The heterochromatic matchings in edge-colored bipartite graphs *

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Abstract

Let (G, C) be an edge-colored bipartite graph with bipartition (X, Y). A heterochromatic matching of G is such a matching in which no two edges have the same color. Let $N^c(S)$ denote a maximum color neighborhood of $S \subseteq V(G)$. We show that if $|N^c(S)| \geq |S|$ for all $S \subseteq X$, then G has a heterochromatic matching with cardinality at least $\lceil \frac{|X|}{3} \rceil$. We also obtain that if $|X| = |Y| = n$ and $|N^c(S)| \geq |S|$ for all $S \subseteq X$ or $S \subseteq Y$, then G has a heterochromatic matching with cardinality at least $\lceil \frac{3n-1}{8} \rceil$.

Keywords: heterochromatic matching, color-neighborhood

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1 Introduction and notation.

We use [3] for terminology and notations not defined here and consider simple undirected graphs only.

Let $G = (V, E)$ be a graph. An *edge-coloring* of G is a function $C : E \to N$(N is the set of nonnegative integers). If G is assigned such a coloring C, then we say that G is an *edge-colored graph*. Denote by (G, C) the graph G together with the coloring C and by $C(e)$ the *color* of the edge $e \in E$. For a subgraph H of G, let $C(H) = \{C(e) : e \in E(H)\}$.

A subgraph H of G is called *heterochromatic*, or *rainbow*, or $colorful$ if its any two edges have different colors. There are many publications studying heterochromatic subgraphs. Very often the subgraphs considered are paths, cycles, trees, etc. The heterochromatic hamiltonian cycle or path problems were studied by Hahn and Thomassen(see [9]), Rödl and Winkler(see [7]), Frieze and Reed, Albert,Frieze and Reed (see [1]), and H. Chen and X. Li (see [5]). For more references, see [2, 6, 9].

For an uncolored graph the following theorems are well known in matching theory and have been widely used.

Theorem 1 [10]. Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex of X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

Theorem 2. A bipartite graph G has a perfect matching if and only if $|N(S)| \geq |S|$ for all $S \subseteq V$ ([3]).

A matching is *heterochromatic* if any two edges of it have different colors. Unlike uncolored matchings for which the maximum matching problem is solvable in polynomial time (see [12]), the maximum heterochromatic matching problem is NP-complete, even for bipartite graphs (see [8]). Heterochromatic matchings have been studied for example in [11] in which by defining $N_c(S)$ (see the definition below) Hu and Li gave some sufficient conditions for the existence of perfect heterochromatic matchings in colored graphs. We have

Let (G, C) be a colored-graph. For a vertex v of G, let $CN(v) = \{C(e) : e$ is incident with $v\}$ and $CN(S) = \cup_{v \in S} CN(v)$ for $S \subseteq V$. For $S \in V(G)$, denote $N_c(S)$ as one of the minimum set(s) W satisfying $W \subseteq N(S) \backslash S$ and $[CN(S) \backslash C(G[S])] \subseteq CN(W)$.

Theorem 3[11]. Let (B, C) be a colored bipartite graph with bipartition X, Y. Then, B contains a heterochromatic matching that saturates every vertex in X, if $|N_c(S)| \geq |S|$, for all $S \subseteq X$.

Theorem 4[11]. A colored graph (G, C) has a perfect heterochromatic matching, if

(1) $o(G - S) \leq |S|$, where $o(G - S)$ denotes the number of odd components in the remaining graph $G - S$, and

(2) $|N_c(S)| \geq |S|$ for all $S \subseteq V$ such that $0 \leq |S| \leq \frac{|G|}{2}$ and $|N(S) \backslash S| \geq |S|$.

We define a maximum color neighborhood and study heterochromatic matchings in edge-colored bipartite graphs under a new condition related to maximum color-neighborhoods of subsets of vertices.

Let (G, C) be a colored bipartite graph with bipartition (X, Y). For a vertex set $S \subseteq X$ or Y, a *color neighbourhood* of S is defined as a set $T \subseteq N_G(S)$ such that there are $|T|$ edges between S and T that are adjacent to distinct vertices of T and have distinct colors. A *maximum color neighborhood* $N^c(S)$ is a color neighborhood of S and $|N^c(S)|$ is maximum. Given an S and a color neighborhood T, denote by $C(S, T)$ a set of $|T|$ distinct colors on the $|T|$ edges between S and distinct vertices of T. Note that there might be more than one such set $C(S, T)$. If it does not cause confusions, let $C(S, T)$ be a fixed color set in the following.

