2}(n-d+1)^r + 1 + 3 > \frac{1}{2}n^r + 3$ (by Lemma (3.1)). Clearly, the desired cycle C can be obtained by modifying C^*.

So we may assume $V(G-y) \neq V(\mathcal{H})$. Note that any vertex of G not contained in any $A(H_i)$, $1 \leq i \leq h$, either is counted in $\sigma(\mathcal{L}') + 2$ for some block chain $\mathcal{L}'$ defined as $\mathcal{L}$ except the maximum requirement (the constant 2 counts the vertices in $V(\mathcal{H} \cap \mathcal{L}')$), or is a degree 2 vertex in $G-y$ (and hence a neighbor of y). Therefore, since $xy \in E(G)$ and $\Delta(G) \leq d$, $\sigma(\mathcal{L}) + 2 \geq \frac{n-\sigma(\mathcal{H})-1}{d-1}$.

(2) There exists a path P in $\mathcal{H}$ from x to $\{c_0, d_0\}$ such that $c_0d_0 \notin E(P)$ and $|E(P)| \geq \frac{1}{2}|H_0|^r + \frac{1}{2}(\sum\{(\frac{|H_i|}{d})^r : i \neq 0 \text{ and } H_i \in B_1(\mathcal{H})\}) + (\sum\{\max\{1, |A(H_i)|-2\} : i \neq 0 \text{ and } H_i \in B_1(\mathcal{H})\}) + 1$. In particular, $|E(P)| \geq \frac{1}{2}(\sigma(\mathcal{H}))^r + 1$.

The first part of (2) follows from Lemma (3.7)(i). The second part of (2) follows from Lemma (3.1). When applying Lemma (3.1), we express $\max\{1, |A(H_i)| - 2\}$ as the sum of 1, and we use $b \geq 4d+1$, $(b-1)(|A(H_i)|-2) \geq |A(H_i)|$ when $|A(H_i)| \geq 3$, and the fact that $|H_0| \geq |H_i|$ for all 3-connected H_i.

(3) We may assume $\sigma(\mathcal{H}) < \frac{n-1}{4}$.

Suppose $\sigma(\mathcal{H}) \geq \frac{n-1}{4}$. Without loss of generality, assume c_0 is an end of the path P in (2). By Lemma (3.6)(i), there is a path Q in $\mathcal{L} - d_0$ from c_0 to some $y' \in N(y) \cap V(L_\ell)$ such that $|E(Q)| \geq \frac{1}{2}(\frac{\sigma(\mathcal{L})}{d})^r + 1$. Let $C^* := (P \cup Q) + \{y, yy', yx\}$. Then

$$|C^*| = |E(P)| + |E(Q)| + 2 \geq \frac{1}{2}(\sigma(\mathcal{H}))^r + 1 + \frac{1}{2}(\frac{\sigma(\mathcal{L})}{d})^r + 3.$$

If $\sigma(\mathcal{H}) \leq \frac{b(b-1)}{4}\frac{\sigma(\mathcal{L})}{d}$, then by Lemmas (3.3) and (3.1),

$$|C^*| \geq \frac{1}{2}(4\sigma(\mathcal{H})+1)^r + 3 \geq \frac{1}{2}n^r + 3.$$

If $\sigma(\mathcal{H}) \geq \frac{b(b-1)}{4}\frac{\sigma(\mathcal{L})}{d}$, then by Lemma (3.2) and since $b \geq 4d+1$,

$$|C^*| \quad > \quad \frac{1}{2}(\sigma(\mathcal{H}) + \frac{b(b-1)}{4}\frac{\sigma(\mathcal{L})}{d} + 2(b-1))^r + 3$$

$$\geq \frac{1}{2}(\sigma(\mathcal{H}) + (4d+1)\sigma(\mathcal{L}) + 8d)^r + 3$$
$$> \frac{1}{2}n^r + 3.$$

The final inequality holds by (1) and $\sigma(\mathcal{H}) < n-1$. Now the desired cycle C can be obtained from C^* by replacing virtual edges in C^* with appropriate paths in G.

(4) We may assume $|H_0|+4(\sigma(\mathcal{H})-|H_0|+\sigma(\mathcal{L})) < n$. In particular, $\sigma(\mathcal{L}) \leq \frac{n-1-|H_0|}{4}$.

Suppose $|H_0| + 4(\sigma(\mathcal{H}) - |H_0| + \sigma(\mathcal{L})) \geq n$. Without loss of generality, assume that the path P in (2) is from x to c_0. By Lemma (3.6)(ii), there is a path Q in $\mathcal{L} - d_0$ from c_0 to some $y' \in N(y) \cap V(L_\ell)$ such that $|E(Q)| \geq \frac{1}{2}(\sum\{(\frac{|A(L_i)|}{d})^r : L_i \in B_1(\mathcal{L})\}) + (\sum\{\max\{1, |A(L_i)|-2\} : L_i \in B_2(\mathcal{L})\}) + 1$.