Let M be a heterochromatic matching of G, we denote $b_M = |\{e \mid e \in E(G - V(M))$ and $C(e) \in C(M)\}|$ and denote by $(X_M \cup Y_M)$ with $X_M \subseteq X$, $Y_M \subseteq Y$, the set of vertices that is incident with the edges in M.

The following main results are obtained in this paper.

Theorem 5. Let (G, C) be a colored bipartite graph with bipartition (X, Y) and $|N^c(S)| \geq |S|$ for all $S \subseteq X$, then G has a heterochromatic matching of cardinality at least $\lceil \frac{|X|}{3} \rceil$.

Theorem 6. Let (G, C) be a colored bipartite graph with bipartition (X, Y) and $|X| = |Y| = n$. If $|N^c(S)| \geq |S|$ for all $S \subseteq X$ or $S \subseteq Y$, then G has a heterochromatic matching of cardinality at least $\lceil \frac{3n-1}{8} \rceil$.

Under the conditions of Theorem 6, the following example shows that the best bound can not be better than $\lceil \frac{n}{2} \rceil$. Let $K^i_{2,2}$ be an edge-colored graph $K_{2,2}$ with two colors $\{c_{1,1}, c_{1,2}\}$ such that $N^c(v) = 2$ for every $v \in V(K_{2,2})$. Let $G_s = K^1_{2,2} \cup K^2_{2,2} \cup \cdots \cup K^s_{2,2}$. Then G_s is a colored bipartite graph with bipartition (X, Y) and $|X| = |Y| = 2s$. And $|N^c(S)| \geq |S|$ for all $S \subseteq X$ or $S \subseteq Y$. Clearly the cardinality of the maximum heterochromatic matching of G_s is $s = \lceil \frac{2s}{2} \rceil$. This example shows that the bound in Theorem 6 is not very far away from the best.

2 Proof of Theorem 5.

Let M be a maximum heterochromatic matching of G. Put $S = X - X_M$. Let $N^c(S)$ be a maximum color neighborhood of S. And write $N^c(S) = Y_P \cup Y_Q (Y_P \cap Y_Q = \phi)$, where $C(S, Y_P) \cap C(M) = \phi$ and $C(S, Y_Q) \subseteq C(M)$. Clearly $|Y_Q| \leq |M|$.

If $Y_P \not\subseteq Y_M$, then there is an edge $e \in E(X - X_M, Y - Y_M)$ and $C(e) \notin C(M)$. Hence $M + e$ is a heterochromatic matching with cardinality $|M|+1$, contrary to the maximality of M.

So $Y_P \subseteq Y_M$. Since $|N^c(S)| = |Y_P| + |Y_Q| \geq |S|$, it follows that $|M| = |Y_M| \geq |Y_P| \geq |S| - |Y_Q| \geq |X| - |M| - |M|$. This gives $|M| \geq \lceil \frac{|X|}{3} \rceil$. □

3 Proof of Theorem 6.

Let M be a maximum heterochromatic matching of G with $t := |M|$ such that b_M is maximum. Assume to the contrary that $t < \frac{3n-1}{8}$.

Let $C(M) = \{c_1, c_2, \cdots, c_t\}$. Put $S_x = X - X_M$ and $S_y = Y - Y_M$. Let $N^c(S_x)$ and $N^c(S_y)$ be a maximum color neighborhood of S_x and S_y respectively. Set $N^c(S_x) = Y_P \cup Y_Q (Y_P \cap Y_Q = \phi)$ where $C(S_x, Y_P) \cap C(M) = \phi$, $C(S_x, Y_Q) \subseteq C(M)$ and let $N^c(S_y) = X_P \cup X_Q (X_P \cap X_Q = \phi)$ where $C(S_y, X_P) \cap C(M) = \phi$, $C(S_y, X_P) \subseteq C(M)$. Clearly $|Y_Q| \leq t$, $|X_Q| \leq t$.

Claim 1. $Y_P \subseteq Y_M$, $X_P \subseteq X_M$.

Proof. Otherwise, there is an edge $e \in E(S_x, S_y)$ and $C(e) \notin C(M)$, then we can obtain a heterochromatic matching $M + e$ with cardinality $t + 1$, a contradiction. □

An *alternating* 4-cycle AC is a cycle $e_1e_2e_3e_4e_1$ such that $e_1 \in E(M)$, $e_3 \in E(G - V(M))$ and $C(e_1) = C(e_3)$, $C(e_2) = C(e_4) \notin C(M)$. Given two alternating 4-cycles $AC = e_1e_2e_3e_4e_1$ and $AC' = e_1'e_2'e_3'e_4'e_1'$, AC is *different* from AC', we mean that $e_1 \neq e_1'$ and $e_3 \neq e_3'$.