Let $C^* = (P \cup Q) + \{y, yy', yx\}$. Then by (2) and above, $|C^*| = |E(P)|+|E(Q)|+2 \geq \frac{1}{2}|H_0|^r + \frac{1}{2}(\sum\{(\frac{|A(H_i)|}{d})^r : i \neq 0 \text{ and } H_i \in B_1(\mathcal{H})\}) + (\sum\{\max\{1, |A(H_i)|-2\} : i \neq 0 \text{ and } H_i \in B_2(\mathcal{H})\}) + \frac{1}{2}(\sum\{(\frac{|A(L_i)|}{d})^r : L_i \in B_1(\mathcal{L})\}) + (\sum\{\max\{1, |A(L_i)|-2\} : L_i \in B_2(\mathcal{L})\}) + 4$. Because $|H_0|$ is maximum among all 3-connected 3-blocks of $G - y$, it follows from Lemma (3.1) and the fact $b \geq 4d+1$ that

$$\begin{aligned} |C^*| &\geq \frac{1}{2}[|H_0| + 4(\sum_{i=1}^{h} |A(H_i)| + \sum_{j=1}^{\ell} |A(L_j)|)]^r + 4 \\ &= \frac{1}{2}[|H_0| + 4(\sigma(\mathcal{H}) - |H_0| + \sigma(\mathcal{L}))]^r + 4 \\ &> \frac{1}{2}n^r + 3. \end{aligned}$$

As before, the desired cycle C can be obtained by modifying C^*. This proves (4).

We need to consider block chains other than $\mathcal{H}$ and $\mathcal{L}$. A block chain $\mathcal{M} := M_1 M_2 \cdots M_m$ is called an $\mathcal{HL}$-*leg* if M_m contains an extreme 3-block of $G - y$ and $V(\mathcal{M} \cap (\mathcal{H} \cup \mathcal{L}))$ consists of two vertices x_0 and y_0 such that $\{x_0, y_0\} \subseteq V(M_1)$ and $\{x_0, y_0\} \neq V(M_1 \cap M_2)$ when $m \geq 2$, and if $M_1 = C_1 \dots C_k$ is a cycle chain with $k \geq 2$ and $V(C_k \cap M_2) = V(M_1 \cap M_2)$ when $m \geq 2$ then $\{x_0, y_0\} \subseteq V(C_1)$ and $\{x_0, y_0\} \neq V(C_1 \cap C_2)$. We view degree 2 vertices of $G - y$ (which are neighbors of y) as trivial $\mathcal{HL}$-legs.

(5) We may assume that there is an $\mathcal{HL}$-leg $\mathcal{M}$ such that $\sigma(\mathcal{M}) > \frac{n}{4(d-2)} > 4d+2$.

Note that each extreme 3-block of $G - y$ contains a neighbor of y. Since $\Delta(G) \leq d$, there are at most $d-2$ $\mathcal{HL}$-legs in $G - y$ (including those trivial ones). Choose an $\mathcal{HL}$-leg $\mathcal{M}$ such that $\sigma(\mathcal{M})$ is maximum. Note that every vertex of $G - y$ either is a degree 2 vertex (hence covered in a trivial $\mathcal{HL}$-leg), or is counted in $\sigma(\mathcal{H})$, or in $\sigma(\mathcal{L}) + 2$, or in $\sigma(\mathcal{M}) + 2$ for some $\mathcal{HL}$-leg $\mathcal{M}$. Hence, because $\sigma(\mathcal{H}) < \frac{n-1}{4}$ (by (3)) and $\sigma(\mathcal{L}) \leq \frac{n-1-|H_0|}{4} \leq \frac{n-5}{4}$ (by (4) and $|H_0| \geq 4$), $\sigma(\mathcal{M}) + 2 \geq \frac{n-1-\sigma(\mathcal{H})-\sigma(\mathcal{L})-2}{d-2} > \frac{n-3}{2(d-2)}$. Since we assume $n > (4d+1)^2$, $\sigma(\mathcal{M}) > \frac{n}{4(d-2)} > 4d+2$.

Let $\mathcal{M}$ be an $\mathcal{HL}$-leg in $G - y$ with $\sigma(\mathcal{M}) \geq \frac{n}{4(d-2)}$. By (5), $\mathcal{M}$ is nontrivial. Let x_0 and y_0 be the vertices in $V(\mathcal{M} \cap (\mathcal{H} \cup \mathcal{L}))$. We consider three cases.

Case 1. $\mathcal{M}$ may be chosen so that $x \notin \{x_0, y_0\} \cap \{c_0, d_0\}$ and $\{x_0, y_0\} \not\subseteq V(\mathcal{H})$.

Then we may assume $\{x_0, y_0\} \subseteq V(L_t)$ with $\{x_0, y_0\} \neq \{c_{t-1}, d_{t-1}\}$.

We claim that there is a path P' in $\mathcal{H}$ from x to $z \in \{c_0, d_0\}$ such that (i) $|E(P')| \geq \frac{1}{2}(\frac{|H_0|}{d})^r + 1$, (ii) $c_0d_0 \notin E(P')$, and (iii) if $z \notin \{c_0, d_0\} \cap \{x_0, y_0\}$ then $\{c_0, d_0\} \cap \{x_0, y_0\} = \emptyset$ or $\{c_0, d_0\} \cap \{x_0, y_0\} \not\subseteq V(P')$. Choose $z' \in \{c_0, d_0\}$ such that, if possible, $z' \in \{c_0, d_0\} \cap \{x_0, y_0\}$. Suppose $c_0d_0 \in E(H_1 \dots H_h)$. Since deleting separating edges of $H_1 \dots H_h$ results in a 2-connected graph, which contains disjoint paths Q_1, Q_2 from x, z' to a_0, b_0, respectively. In H_0 we use Theorem (1.2)(c) to find a cycle C_0 through a_0b_0 such that $|C_0| \geq \frac{1}{2}|H_0|^r + 3$. If $c_0d_0 \in E(Q_2)$ then $P' := (C_0 - a_0b_0) \cup Q_1 \cup (Q_2 - z')$ gives the desired path; otherwise, $P' := (C_0 - a_0b_0) \cup Q_1 \cup Q_2$ gives the desired path. So we may assume $c_0d_0 \notin E(H_1 \dots H_h)$. Suppose $h = 0$. We apply Theorem (1.2)(a) to find a cycle C_0 in $(H_0 + \{xc_0, xd_0\}) - (\{c_0, d_0\} - \{z'\})$ through xz' such that $|C_0| \geq \frac{1}{2}(\frac{|H_0|}{d})^r + 2$; then $P' := C_0 - xz'$ gives the desired path. So let $h \geq 1$. Then $c_0d_0 \in E(H_0)$ and $\{c_0, d_0\} \neq \{a_0, b_0\}$. Without loss of generality, we may assume $a_0 \notin \{c_0, d_0\}$. Let Q' be a path in $(H_1 \dots H_h) - b_0$ from x to a_0 and not containing any separating edge of $\mathcal{H}$. If $z' = b_0$ we use Theorem (1.2)(c) to find a cycle C_0 through a_0b_0 in H_0 such that $|C_0| \geq \frac{1}{2}|H_0|^r + 3$, and $P' := (C_0 - a_0b_0) \cup Q'$ (when $c_0d_0 \notin E(C_0)$) or $P' := (C_0 - b_0) \cup Q'$ (when $c_0d_0 \in E(C_0)$) gives the desired path. So assume $z' \neq b_0$. If $z' \in \{c_0, d_0\} \cap \{x_0, y_0\}$, then z' has at most $d - 1$ neighbors in H_0 (since $\{x_0, y_0\} \neq \{c_0, d_0\}$), and in $(H_0 + \{a_0z', z'b_0\}) - b_0$ we apply Theorem (1.2)(a) to find a cycle C_0 through a_0z' such that $|C_0| \geq \frac{1}{2}(\frac{|H_0|}{d})^r + 2$; then $P' := (C_0 - a_0z') \cup Q'$ gives the desired path. So we may assume $z' \notin \{c_0, d_0\} \cap \{x_0, y_0\}$. By the choice of z', $\{c_0, d_0\} \cap \{x_0, y_0\} = \emptyset$ (so (iii) is automatic). Let H_0' be obtained from H_0 by an H-transform at $\{a_0b_0, c_0d_0\}$, and let a', x' denote the new vertices. By applying Theorem (1.2)(c) we find a cycle C_0 through $a'x'$ in H_0' such that $|C_0| \geq \frac{1}{2}|H_0|^r + 3$. Now $C_0 - \{a', x'\}$ is a path from some $z \in \{c_0, d_0\}$ to some $b' \in \{a_0, b_0\}$. Then $C_0 - \{a', x'\}$ and a path in $(H_1 \dots H_h) - (\{a_0, b_0\} - \{b'\})$ from x to b' (not containing any separating edge of $\mathcal{H}$) gives the desired path P'.

Without loss of generality, we may assume that P' is from x to c_0. Then $d_0 \notin V(P')$ or $d_0 \notin \{x_0, y_0\}$. Therefore, since each L_i is 3-connected or is a cycle chain, there exists a path Q in $\bigcup_{i=0}^t L_i$ from c_0 to some $z \in \{c_t, d_t\} \cup \{x_0, y_0\}$ such that (i) Q contains no separating edge of $\mathcal{L}$ except possibly c_td_t and x_0y_0, (ii) Q avoids d_0 if $d_0 \in V(P')$ (since in that case $\{x_0, y_0\} \cap \{c_0, d_0\} = \emptyset$), (iii) if $z \in \{c_t, d_t\}$ then $x_0y_0 \in E(Q)$, and $c_td_t \notin E(Q)$ unless $x_0y_0 = c_td_t$, and (iv) if $z \in \{x_0, y_0\}$ then $c_td_t \in E(Q)$, and $x_0y_0 \notin E(Q)$ unless $x_0y_0 = c_td_t$.