Claim 2. There exists an alternating 4-cycle in G.

Proof. Since $|N^c(S_x)| = |Y_P| + |Y_Q| \geq |S_x| = n - t$, it follows that $|Y_P| \geq n - t - |Y_Q| \geq n - 2t$. Similarly $|X_P| \geq n - t - |X_Q| \geq n - 2t$. Hence $|X_P| + |Y_P| \geq 2(n - 2t) = 2n - 4t > t = |X_M| = |Y_M|$. Then there exists an edge $xy \in E(M)$ such that x is adjacent with a vertex $y' \in S_y$, $C(xy') \notin C(M)$ and y is adjacent with a vertex $x' \in S_x$, $C(x'y) \notin C(M)$. Clearly $C(xy') = C(x'y)$, otherwise we obtain a new heterochromatic matching $M' = M \cup xy' \cup x'y - xy$ with $|M'| = |M| + 1 > M$, a contradiction.

Then there exists an edge $e \in E(G - V(M))$ such that $C(e) = C(xy)$. Otherwise $M'' = M \cup xy' - xy$ is a heterochromatic matching with $|M''| = |M|$ and $b_{M''} \geq b_M + 1$, contradicting with the choice of M. If $e \neq x'y'$, without loss of generality, assume that y' is not incident with e, then $M''' = M \cup e \cup xy' - xy$ is a heterochromatic matching with $|M'''| = |M| + 1$, a contradiction. □

Suppose that the maximum number of the vertex-disjoint pairwise different alternating 4-cycles in G is l. Clearly $1 \leq l \leq t$. Assume that the alternating 4-cycle AC_i has edges

$\{x_iy_i', y_i'x_i', x_i'y_i, y_ix_i\}$ and $C(xy) = C(x_i'y_i') = c_i \in C(M), C(xy_i') = C(x_i'y) = c_i' \notin C(M)$, where $xy \in E(M)$, and $y_i' \in S_y, x_i' \in S_x$.

Denote

$$\begin{aligned} X_L &= \{x_1', x_2', \cdots, x_l'\}, Y_L = \{y_1', y_2', \cdots, y_l'\}, \\ X_{M_l} &= \{x_1, x_2, \cdots, x_l\} \in X_M, \\ Y_{M_l} &= \{y_1, y_2, \cdots, y_l\} \in Y_M, \end{aligned}$$

where $\{x_1y_1, x_2y_2, \cdots, x_ly_l\} = E(M_l) \in E(M)$. We abbreviate $C(M_l) = \{c_1, c_2, \cdots, c_l\}$ and $C_L = \{c_1', c_2', \cdots, c_l'\}$, where $c_i' \notin C(M)$ and $c_i' \neq c_j'$ if $i \neq j$. Clearly $C(M) - C(M_l) = C(M - M_l)$.

Then put $S_x' = X - X_M - X_L$ and $S_y' = Y - Y_M - Y_L$. Let $N^c(S_x')$ and $N^c(S_y')$ be a maximum color neighborhood of S_x' and S_y' respectively. Write $N^c(S_x') = Y_P' \cup Y_Q'(Y_P' \cap Y_Q' = \phi)$, where $C(S_x', Y_P') \cap C(M - M_l) = \phi$ and $C(S_x', Y_Q') \subseteq C(M - M_l)$. And let $N^c(S_y') = X_P' \cup X_Q'(X_P' \cap X_Q' = \phi)$, where $C(S_y', X_P') \cap C(M - M_l) = \phi$ and $C(S_y', X_Q') \subseteq C(M - M_l)$. Clearly $|Y_Q'| \leq t - l$ and $|X_Q'| \leq t - l$.

Claim 3. $Y_P' \in Y_M - Y_{M_l}$.

Proof. By contradiction. Then there exists an edge $e \in [S_x', Y - (Y_M - Y_{M_l})]$ with $C(e) \notin C(M - M_l)$.

We distinguish the following three cases.

Case 1. $e \in E(S_x', S_y')$. Let

$$M^1 = \begin{cases} M \cup e & C(e) \notin C(M_l); \\ M \cup e \cup x_iy_i' - x_iy_i & C(e) \in C(M_l), \text{w.l.o.g, suppose } C(e) = c_i. \end{cases}$$

Then we get a heterochromatic matching M^1 with $|M^1| > |M|$, a contradiction.