Suppose $z \in \{c_t, d_t\}$, and assume the notation is chosen so that $z = c_t$. By Lemma (3.6)(ii) there is a path P_1 in $(L_{t+1} \dots L_\ell) - d_t$ from z to some $y' \in N(y) \cap V(L_\ell)$ and containing no separating edge of $\mathcal{H}$ such that $|E(P_1)| \geq \frac{1}{2}(\sum\{(\frac{|A(L_i)|}{d})^r : t+1 \leq i \leq \ell$ and $L_i \in B_1(\mathcal{L})\}) + (\sum\{\max\{1, |A(L_i)| - 2\} : t+1 \leq i \leq \ell$ and $L_i \in B_2(\mathcal{L})\}) + 1$. By Lemma (3.5), let P_2 be a path from x_0 to y_0 in $\mathcal{M}$ such that $|E(P_2)| \geq \frac{1}{2}(\frac{(d-1)\sigma(\mathcal{M})}{d})^r + 1$. Let $n^* := \sum_{i=t+1}^{\ell} |A(L_i)|$; then by the choice of $\mathcal{L}$, $n^* \geq \sigma(\mathcal{M}) - 2$. Let $C^* := (P' \cup (Q - x_0y_0) \cup P_1 \cup P_2) + \{y, yy', yx\}$. As in the proof of (4),

$$\begin{aligned}
|C^*| &\geq |E(P')| + |E(P_1)| + |E(P_2)| + 2 \\
&\geq \frac{1}{2}[(\frac{|H_0|}{d})^r + 2 + \sum(\frac{|A(L_i)|}{d})^r + sum \max\{1, |A(L_i)| - 2\} + (\frac{(d-1)\sigma(\mathcal{M})}{d})^r] + 4 \\
&> \frac{1}{2}[(2 + (b-1)n^*/d)^r + ((d-1)\sigma(\mathcal{M})/d)^r] + 4 \quad \text{(by Lemma (3.1))} \\
&> \frac{1}{2}[2 + n^* + (b-1)(d-1)\sigma(\mathcal{M})/d]^r + 4 \quad \text{(by Lemma (3.1))}
\end{aligned}$$

$$
\begin{aligned}
&> \frac{1}{2}(4(d-1)\sigma(\mathcal{M}))^r + 3 \quad (\text{since } b \geq 4d+1) \\
&> \frac{1}{2}n^r + 3 \quad (\text{by } (5)).
\end{aligned}
$$

As before, the desired cycle C may be obtained by modifying C^*.

Now assume $z \in \{x_0, y_0\}$, and that the notation is chosen so that $z = x_0$. By Lemma (3.6)(ii), there is a path P_2 in $\mathcal{M} - y_0$ from x_0 to some $y'' \in N(y) \cap V(M_m)$ and containing no separating edge of $\mathcal{M}$ such that $|E(P_2)| \geq \frac{1}{2}(\sum\{(\frac{|A(M_i)|}{d})^r : M_i \in B_1(\mathcal{M})\}) + (\sum\{\max\{1, |A(M_i)| - 2\} : M_i \in B_2(\mathcal{M})\}) + 1$. By Lemma (3.5) there is a path P_1 in $L_{t+1} \dots L_\ell$ from c_t to d_t such that $|E(P_1)| \geq \frac{1}{2}((d-1)n^*/d)^r$. Let $C^* := (P' \cup (Q - c_t d_t) \cup P_1 \cup P_2) + \{y, yx, yy''\}$. Then by applying Lemma (3.1) as in the above paragraph (by swapping the roles of L_i and M_i), we have

$$
|C^*| \geq \frac{1}{2}[(2 + \sigma(\mathcal{M}))^r + ((d-1)n^*/d)^r] + 4.
$$

If $(d-1)n^*/d \leq 2 + \sigma(\mathcal{M})$, then by Lemma (3.1) and because $n^* \geq \sigma(\mathcal{M}) - 2$,

$$
\begin{aligned}
|C^*| &\geq \frac{1}{2}(2 + \sigma(\mathcal{M}) + (b-1)(d-1)n^*/d + 2(b-1))^r + 3 \\
&> \frac{1}{2}(4(d-1)\sigma(\mathcal{M}))^r + 3 \quad (\text{since } b \geq 4d+1) \\
&> \frac{1}{2}n^r + 3 \quad (\text{by } (5)).
\end{aligned}
$$

So assume $(d-1)n^*/d \geq 2 + \sigma(\mathcal{M})$. Applying Lemma (3.1) and (5) again, we have

$$
|C^*| \geq \frac{1}{2}((d-1)n^*/d + (b-1)(2 + \sigma(\mathcal{M})))^r + 4 > \frac{1}{2}(4d\sigma(\mathcal{M}))^r + 3 > \frac{1}{2}n^r + 3.
$$

As before, the desired cycle C can be obtained by modifying C^*.

Case 2. $\mathcal{M}$ may be chosen so that $x \notin \{x_0, y_0\} \cap \{c_0, d_0\}$, $\{c_0, d_0\} \neq \{x_0, y_0\}$, and $\{x_0, y_0\} \subseteq V(\mathcal{H})$.

We may assume that $\{c_0, d_0\} \subseteq V(H_s)$ and $\{c_0, d_0\} \neq \{a_{s-1}, b_{s-1}\}$, and $\{x_0, y_0\} \subseteq V(H_t)$ and $\{x_0, y_0\} \neq \{a_{t-1}, b_{t-1}\}$. Note that a_{-1} and b_{-1} are not defined. We only consider the case $s \leq t$; since the case $s \geq t$ is similar. By the choice of $\mathcal{L}$ and by (5), $\sigma(\mathcal{L}) \geq \sigma(\mathcal{M}) \geq \frac{n}{4(d-2)}$.

We claim that there is a path P_0 in $\mathcal{H}$ from x to some $z \in \{c_0, d_0\} \cup \{x_0, y_0\}$ and containing no separating edge of $\mathcal{H}$ (except possibly $c_0 d_0$ or $x_0 y_0$) such that (a) $|E(P_0)| \geq \frac{1}{2}(|H_0|/d)^r + 1$, (b) $c_0 d_0 \in E(P_0)$ or $x_0 y_0 \in E(P_0)$, (c) if $c_0 d_0 \in E(P_0)$ then $z \in \{x_0, y_0\}$ and $x_0 y_0 \notin E(P_0)$, and (d) if $x_0 y_0 \in E(P_0)$ then $z \in \{c_0, d_0\}$ and $c_0 d_0 \notin E(P_0)$.

Suppose $h = 0$. We may assume $x \notin \{x_0, y_0\}$ (the case $x \notin \{c_0, d_0\}$ is symmetric). Let H_0' be obtained from H_0 by a T-transform at $\{x, x_0 y_0\}$, and let x' denote the new vertex. By Theorem (1.2)(b) we find a cycle C_0 in H_0' through $c_0 d_0$ and xx' such that $|C_0| \geq \frac{1}{2}(|H_0|/d)^r + 3$. Now $P_0 := C_0 - x'$ gives the desired path.

So we may assume $h \geq 1$. Let H' be obtained from $H_1 \dots H_h$ by deleting all separating edges of $\mathcal{H}$ different from $a_0 b_0, c_0 d_0$ and $x_0 y_0$. Note that H' is 2-connected.

Assume $s = t = 0$. Since $c_0 d_0 \neq x_0 y_0$, we may assume $x_0 y_0 \neq a_0 b_0$ (the case $c_0 d_0 \neq a_0 b_0$ is the same). Suppose $c_0 d_0 = a_0 b_0$. By Theorem (1.2)(b) we find a cycle

C_0 in H_0 through a_0b_0 and x_0y_0 such that $|C_0| \geq \frac{1}{2}(|H_0|/d)^r + 3$. In $H' - b_0$ we find a path P' from x to a_0. Then $P_0 := (C_0 - a_0b_0) \cup P'$ gives the desired path. So assume $c_0d_0 \neq a_0b_0$. Let H_0' be obtained from H_0 by an H-transform at $\{x_0y_0, a_0b_0\}$, and let x', a' denote the new vertices with a' subdividing a_0b_0. By Theorem (1.2)(b), there is a cycle C_0 in H_0' through c_0d_0 and $a'x'$ such that $|C_0| \geq \frac{1}{2}(|H_0|/d)^r + 3$. Let P' be a path in $H' - a_0b_0$ from x to the end, say v, of $C_0 - \{a', x'\}$ adjacent to a' and avoiding $\{a_0, b_0\} - \{v\}$. Then $P_0 := (C_0 - \{a', x'\}) \cup P'$ gives the desired path.

Now assume $s = 0 < t$. Then $x_0y_0 \neq a_0b_0$. By 2-connectivity of H', P' be a path in $H' - a_0b_0$ from x to $z \in \{a_0, b_0\}$ through x_0y_0. By choosing appropriate notation, we may let $z = a_0$. Suppose $a_0b_0 = c_0d_0$. By Theorem (1.2)(c), we find a cycle C_0 in H_0 through a_0b_0 such that $|C_0| \geq \frac{1}{2}|H_0|^r + 3$. Now $P_0 := (C_0 - a_0b_0) \cup P'$ gives the desired path. So we may assume $a_0b_0 \neq c_0d_0$, and let $c_0 \notin \{a_0, b_0\}$ (by choosing appropriate notation). If $d_0 \in \{a_0, b_0\}$ then let $z' \in \{a_0, b_0\} - \{d_0\}$, and apply Theorem (1.2)(a) to find cycle C_0 in $(H_0 + z'c_0) - b_0$ through a_0c_0 such that $|C_0| \geq \frac{1}{2}(|H_0|/d)^r + 2$; then $P_0 := (C_0 - a_0c_0) \cup P'$ gives the desired path. Now assume $d_0 \notin \{a_0, b_0\}$. Let H_0' be obtained from H_0 by a T-transform at $\{a_0, c_0d_0\}$, and let c' denote the new vertex. We apply Theorem (1.2)(a) to find a cycle C_0 in $(H_0' + b_0c') - b_0$ through a_0c' such that $|C_0| \geq \frac{1}{2}(|H_0|/d)^r + 2$. Now $P_0 := (C_0 - c') \cup P'$ gives the desired path.