Case 2. $e \in E(S_x', Y_{M_l})$. Without loss of generality, suppose e is incident with y_i. Let

$$M^1 = \begin{cases} M \cup e \cup x_iy_i' - x_iy_i & C(e) \notin C(M_l) \cup C_L; \\ M \cup e \cup x_i'y_i' - x_iy_i & C(e) \in C_L; \\ M \cup e \cup x_iy_i' - x_iy_i & C(e) = c_i \in C(M_l); \\ M \cup e \cup x_iy_i' \cup x_jy_j' - x_iy_i - x_jy_j & C(e) = c_j \in C(M_l) \text{ and } c_j \neq c_i. \end{cases}$$

Then we obtain a heterochromatic matching M^1 and $|M^1| > |M|$, a contradiction.

Case 3. $e \in E(S_x', Y_L)$. Without loss of generality, suppose e is incident with y_i'. Let

$$M^1 = \begin{cases} M \cup e & C(e) \notin C(M_l); \\ M \cup e \cup x_i'y_i - x_iy_i & C(e) = c_i \in C(M_l); \\ M \cup e \cup x_jy_j' - x_jy_j & C(e) = c_j \in C(M_l) \text{ and } c_j \neq c_i. \end{cases}$$

Then we obtain a heterochromatic matching M^1 and $|M^1| > |M|$, a contradiction.

This completes the proof of the claim. $\square$

Since $|N^c(S_x^{'})| = |Y_P^{'}| + |Y_Q^{'}| \geq |S_x^{'}|$, it follows that $|Y_P^{'}| \geq n - t - l - |Y_Q^{'}| \geq n - t - l - (t - l) \geq n - 2t$.

Similarly it holds that $X_P^{'} \in X_M - X_{M_l}$ and hence $|X_P^{'}| \geq n - 2t$.

Since $Y_P^{'} \in Y_M - Y_{M_l}$ and $X_P^{'} \in X_M - X_{M_l}$, it holds that

$$2(t - l) = |X_M - X_{M_l}| + |Y_M - Y_{M_l}| \geq |X_P^{'}| + |Y_P^{'}| \geq 2n - 4t.$$

That is

$$l \leq 3t - n.$$

Then

$$l \leq 3t - n \leq 3 \times \frac{3n-1}{8} - n \leq \frac{n-3}{8}.$$

If follows that

$$\begin{aligned}
&|X_P^{'}| + |Y_P^{'}| - |X_M - X_{M_l}| \\
&\geq 2n - 4t - (t - l) \\
&\geq 2n - 5t + l. \\
&\geq 2n - 5 \times \frac{3n-1}{8} + l \\
&\geq \frac{n-3}{8} + l + 1 \\
&\geq 2l + 1.
\end{aligned}$$

So there exists an edge $x_0 y_0 \in E(M - M_l)$, where x_0 is adjacent with a vertex $y_0^{'} \in S_y^{'}$ and y_0 is adjacent with a vertex $x_0^{'} \in S_x^{'}$ such that at least one of $C(x_0 y_0^{'}), C(x_0^{'} y_0)$ is not in $C(M_l) \cup C_L$. Without loss of generality, suppose $C(x_0 y_0^{'}) \notin C(M_l) \cup C_L$. Note that $C(x_0^{'} y_0) \notin C(M - M_l)$.

If $C(x_0^{'} y_0) \in C(M_l)$, suppose $C(x_0^{'} y_0) = c_i$. Then $M^1 = M \cup x_0 y_0^{'} \cup x_0^{'} y_0 \cup x_i y_i^{'} - x_i y_i - x_0 y_0$ is a heterochromatic matching and $|M^1| > |M|$, a contradiction with the maximality of M.

If $C(x_0^{'} y_0) \in C_L$ or $C(x_0^{'} y_0) \notin C(M_l) \cup C_L$ and $C(x_0^{'} y_0) \neq C(x_0 y_0^{'})$. Then $M^1 = M \cup x_0 y_0^{'} \cup x_0^{'} y_0 - x_0 y_0$ is a heterochromatic matching and $|M^1| > |M|$, a contradiction.

If $C(x_0^{'} y_0) = C(x_0 y_0^{'})$. By the same proof in Claim 2, it holds that $C(x_0 y_0) = C(x_0^{'} y_0^{'})$. Then we obtain an alternating 4-cycle with edges $\{x_0 y_0, x_0^{'} y_0, x_0^{'} y_0^{'}, x_0 y_0^{'}\}$ and $C(x_0 y_0) = C(x_0^{'} y_0^{'}), C(x_0^{'} y_0) = C(x_0 y_0^{'}) \notin C(M) \cup C_L$, where $x_0 y_0 \in E(M - M_l)$ and $y_0^{'} \in S_y^{'}, x_0^{'} \in S_x^{'}$. So the number of vertex-disjoint pairwise different alternating 4-cycles is at least $l + 1$, a contradiction.

The proof of Theorem 6 is complete. $\square$

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