Finally, we may assume $s \geq 1$. By exactly the same argument as for (1) of Case 2 in the proof of Lemma (4.2), with a_0b_0, c_0d_0, x_0y_0 playing the roles of f, pq, vw, respectively, we find a path P' through a_0b_0 in H' from x to $z \in \{c_0, d_0\} \cup \{x_0, y_0\}$ such that P' satisfies (b), (c) and (d). By Theorem (1.2)(c), we find a cycle C_0 in H_0 through a_0b_0 such that $|C_0| \geq \frac{1}{2}|H_0|^r + 3$. Now $P_0 := (C_0 - a_0b_0) \cup (P' - a_0b_0)$ gives the desired path.

Suppose $x_0y_0 \in E(P_0)$. Without loss of generality, assume $z = c_0$. By Lemma (3.6)(ii) there is a path P_1 in $\mathcal{L} - d_0$ from c_0 to some $y' \in N(y) \cap V(L_\ell)$ and containing no separating edge of $\mathcal{L}$ such that $|E(P_1)| \geq \frac{1}{2}(\sum\{(|A(L_i)|/d)^r : L_i \in B_1(\mathcal{L})\}) + (\sum\{\max\{1, |A(L_i)| - 2\} : L_i \in B_2(\mathcal{L})\}) + 1$. By Lemma (3.5), there is a path P_2 from x_0 to y_0 in $\mathcal{M}$ such that $|E(P_2)| \geq \frac{1}{2}((d-1)\sigma(\mathcal{M})/d)^r + 2$. Let C^* be the cycle obtained from $(P_0 \cup P_1) + \{y, yy', yx\}$ by replacing x_0y_0 with P_2. Then, as in Case 1 (with n^* playing the role of $\sigma(\mathcal{M})$), by Lemma (3.1) and since $\sigma(\mathcal{L}) \geq \sigma(\mathcal{M})$, we have

$$
\begin{aligned}
|C^*| &\geq |E(P_0)| + |E(P_1)| + |E(P_2)| + 1 \\
&> \frac{1}{2}[(2 + \sigma(\mathcal{L}))^r + ((d-1)\sigma(\mathcal{M})/d)^r] + 4 \\
&> \frac{1}{2}[(b-1)(d-1)\sigma(\mathcal{M})/d]^r + 3 \\
&> \frac{1}{2}n^r + 3 \quad \text{(by (5))}.
\end{aligned}
$$

Now assume $c_0d_0 \in E(P_0)$. Without loss of generality, assume $z = x_0$. By Lemma (3.6)(ii), there is a path P_1 from x_0 to some $y' \in N(y) \cap V(M_m)$ in $\mathcal{M} - y_0$ such that $|E(P_1)| \geq \frac{1}{2}(\sum\{(|A(M_i)|/d)^r : M_i \in B_1(\mathcal{M})\}) + (\sum\{\max\{1, |A(M_i)| - 2\} : M_i \in B_2(\mathcal{M})\}) + 1$. By Lemma (3.5), there is a path P_2 from c_0 to d_0 in $\mathcal{L}$ such that $|E(P_2)| \geq \frac{1}{2}(\frac{(d-1)\sigma(\mathcal{L})}{d})^r + 2$. Let C^* be the cycle obtained from $(P_0 \cup P_1) + \{y, yy', yx\}$ by replacing c_0d_0 with P_2. Then as in Case 1 and by Lemma (3.1),

$$
\begin{aligned}
|C^*| &\geq |E(P_0)| + |E(P_1)| + |E(P_2)| + 1 \\
&\geq \frac{1}{2}[(2 + \sigma(\mathcal{M}))^r + ((d-1)\sigma(\mathcal{L})/d)^r] + 4.
\end{aligned}
$$

If $(d-1)\sigma(\mathcal{L})/d \geq 2+\sigma(\mathcal{M})$, then by Lemma (3.1) and by (5),

$$|C^*| > \frac{1}{2}((b-1)\sigma(\mathcal{M}))^r + 4 > \frac{1}{2}(4d\sigma(\mathcal{M}))^r + 3 > \frac{1}{2}n^r + 3.$$

Now assume $(d-1)\sigma(\mathcal{L})/d \leq 2+\sigma(\mathcal{M})$. Then by Lemma (3.1) and (5) and because $\sigma(\mathcal{L}) \geq \sigma(\mathcal{M})$,

$$|C^*| > \frac{1}{2}((b-1)(d-1)\sigma(\mathcal{L})/d)^r + 4 > \frac{1}{2}(4(d-1)\sigma(\mathcal{M}))^r + 3 > \frac{1}{2}n^r + 3.$$

As before, the desired cycle C can be obtained by modifying C^*.

Case 3. For every choice of $\mathcal{M}$ with $\sigma(\mathcal{M}) \geq \frac{n}{4(d-2)}$, we have $x \in \{c_0, d_0\} \cap \{x_0, y_0\}$ or $\{c_0, d_0\} = \{x_0, y_0\}$.

Let $\mathcal{M}_i$, $1 \leq i \leq k$, denote the $\mathcal{HL}$-legs with $\sigma(\mathcal{M}_i) \geq \frac{n}{4(d-2)}$. Since $\sum\{\sigma(\mathcal{M}) : \mathcal{M}$ is an $\mathcal{HL}$-leg as in Case 1 or Case 2$\} \leq \frac{(d-2-k)n}{4(d-2)}$ and because $n > (4d+1)^2$, it follows from (3) and (4) that

$$\sum_{i=1}^{k} \sigma(\mathcal{M}_i) \geq (n-1) - \frac{n-1}{4} - \frac{n-1}{4} - \frac{(d-2-k)n}{4(d-2)} - 2d > \frac{n}{4}.$$

Let G_i denote the graph obtained from G by deleting all components of $(G-y) - V(\mathcal{M}_i \cap (\mathcal{H} \cup \mathcal{L}))$ not containing any vertex of $\mathcal{M}_i$. Let $z \in \{c_0, d_0\}$ such that $z = x$ if $x \in \{c_0, d_0\}$. Since we are in Case 3, $z \in V(\mathcal{M}_i)$ for $1 \leq i \leq k$. Let t_i be the number of neighbors of z in G_i different from y and not in $V(\mathcal{M}_i \cap (\mathcal{H} \cup \mathcal{L}))$. Then $t_i \geq 1$. We claim that $\sum_{i=1} t_i \leq d-1$. This is clear when $z = x$ because yx is an edge of G. Now suppose $z \neq x$. Then $\{c_0, d_0\} \subseteq V(\mathcal{M}_i)$ for all $1 \leq i \leq k$. Since z is incident with edges in both $\mathcal{H} - c_0d_0$ and $\mathcal{L} - c_0d_0$, we have $\sum_{i=1}^{k} t_i \leq d-1$.

Let $1 \leq s \leq k$ such that $\frac{|G_s|}{t_s}$ is maximum. Then $\frac{|G_s|}{t_s} \geq \frac{n}{4(d-1)}$. This follows from the following result (which can be proved by induction on k): If $\alpha_1 + \ldots + \alpha_k \geq \alpha$ and $t_1 + \ldots + t_k = m$, then $\max\{\frac{\alpha_i}{t_i} : 1 \leq i \leq k\} \geq \frac{\alpha}{m}$.

For convenience, let $\{x_s, y_s\} = V(\mathcal{M}_s \cap (\mathcal{H} \cup \mathcal{L}))$, and assume, without loss of generality, $z = x_s = c_0$. Note that $G_s^* := G_s + \{yx_s, yy_s, x_sy_s\}$ is 3-connected, $\Delta(G_s^*) \leq d+1$, and any vertex of degree $d+1$ must be incident with x_sy or x_sy_s. By Theorem (1.2)(a), there is a path Q_2 from y_s to y in $G_s^* - x_s$ such that

$$|E(Q_2)| \geq \frac{1}{2}(\frac{(d-1)|G_s|}{dt_s})^r + 1 \geq \frac{1}{2}(\frac{n}{4d})^r + 1.$$

Suppose $y_s \in V(\mathcal{H})$. By 2-connectivity of $\mathcal{H}$, there is a path Q_0 from x to y_s in $\mathcal{H}$ through c_0d_0. By Lemma (3.5), there is a path Q_1 from c_0 to d_0 in $\mathcal{L}$ such that $|E(Q_1)| \geq \frac{1}{2}(\frac{(d-1)\sigma(\mathcal{L})}{d})^r + 2$. Let $C^* := ((Q_0 - c_0d_0) \cup Q_1 \cup Q_2) + yx$. Then

$$|C^*| \geq |E(Q_1)| + |E(Q_2)| \geq \frac{1}{2}[(\frac{(d-1)\sigma(\mathcal{L})}{d})^r + (\frac{n}{4d})^r] + 3.$$

If $\frac{(d-1)\sigma(\mathcal{L})}{d} \geq \frac{n}{4d}$ then by Lemma (3.1) and since $b \geq 4d+1$,

$$|C^*| \geq \frac{1}{2}(\frac{(b-1)n}{4d})^r + 3 \geq \frac{1}{2}n^r + 3.$$

Now assume $\frac{(d-1)\sigma(\mathcal{L})}{d} \leq \frac{n}{4d}$. By Lemma (3.1) and since $\sigma(\mathcal{L}) \geq \sigma(\mathcal{M}) > \frac{n}{4(d-2)}$ (by (5)),

$$|C^*| \geq \frac{1}{2}[\frac{(b-1)(d-1)\sigma(\mathcal{L})}{d}]^r + 3 > \frac{1}{2}n^r + 3.$$

As before, the desired cycle C can be obtained by modifying C^*.

Thus, we may assume $y_s \notin V(\mathcal{H})$. Then $z = x$ and $y_s \in V(L_t)$ for some $1 \leq t < \ell$ ($t \neq \ell$ by the choice of $\mathcal{L}$). Let $n^* := \sum_{i=t+1}^{\ell} |A(L_i)|$. Note that $n^* \leq \sigma(L_{t+1} \dots L_\ell)$. By our choice of $\mathcal{L}$, $n^* \geq \sigma(\mathcal{M}) - 2$. By 2-connectivity, let Q_0 be a path from x to y_s through $c_t d_t$ in $L_1 \dots L_t$. Note, $|E(Q_0)| \geq 2$. By Lemma (3.5) there is a path Q_1 from c_t to d_t in $L_{t+1} \dots L_\ell$ such that $|E(Q_1)| \geq \frac{1}{2}(\frac{(d-1)n^*}{d})^r + 1$. Let $C^* := ((Q_0 - c_t d_t) \cup Q_1 \cup Q_2) + yx$. Then

$$|C^*| \geq |E(Q_1)| + |E(Q_2)| + 2 \geq \frac{1}{2}[(\frac{(d-1)n^*}{d})^r + 2 + (\frac{n}{4d})^r] + 3.$$

If $\frac{(d-1)n^*}{d} \geq \frac{n}{4d}$ then by Lemma (3.1) and since $b \geq 4d + 1$,

$$|C^*| \geq \frac{1}{2}(\frac{(b-1)n}{4d})^r + 3 \geq \frac{1}{2}n^r + 3.$$

Now assume $\frac{(d-1)n^*}{d} \leq \frac{n}{4d}$. Then by Lemma (3.1) and since $n^* \geq \sigma(\mathcal{M}) - 2 > \frac{n}{4(d-1)} - 2$ (by (5)),

$$|C^*| > \frac{1}{2}(\frac{(b-1)(d-1)n^*}{d} + 2(b-1))^r + 3 > \frac{1}{2}[4(d-1)n^* + 2(b-1)]^r + 3 > \frac{1}{2}n^r + 3.$$

Again, the desired cycle C can be obtained by modifying C^*. □

6 Conclusions

We now complete the proof of Theorem (1.2). Let n, d, r, G be given as in Theorem (1.2). We apply induction on n. When $n = 5$, G is isomorphic to one of the following three graphs: K_5, K_5 minus an edge, or the wheel on five vertices. In each case, we can verify that Theorem (1.2) holds. So assume that $n \geq 6$ and Theorem (1.2) holds for all 3-connected graphs with at most $n-1$ vertices. Then Theorem (1.2)(a) holds by Lemma (4.1), Theorem (1.2)(b) holds by Lemma (4.2), and Theorem (1.2)(c) holds by Lemma (5.1). This completes the proof of Theorem (1.2). □

Our proof of Theorem (1.2) implies a polynomial time algorithm which, given a 3-connected n-vertex graph, finds a cycle of length $\frac{1}{2}n^r + 3$. When combined with the next two results [8], our proof implies a cubic algorithm.

(6.1) Lemma. *Let G be a k-connected graph, where k is a positive integer. Then G contains a k-connected spanning subgraph with $O(V)$ edges, and such a subgraph can be found in $O(E)$ time.*

(6.2) Lemma. *Let G be a 2-connected graph and let $e, f \in E(G)$. Then there is a cycle through e and f in G, and such a cycle can be found in $O(V)$ time.*

Lemma (6.2) is actually an easy consequence of a result in [8], which states that, in a 2-connected graph G, one can find, in $O(V)$ time, two disjoint paths linking two given vertices. Our algorithm is similar to that in [3]. Therefore, we only give an outline and omit complexity analysis.

Algorithm: Let G be a 3-connected graph with $\Delta(G) \leq d$, and assume $|G| \geq 5$. The following procedure finds a cycle C in G with $|C| \geq \frac{1}{2}|G|^r + 3$.

1. **Preprocessing** Replace G with a 3-connected spanning subgraph of G with $O(|G|)$ edges.
2. We either find the desired cycle C, or we reduce the problem to Theorem (1.2) for some 3-connected graphs G_i, for which $|G_i| < |G|$ and each G_i contains a vertex which does not belong to any other G_i.
3. Replace each G_i with a 3-connected spanning subgraph of G_i with $O(|G_i|)$ edges.
4. Apply Lemma (4.1) to those G_i for which Theorem (1.2)(a) needs to be applied. Apply Lemma (4.2) to those G_i for which Theorem (1.2)(b) needs to be applied. Apply Lemma (5.1) to those G_i for which Theorem (1.2)(c) needs to be applied.
5. Repeat step 3 and step 4 for new 3-connected graphs.
6. In the final output, replace all virtual edges by appropriate paths in G to complete the desired cycle C.

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